Istituto di Analisi dei Sistemi ed Informatica "Antonio Ruberti"
Consiglio Nazionale delle Ricerche

colloquia@iasi

La complessa dinamica del modello di Gurtin e MacCamy

"La mente è come un paracadute.
Funziona solo se si apre"
A. Einstein

Mimmo Iannelli

Università di Trento

Outline of the talk

- **A chapter from the theory of age-structured populations**:
  - Gurtin-McCamy model
  - Structured logistic growth
  - Juveniles-adults dynamics
- **Some recent results**:
  - A numerical method for the analysis
  - Exploration of the models

Outline of the talk

- **A collaboration with:**
  - F. Milner, Arizona University, Tempe, Mathematics Department
  - C. Cusulin, Vienna University, Mathematics Department
  - S. Maset, Trieste University, Mathematics Department
  - D. Breda and R. Vermiglio, Udine University, Mathematics Department
  - + . . .

focused on numerical treatment of the Gurtin-McCamy model

Gurtin-MacCamy

The Gurtin-MacCamy system

$$\frac{\partial p}{\partial t}(a,t) + \frac{\partial p}{\partial a}(a,t) + \mu(a, S_1(t), \ldots, S_n(t))p(a,t) = 0,$$

$$p(0,t) = \int_0^{a_\dagger} \beta(a, S_1(t), \ldots, S_n(t))p(a,t)\, da,$$

$$S_i(t) = \int_0^{a_\dagger} \gamma_i(a)p(a,t)\, da, \quad i = 1, \ldots, n,$$

$$p(a,0) = p_0(a).$$

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age-distribution

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mortality

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fertility

The Gurtin-MacCamy system

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The basic ingredients

- $p(a,t)$ **age-distribution of the population**
- $S_i(t) = \int_0^{a_\dagger} \gamma_i(a) p(a,t)\, da$ **weighted selection of the population**
- $\beta(a, S_1(t), \dots, S_n(t))$ **fertility**
- $\mu(a, S_1(t), \dots, S_n(t))$ **mortality**

Structured logistic growth

Logistic growth

- **one single size:** $S(t) = \int_0^{a_\dagger} \gamma(a) p(a,t)\,da$
- **fertility:** $\beta(a,x) = R_0 \beta_0(a) \Phi(x)$
- **mortality:** $\mu(a,x) = \mu_0(a) + m(a)\Psi(x)$

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- $\beta_0(a)$ **and** $m(a)$ **describe how crowding impacts on different ages**

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- $R_0 =$ **basic reproduction number**

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- $\beta_0(a)$ **and** $m(a)$ **describe how crowding impacts on different ages**
- $R_0 =$ **the number of off-springs produced during the whole life**

Structured logistic growth

The search for a stationary state $p^*(a)$

- $$\frac{\partial p^*}{\partial a}(a) + \mu(a, S^*)p^*(a) = 0$$

  $$\Longrightarrow p^*(a) = \Pi(a, S^*)p^*(0), \quad \Pi(a, S) = e^{-\int_0^a \mu(\sigma, S)d\sigma}$$

- $$1 = \int_0^{a_\dagger} \beta(a, S^*)\Pi(a, S^*)da, \quad p^*(0) = \frac{S^*}{\int_0^{a_\dagger} \gamma_i(a)\Pi(a, S^*)da}$$

Structured logistic growth

$$1 = \int_0^{a_\dagger} \beta(a, S^*)\Pi(a, S^*)da$$

Structured logistic growth

$$1 = R_0 \Phi(S^*) \int_0^{a_\dagger} \beta_0(a) e^{-\int_0^a \mu_0(\sigma) d\sigma} e^{-\Psi(S^*) \int_0^a m(\sigma) d\sigma} da$$

Structured logistic growth

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↑

decreasing as a function of S^*

Structured logistic growth

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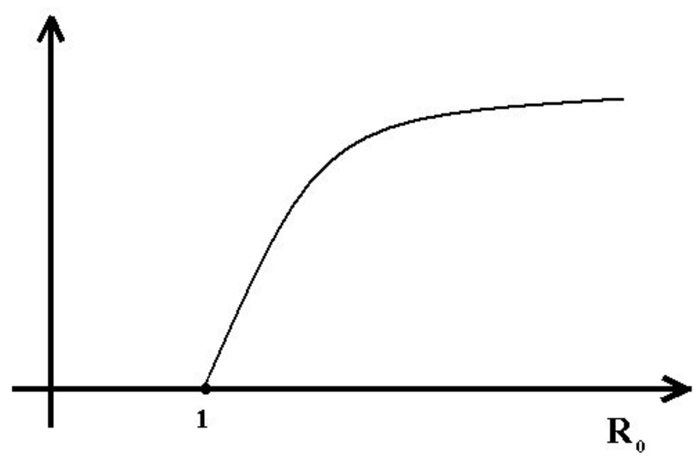
bifurcation graph



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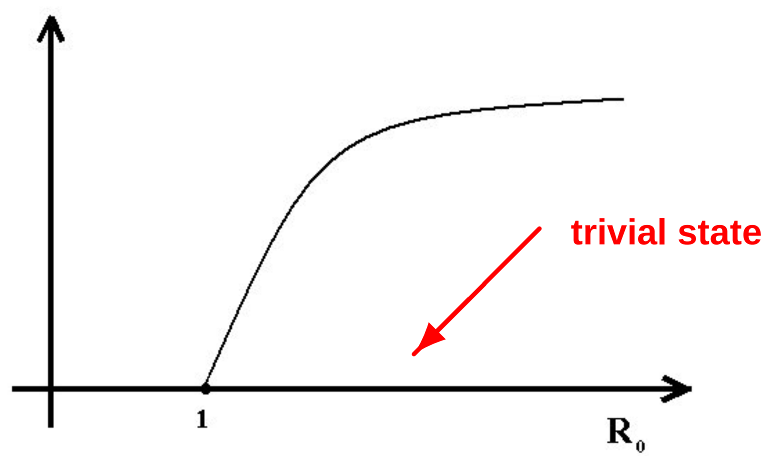

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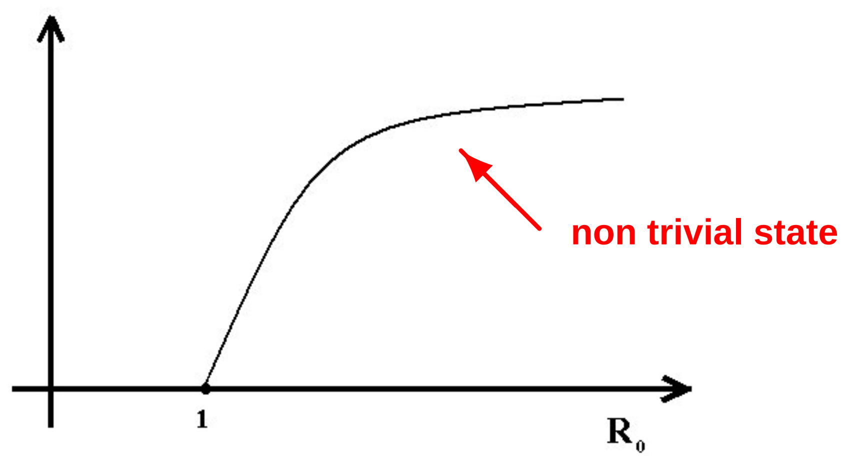

Structured logistic growth

Stability by linearization at $p^*(a)$

deviation from the steady state $v(a,t) = p(a,t) - p^*(a)$

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\begin{cases}
\dfrac{\partial v}{\partial t}(a,t) + \dfrac{\partial v}{\partial a}(a,t) + \mu(a,S^*)v(a,t) + \\
\qquad + p^*(a)\dfrac{\partial \mu}{\partial S}(a,S^*)\displaystyle\int_0^{a_\dagger}\gamma(a)v(a,t)da = 0 \\
v(0,t) = \displaystyle\int_0^{a_\dagger}\beta(a,S^*)v(a,t)da + \\
\qquad + \displaystyle\int_0^{a_\dagger}p^*(\sigma)\dfrac{\partial \beta}{\partial S}(\sigma,S^*)d\sigma\displaystyle\int_0^{a_\dagger}\gamma(a)v(a,t)da
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Structured logistic growth

Characteristic equation

$$\det \begin{vmatrix} 1 - \widehat{K}_{00}(\lambda) & -b^* - \widehat{K}_{01}(\lambda) \\ -\widehat{K}_{10}(\lambda) & 1 - \widehat{K}_{11}(\lambda) \end{vmatrix} = 0$$

$$K_{00}(t) = \beta(t, S^*)\Pi(t, S^*) \quad K_{10}(t) = \gamma(t)\Pi(t, S^*)$$

$$K_{01}(t) = -p^*(0) \int_0^{a_\dagger} \frac{\partial \mu}{\partial S}(\sigma, S^*) K_{00}(t + \sigma) d\sigma$$

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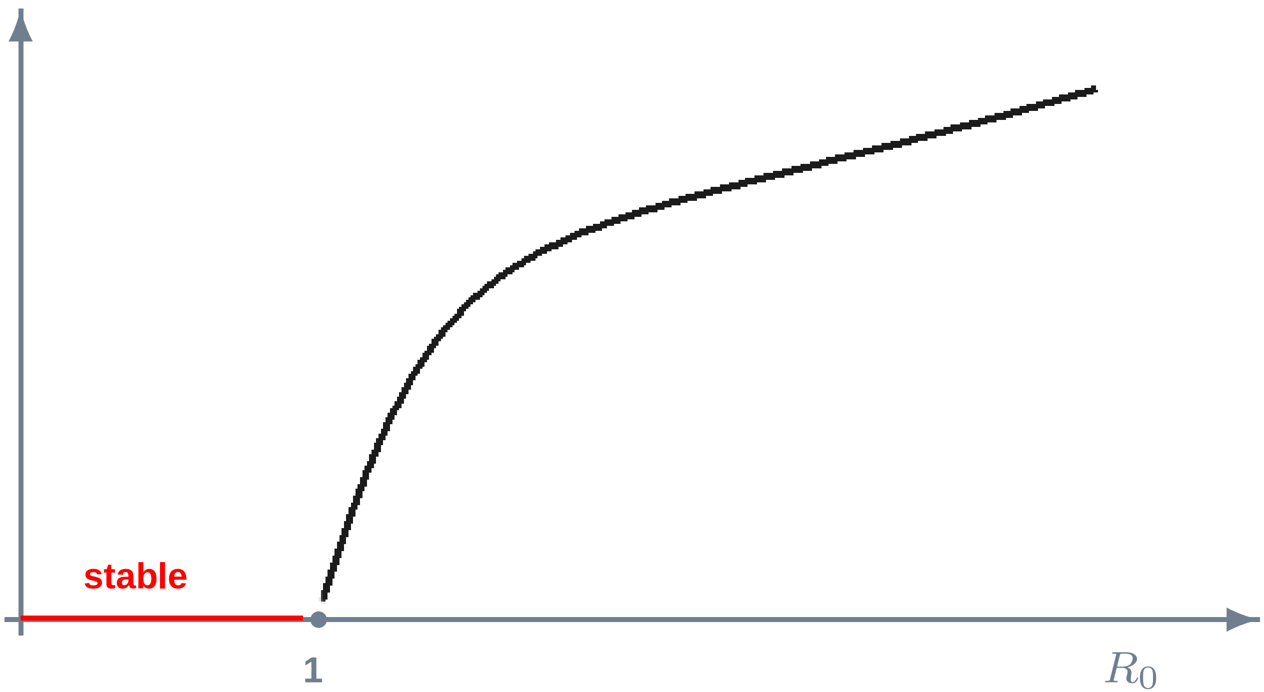
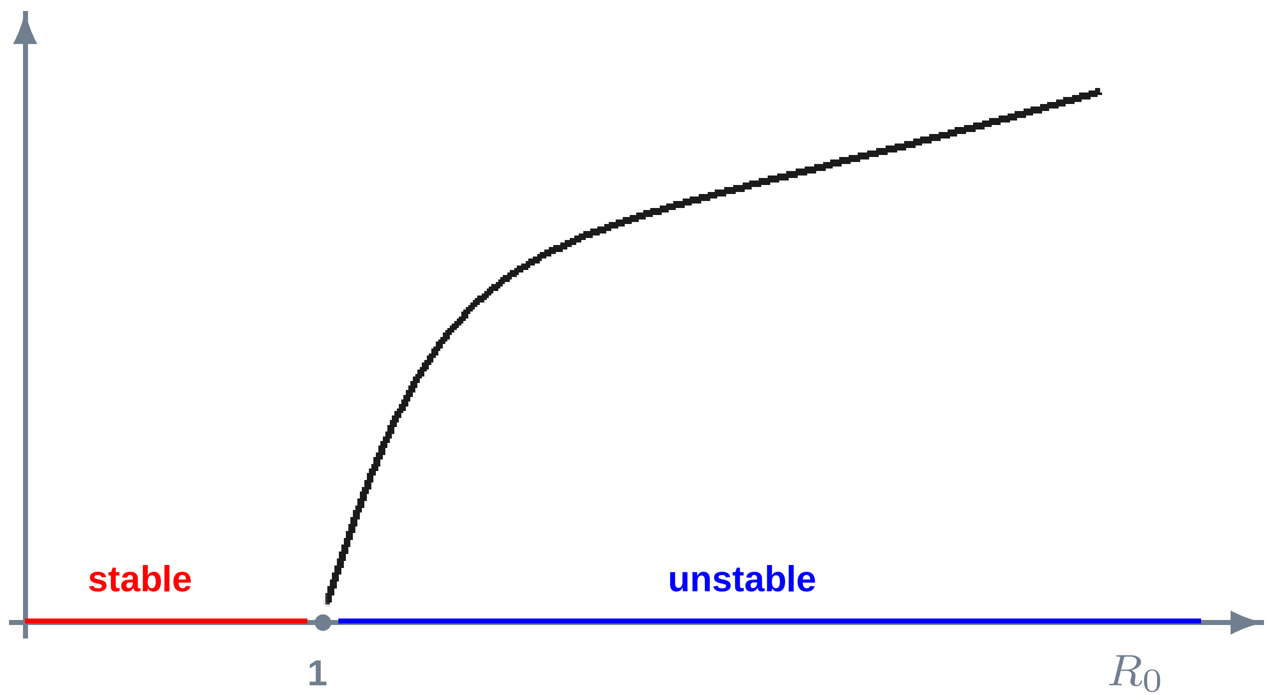
If all characteristic roots have negative real part then the steady state $p^*(a)$ is stable.
If at least one of the characteristic roots has a positive real part then the state is unstable.

Structured logistic growth



Structured logistic growth



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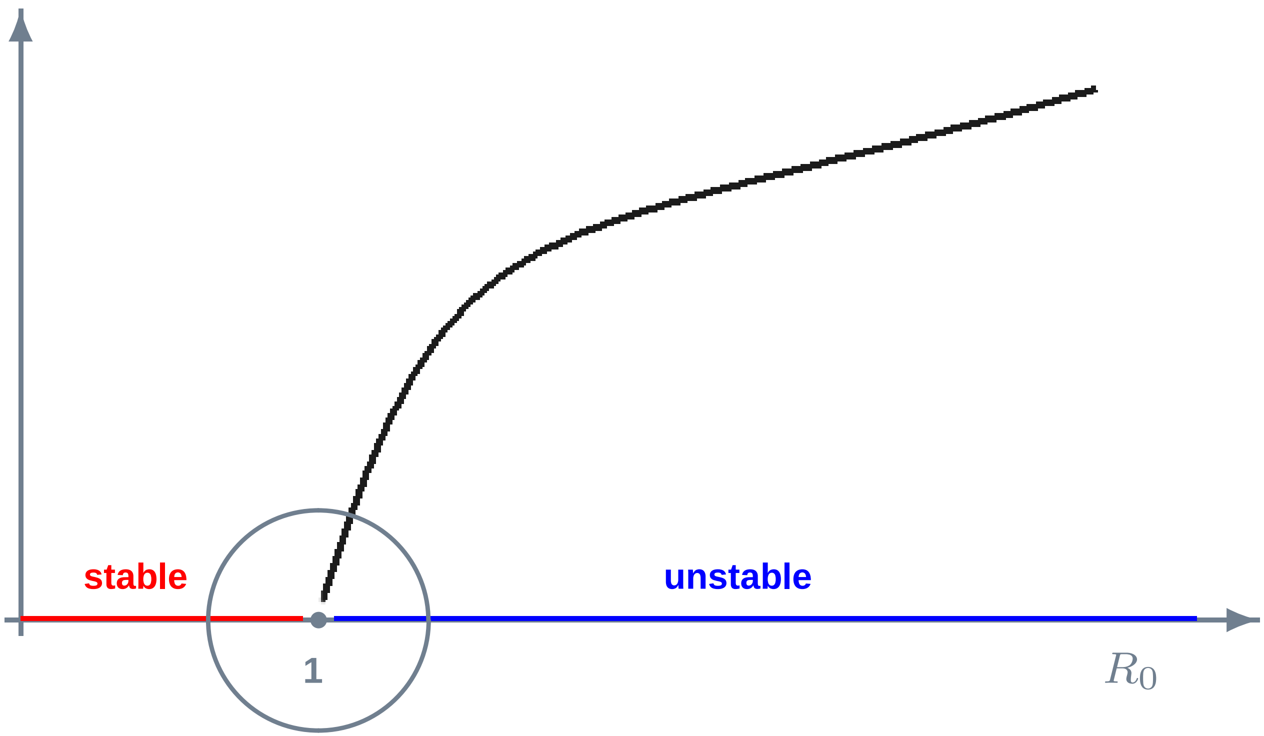

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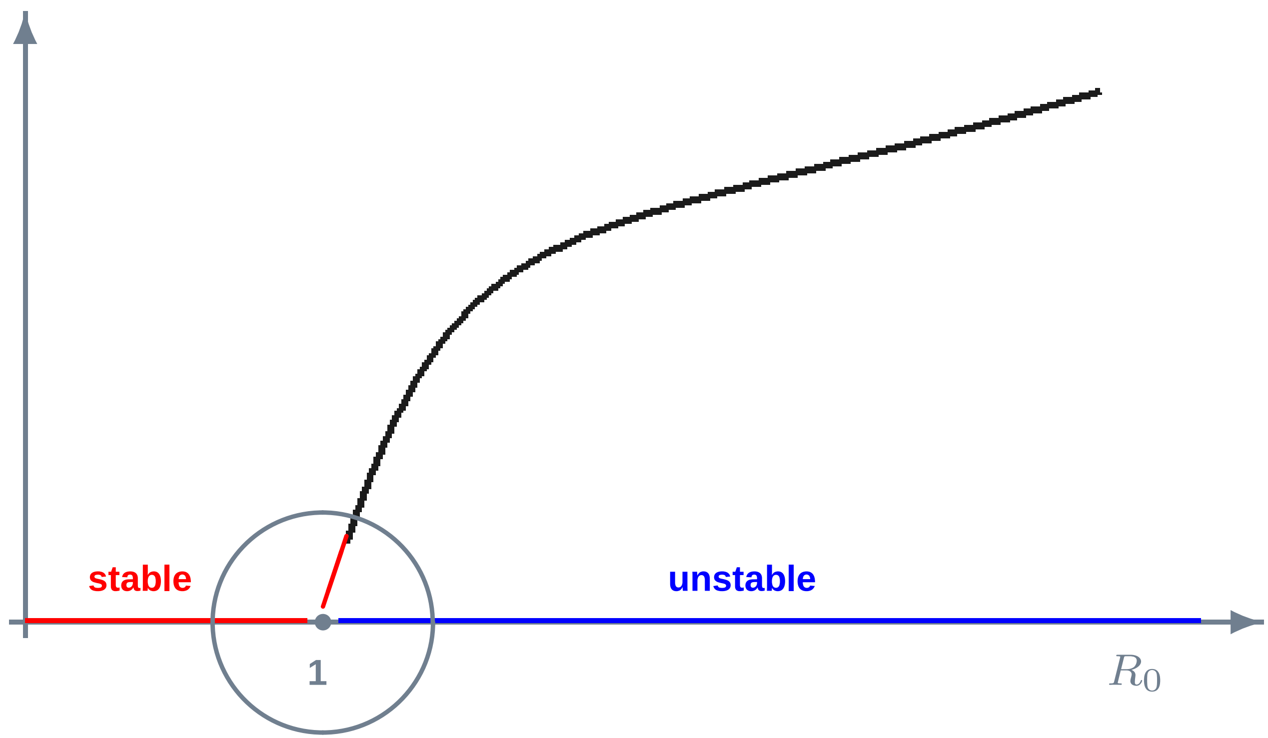

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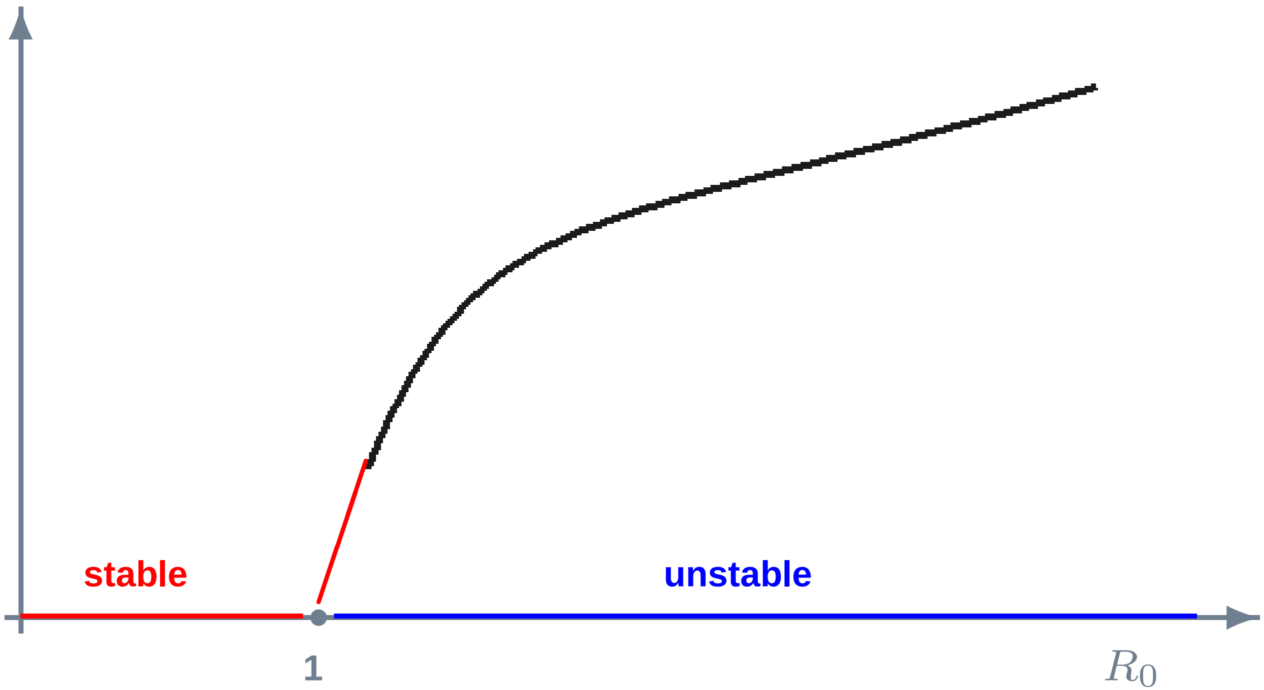

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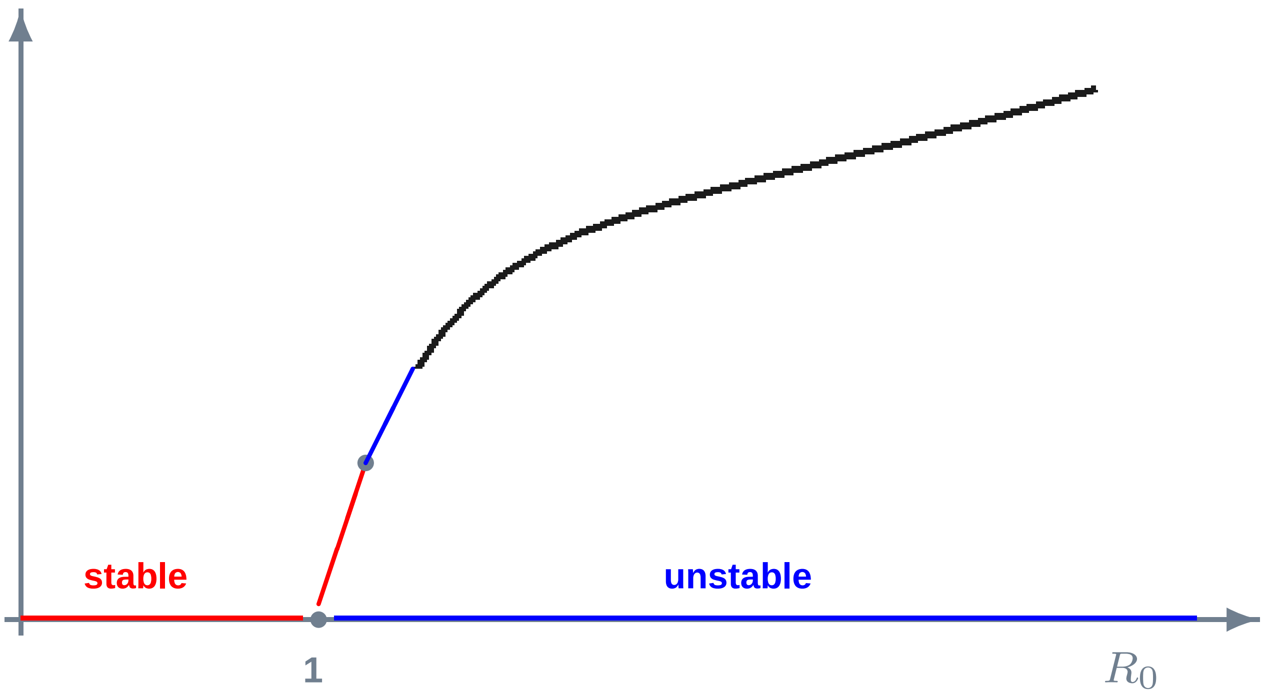

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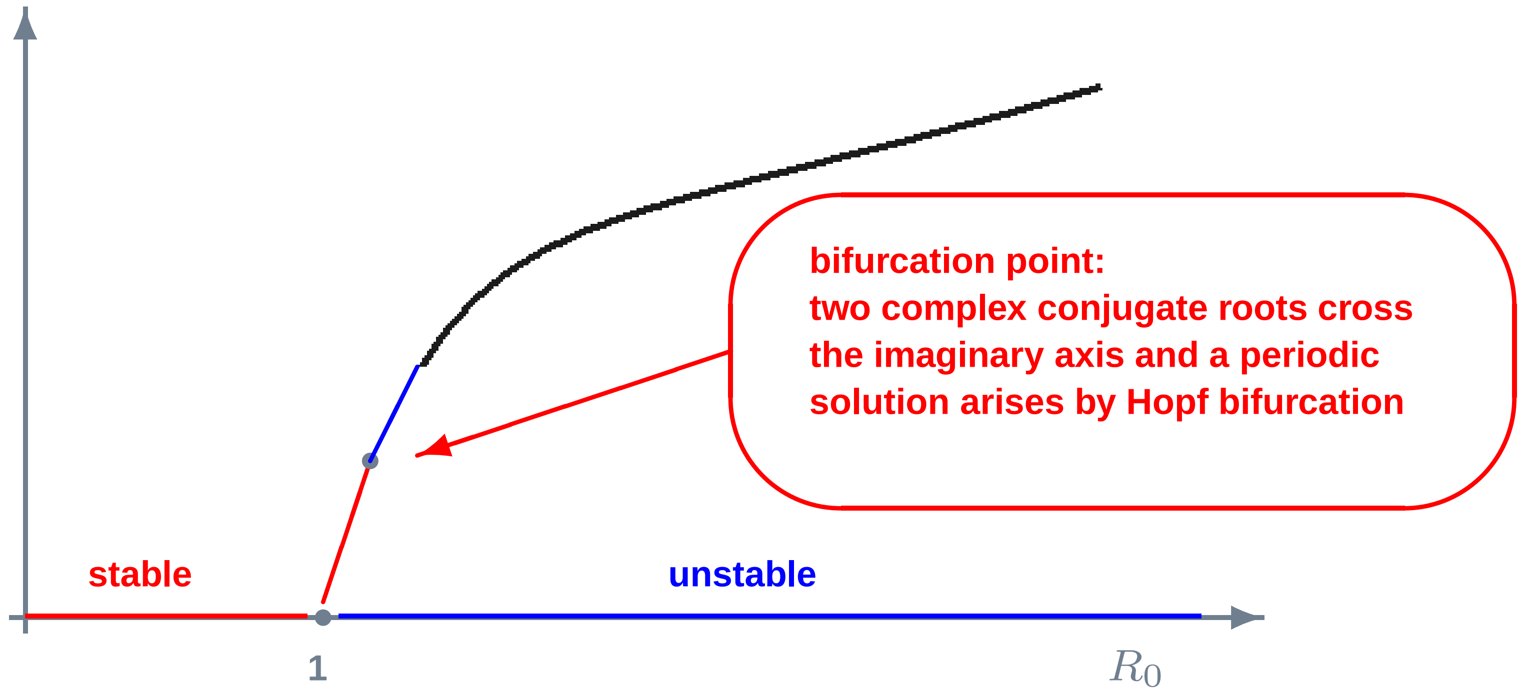

Juveniles-adult dynamics

The example of juveniles-adults dynamics

two selected groups

$$
\begin{cases}
J(t) = \displaystyle\int_0^{a^*} p(a,t)\, da, & \textit{juveniles} \\[2ex]
A(t) = \displaystyle\int_{a^*}^{a_\dagger} p(a,t)\, da, & \textit{adults}
\end{cases}
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a^* **is the maturation age**

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- **separated niches**
- **Allee effect**
- **cannibalism**

Juveniles-adult dynamics

The case of two different ecological niches

$$\beta(a, J, A) = R_0 b \chi_{[a^*, a_\dagger]}(a) e^{-(b_1 J + b_2 A)}$$

$$\mu(a, J, A) = \mu_0(a) + m_1 \chi_{[0, a^*]}(a) J + m_2 \chi_{[a^*, a_\dagger]}(a) A$$

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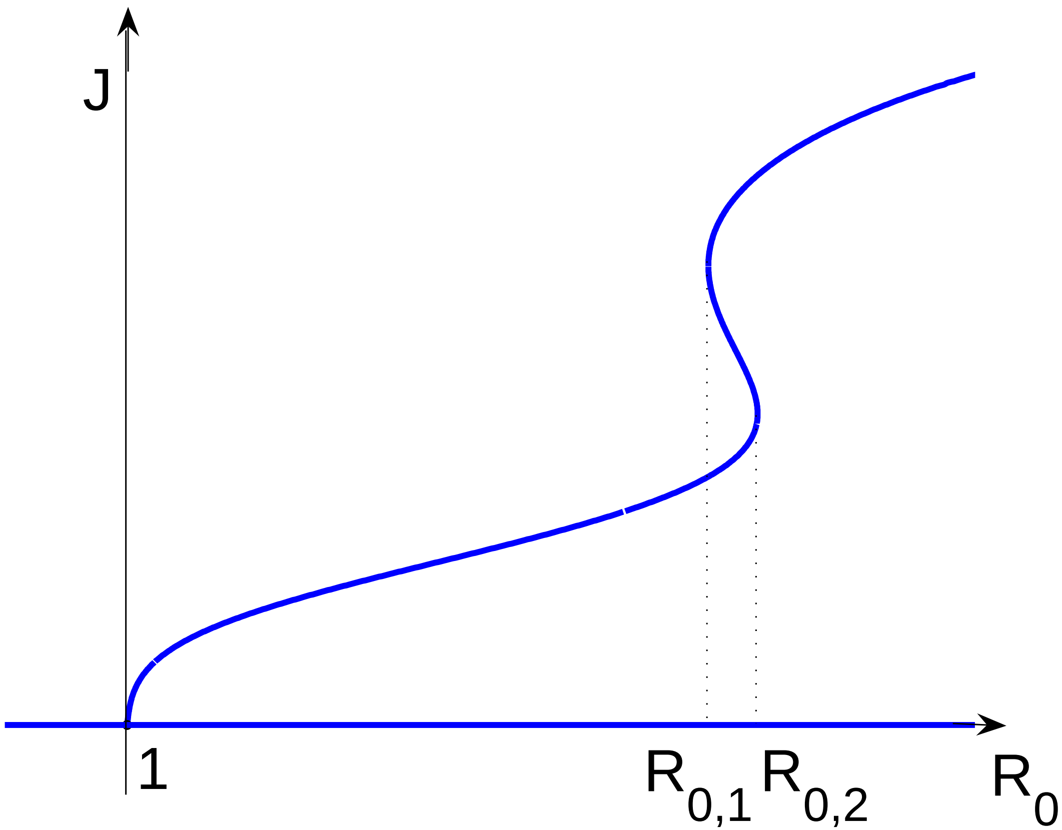

Juveniles-adult dynamics

The Allee effect

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$$\mu(a,J,A) = \mu_0(a) + m_1 \chi_{[0,a^*]}(a) J + m_2 \chi_{[a^*,a_\dagger]}(a) A + \\ - \left[\theta_1 \mu_0(a) + \theta_2 m_1 J\right] \chi_{[0,a^*]}(a) \alpha(A)$$

A positive effect (a decrease of mortality) on juveniles, due to adults presence

Juveniles-adult dynamics

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The Allee effect

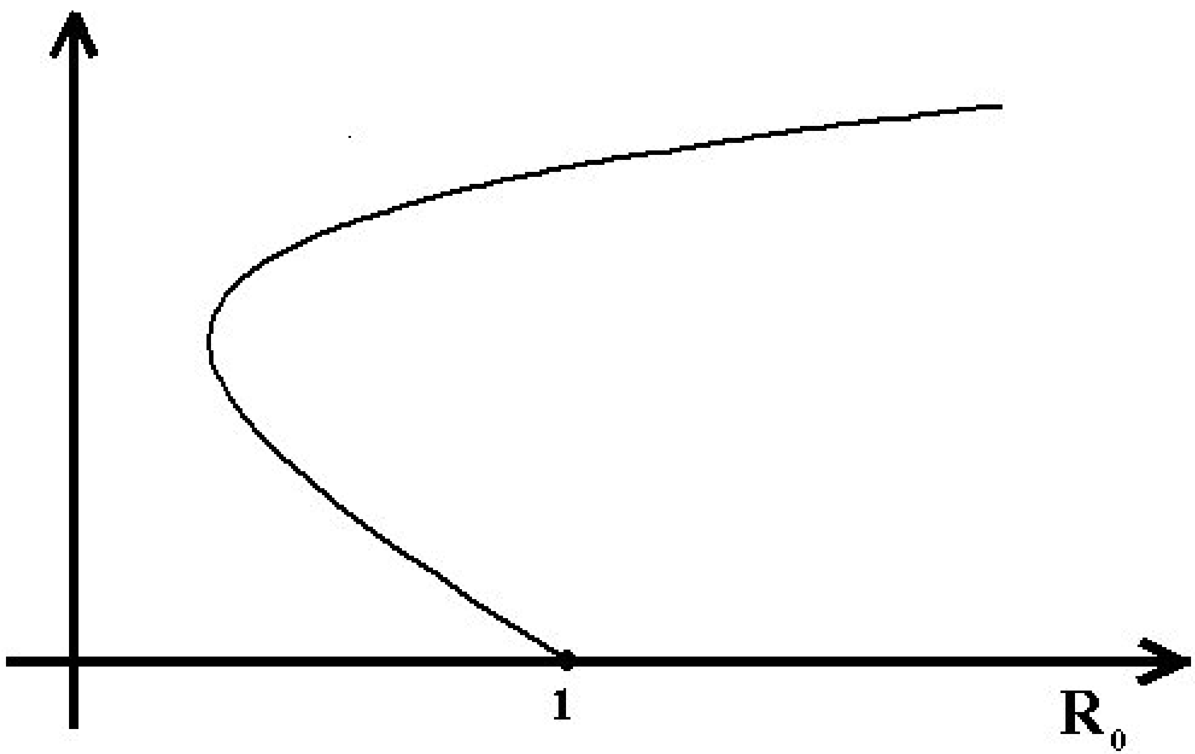
$$\beta(a,J,A) = R_0 b \chi_{[a^*,a_\dagger]}(a) e^{-(b_1 J + b_2 A)}$$

$$\mu(a,J,A) = \mu_0(a) + m_1 \chi_{[0,a^*]}(a) J + m_2 \chi_{[a^*,a_\dagger]}(a) A + \\ - \left[\theta_1 \mu_0(a) + \theta_2 m_1 J\right] \chi_{[0,a^*]}(a) \alpha(A)$$

A positive effect (a decrease of mortality) on juveniles, due to adults presence

Juveniles-adult dynamics

The Allee effect



Juveniles-adult dynamics

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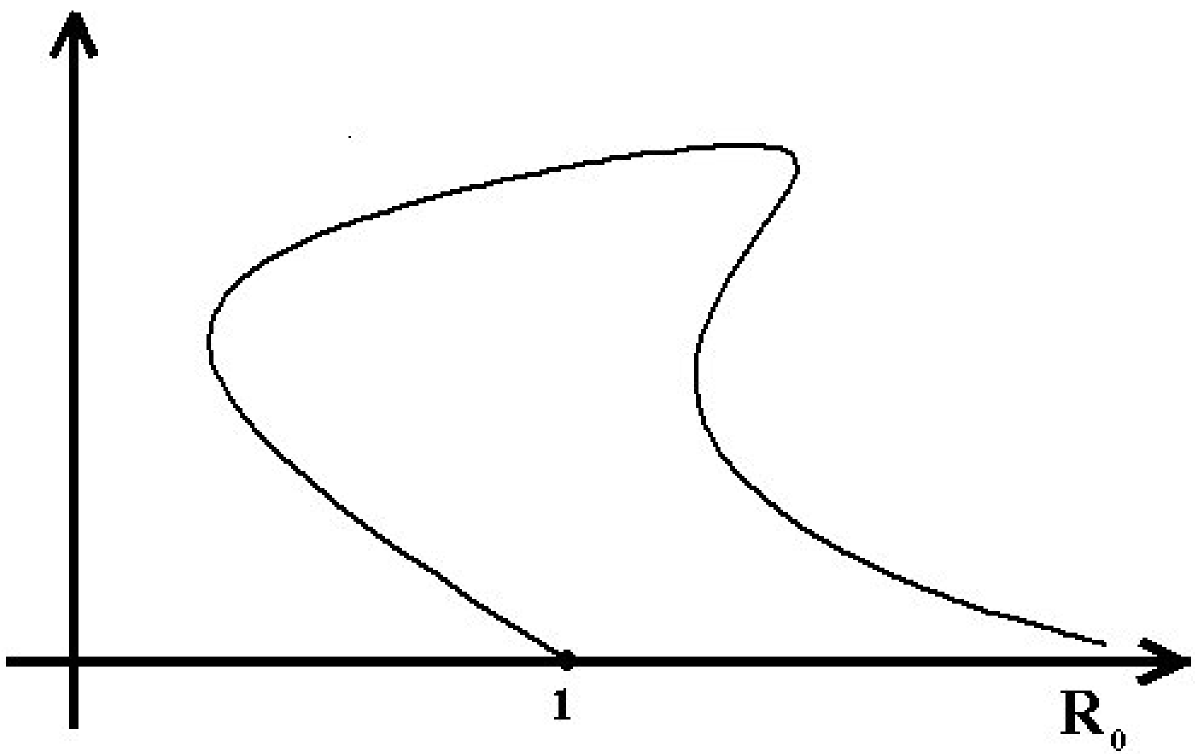

Juveniles-adult dynamics

Cannibalism (of adults on juveniles)

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A negative effect (increase of mortality) on juveniles, due to predation by adults, regulated by a functional response of Holling type

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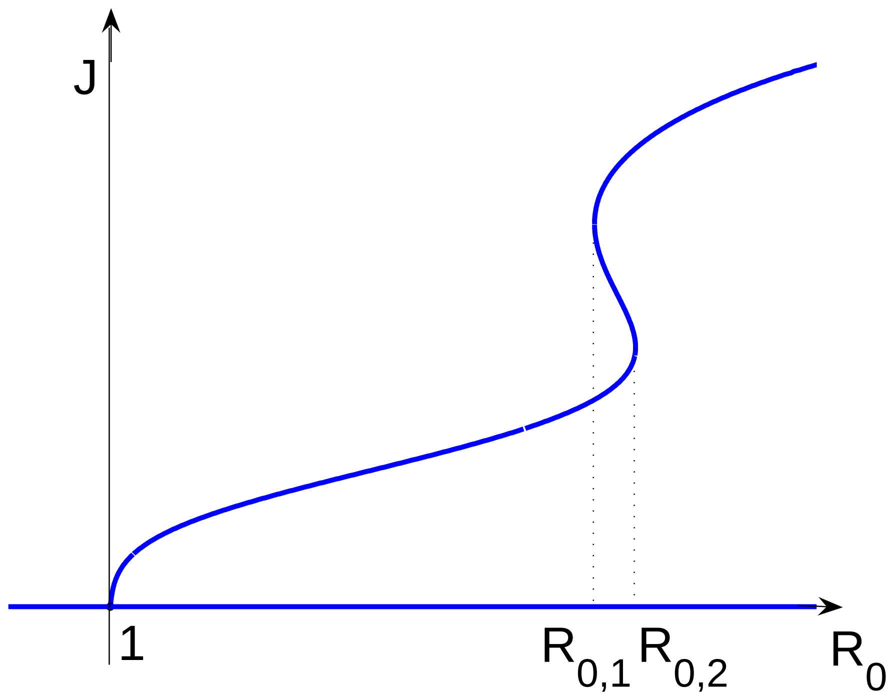

A numerical method for stability analysis

The starting point: linearization at a steady state $p^*(a)$

$$
\begin{cases}
\dfrac{\partial v}{\partial t}(a,t) + \dfrac{\partial v}{\partial a}(a,t) + \mu(a,S^*)v(a,t) + \\
\qquad\qquad + p^*(a)\dfrac{\partial \mu}{\partial S}(a,S^*)\displaystyle\int_0^{a_\dagger}\gamma(a)v(a,t)da = 0 \\
v(0,t) = \displaystyle\int_0^{a_\dagger}\beta(a,S^*)v(a,t)da + \\
\qquad\qquad + \displaystyle\int_0^{a_\dagger} p^*(\sigma)\dfrac{\partial \beta}{\partial S}(\sigma,S^*)d\sigma \int_0^{a_\dagger}\gamma(a)v(a,t)da
\end{cases}
$$

A numerical method for stability analysis

The starting point: the resulting characteristic equation

$$\det \begin{vmatrix} 1-\widehat{K}_{00}(\lambda) & -b^{*}-\widehat{K}_{01}(\lambda) \\ \\ -\widehat{K}_{10}(\lambda) & 1-\widehat{K}_{11}(\lambda) \end{vmatrix} = 0$$

The goal: to approximate the roots

A numerical method for stability analysis

The starting point: **the resulting characteristic equation**

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The goal: **to approximate the roots**

- reformulation of the linearization as an abstract Cauchy problem
- discrete approximation of the generator
- computation of the spectrum of the approximated generator

A numerical method for stability analysis

Reformulation as an abstract Cauchy problem

$$
\begin{cases}
\dfrac{\partial v}{\partial t}(a,t) + \dfrac{\partial v}{\partial a}(a,t) + \mu(a,S^*)v(a,t) + \\
\qquad\qquad + p^*(a)\dfrac{\partial \mu}{\partial S}(a,S^*) \displaystyle\int_0^{a_\dagger} \gamma(a)v(a,t)da = 0 \\
v(0,t) = \displaystyle\int_0^{a_\dagger} \beta(a,S^*)v(a,t)da + \\
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v(a,0) = v_0(a)
\end{cases}
$$

A numerical method for stability analysis

Reformulation as an abstract Cauchy problem

$$
\begin{cases}
\dfrac{\partial v}{\partial t}(a,t) + \dfrac{\partial v}{\partial a}(a,t) + \mu(a,S^*)v(a,t) + \\
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v(a,0) = v_0(a)
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$$

A numerical method for stability analysis

Reformulation as an abstract Cauchy problem

$$
\begin{cases}
\dfrac{\partial v}{\partial t}(a,t) + \dfrac{\partial v}{\partial a}(a,t) + (\mathcal{H}v(\cdot,t))(a) = 0 \\
v(0,t) = K_0 v(\cdot,t) \\
v(a,0) = v_0(a)
\end{cases}
$$

A numerical method for stability analysis

Reformulation as an abstract Cauchy problem

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\begin{cases}
\dfrac{\partial v}{\partial t}(a,t) + \dfrac{\partial v}{\partial a}(a,t) + (\mathcal{H}v(\cdot,t))(a) = 0 \\
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$$

$$
\begin{cases}
\dfrac{d}{dt}u(t) = \mathcal{A}u(t),\ t \geq 0, \\
u(0) = u_0 \in X.
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$$

A numerical method for stability analysis

Reformulation as an abstract Cauchy problem

$$\begin{cases} \dfrac{\partial v}{\partial t}(a,t) + \dfrac{\partial v}{\partial a}(a,t) + (\mathcal{H}v(\cdot,t))(a) = 0 \\ v(0,t) = K_0 v(\cdot,t) \\ v(a,0) = v_0(a) \end{cases}$$

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$$L^1([0, a_\dagger], \mathbf{R})$$

A numerical method for stability analysis

Reformulation as an abstract Cauchy problem

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$$u(t) \equiv v(\cdot,t)$$

$$u_0 \equiv v_0(\cdot)$$

A numerical method for stability analysis

Reformulation as an abstract Cauchy problem

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$$\begin{cases} \dfrac{d}{dt}u(t) = \mathcal{A}u(t), \ t \geq \\ u(0) = u_0 \in X. \end{cases}$$

$$\mathcal{A}\varphi = -\varphi' - \mathcal{H}\varphi$$

$$D(\mathcal{A}) = \{\varphi \in X \mid \varphi' \in X, \ \varphi(0) = K_0\varphi\}$$

A numerical method for stability analysis

Discrete approximation of the generator

Recent results: a numerical method for stability an

Discrete approximation of the generator

- $[0, a_\dagger] \rightsquigarrow \Omega_N = \left\{\theta_i = \frac{a_\dagger}{2}\cos\left(\frac{N-i}{N}\pi\right) + \frac{a_\dagger}{2} \,:\, i = 0, \ldots, N\right\}$

A numerical method for stability analysis

Discrete approximation of the generator

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- $\varphi \in X \rightsquigarrow y \in X_N \cong \mathbb{C}^N$
  - **set** $y_i = \varphi(\theta_i)$**,** $i = 1, \ldots, N$

A numerical method for stability analysis

Discrete approximation of the generator

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- $\varphi \in X \rightsquigarrow y \in X_N \cong \mathbb{C}^N$
  - **set** $y_i = \varphi(\theta_i)$, $i = 1, \ldots, N$
- $\mathcal{A} \rightsquigarrow \mathcal{A}_N : X_N \to X_N$,
  - **build** φ_N **an interpolating polynomial through** y_i **such that** $\varphi_N(0) = K_0\varphi_N$
  - **compute** $z_i = -\varphi_N'(\theta_i) - (\mathcal{H}\varphi_N)(\theta_i)$, $i = 1, \ldots, N$
  - **set** $(\mathcal{A}_N y)_i = z_i$

A numerical method for stability analysis

Discrete approximation of the generator

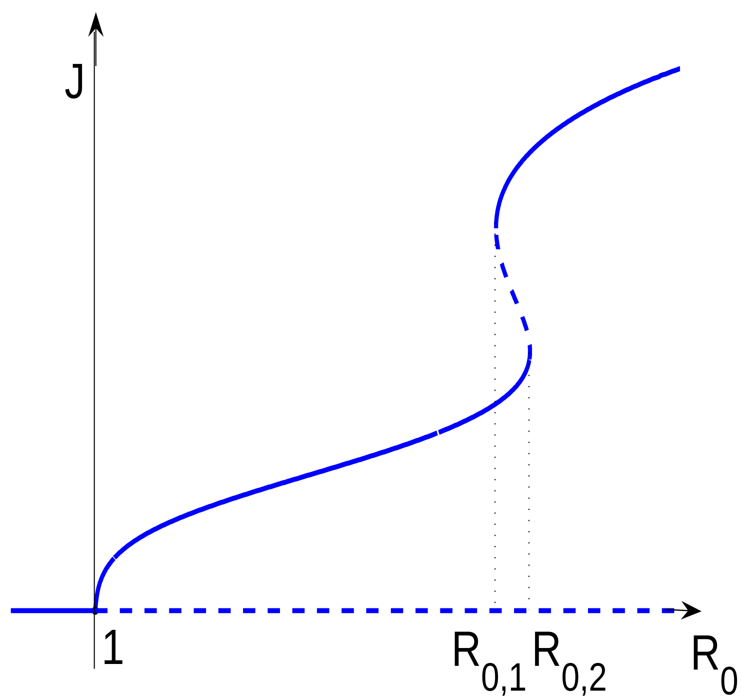
the eigenvalues of $\mathcal{A}_N$ approximate the eigenvalues of $\mathcal{A}$

If λ is an eigenvalue of $\mathcal{A}$ with multiplicity ν, then for N sufficiently large, $\mathcal{A}_N$ has exactly ν eigenvalues λ_i, $i = 1, \ldots, \nu$, such that

$$\max_{1 \le i \le \nu} |\lambda - \lambda_i| \le \left(\frac{C_2}{C_3} \right)^{1/\nu} \left(\varepsilon_N + \frac{1}{\sqrt{N}} \left(\frac{C_1}{N} \right)^N \right)^{1/\nu}$$

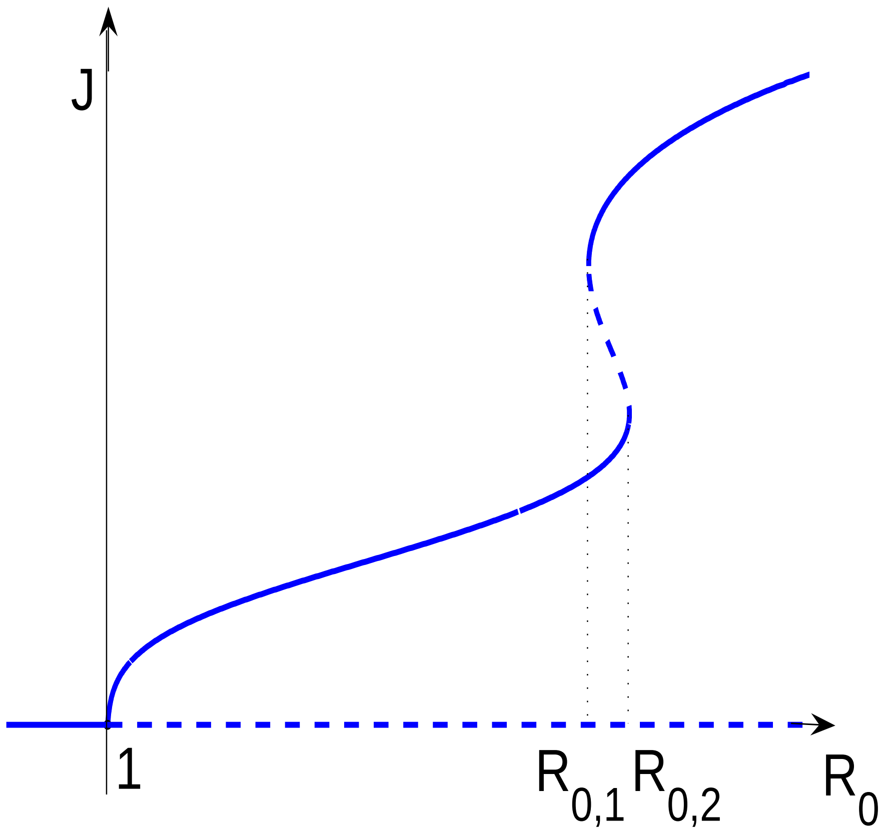
Exploration of juveniles-adults dinamics

Back to adults-juveniles competition: the case of separate niches



Exploration of juveniles-adults dinamics

Separate niches: exploring the bifurcation graph



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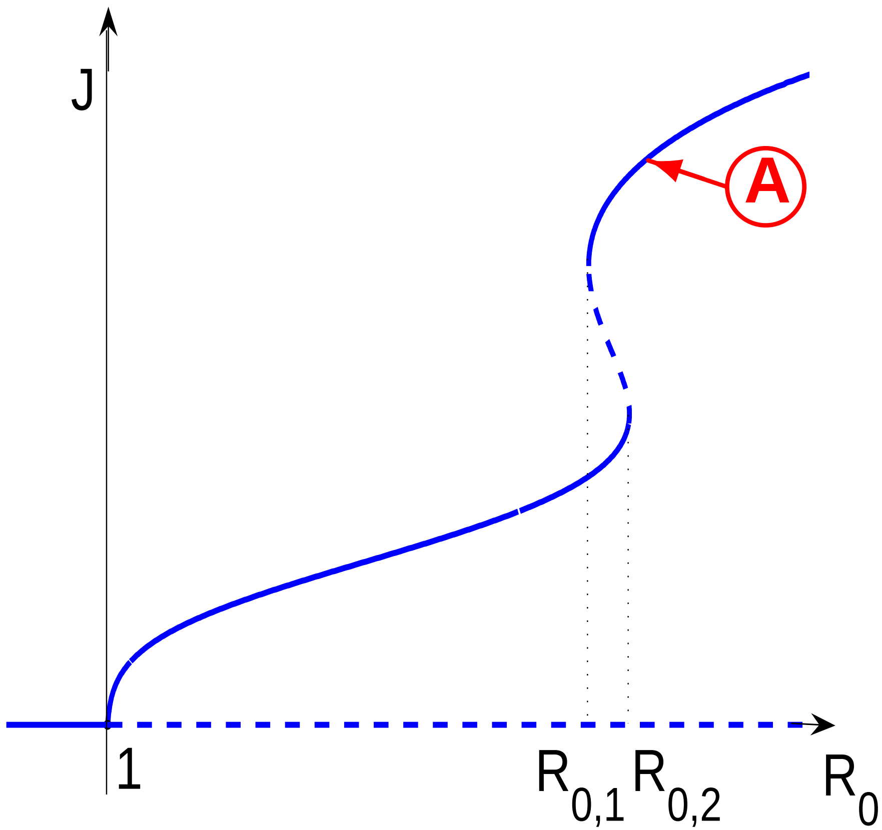

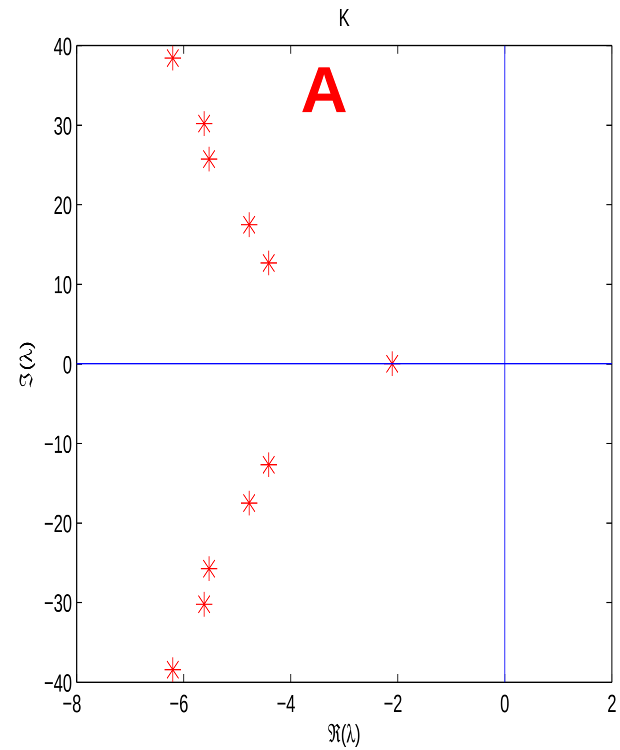

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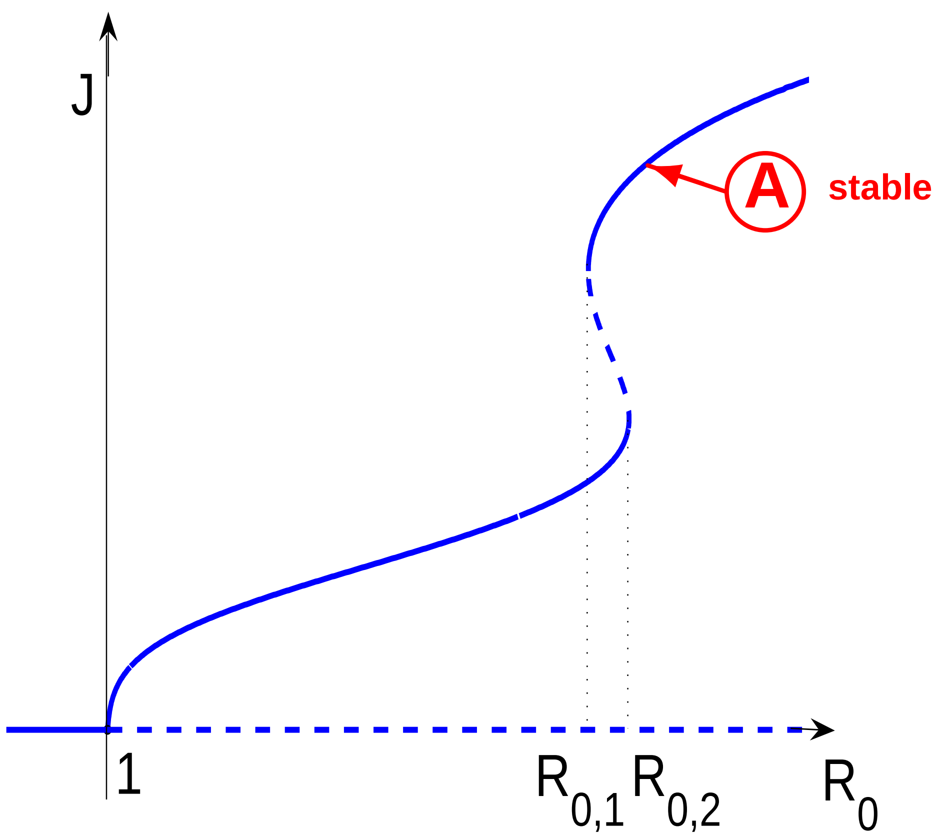

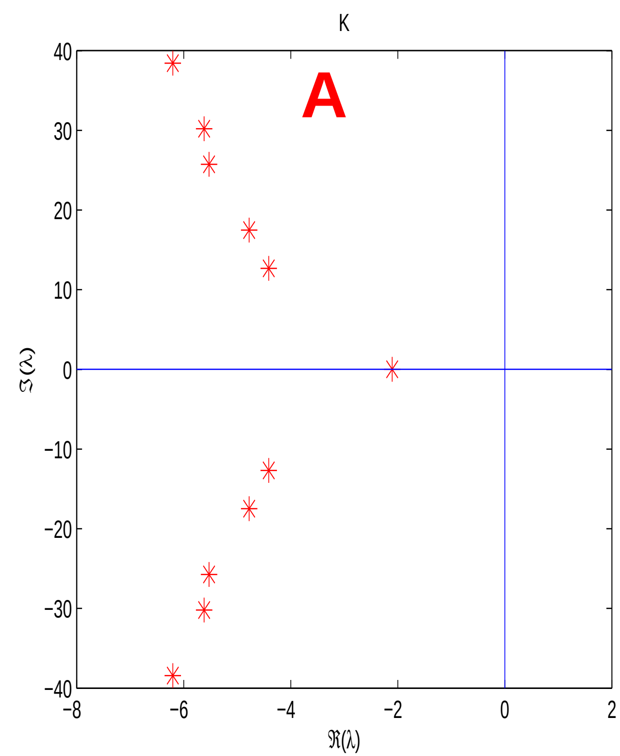

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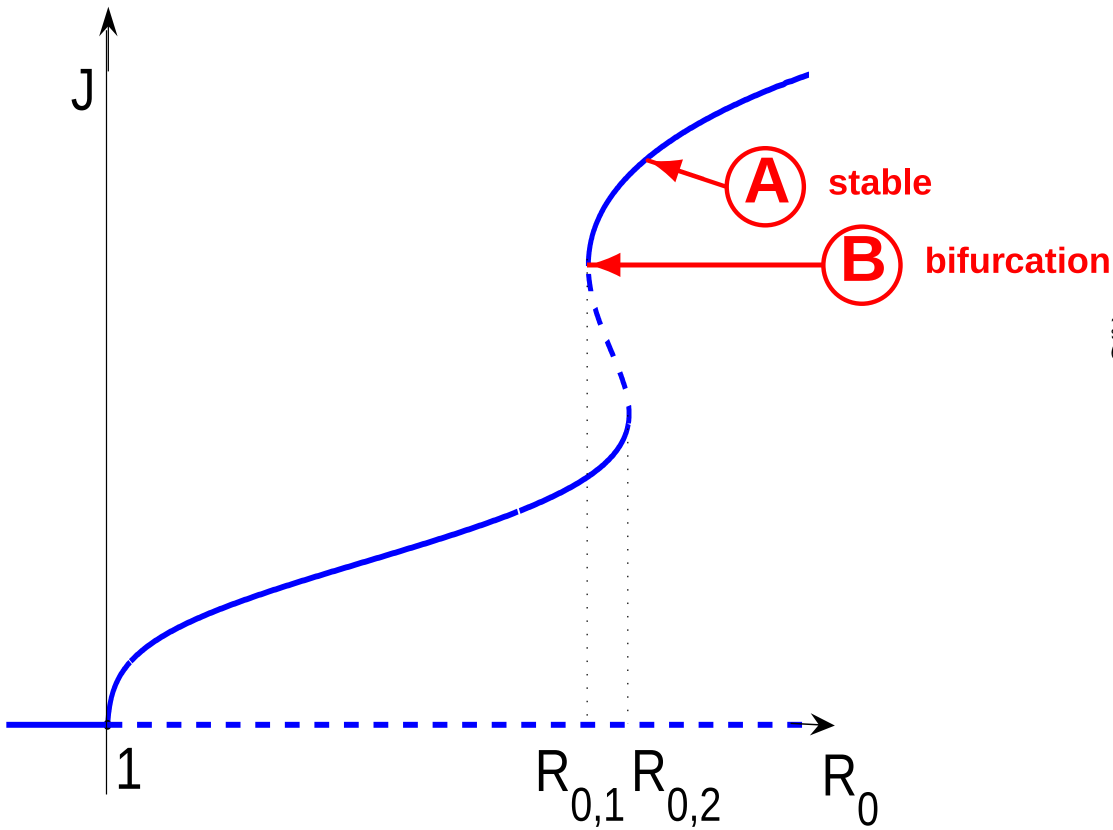

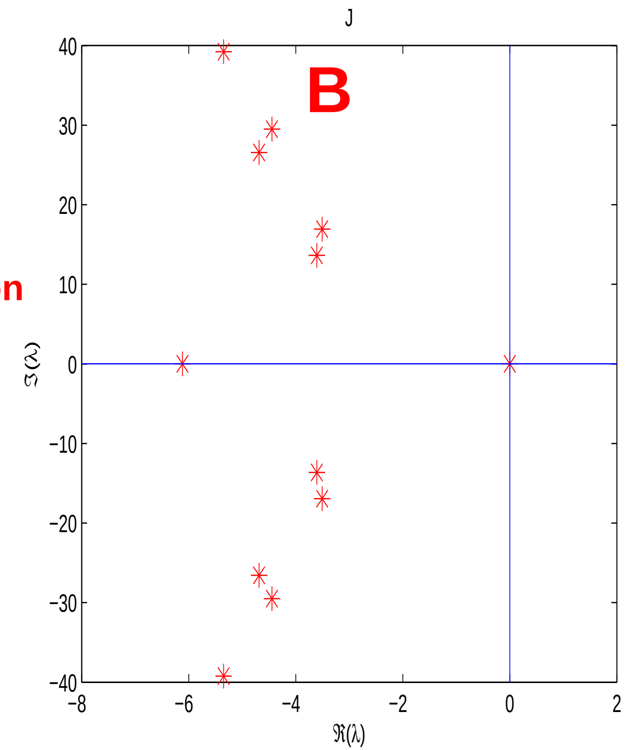

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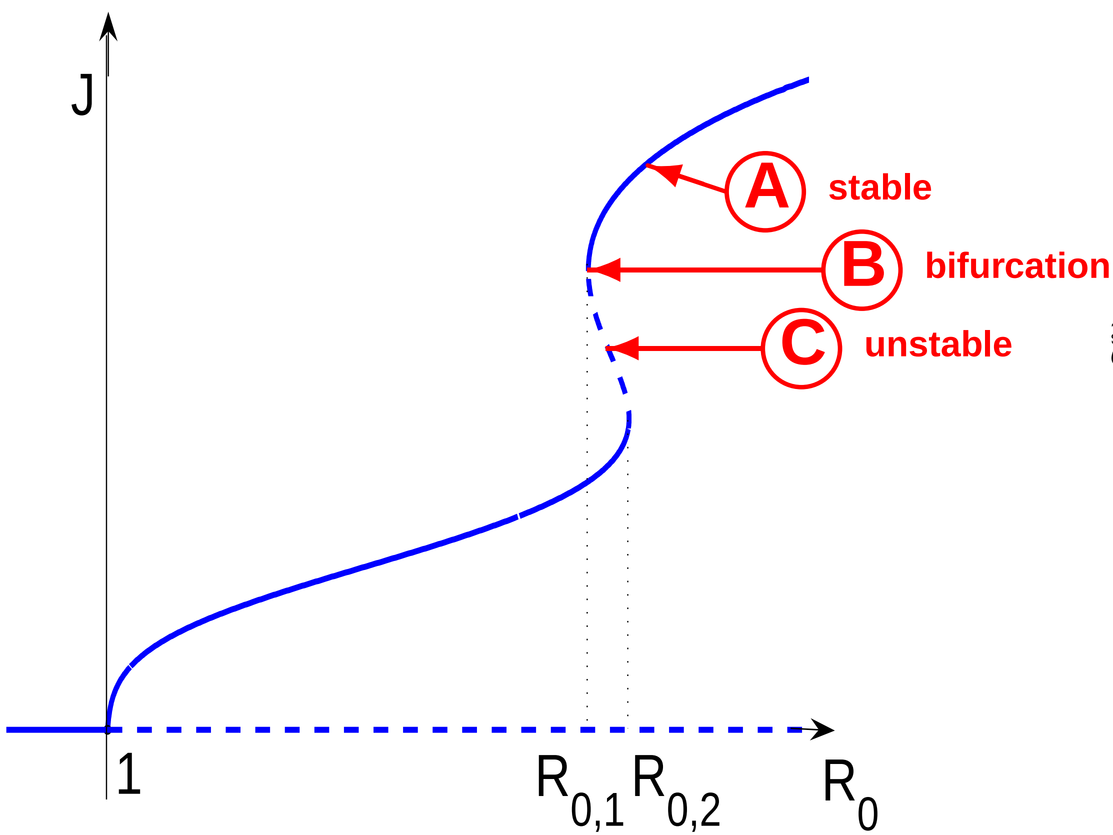

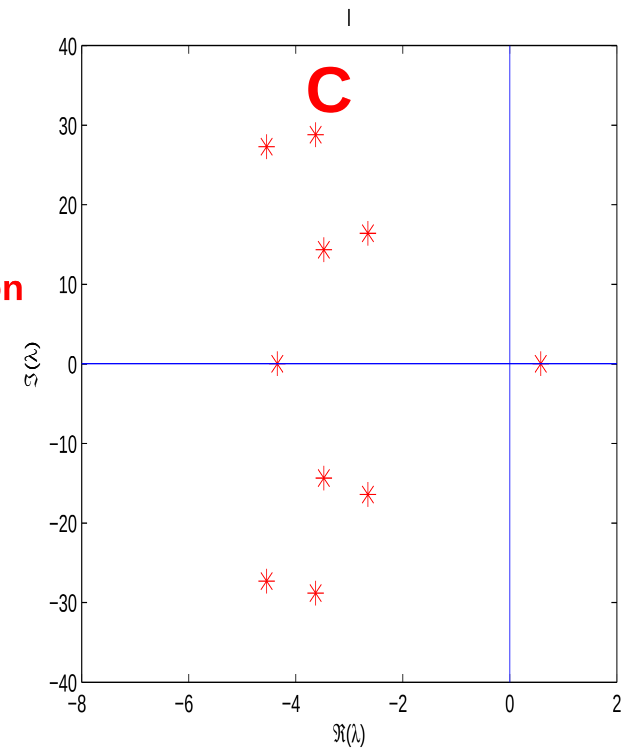

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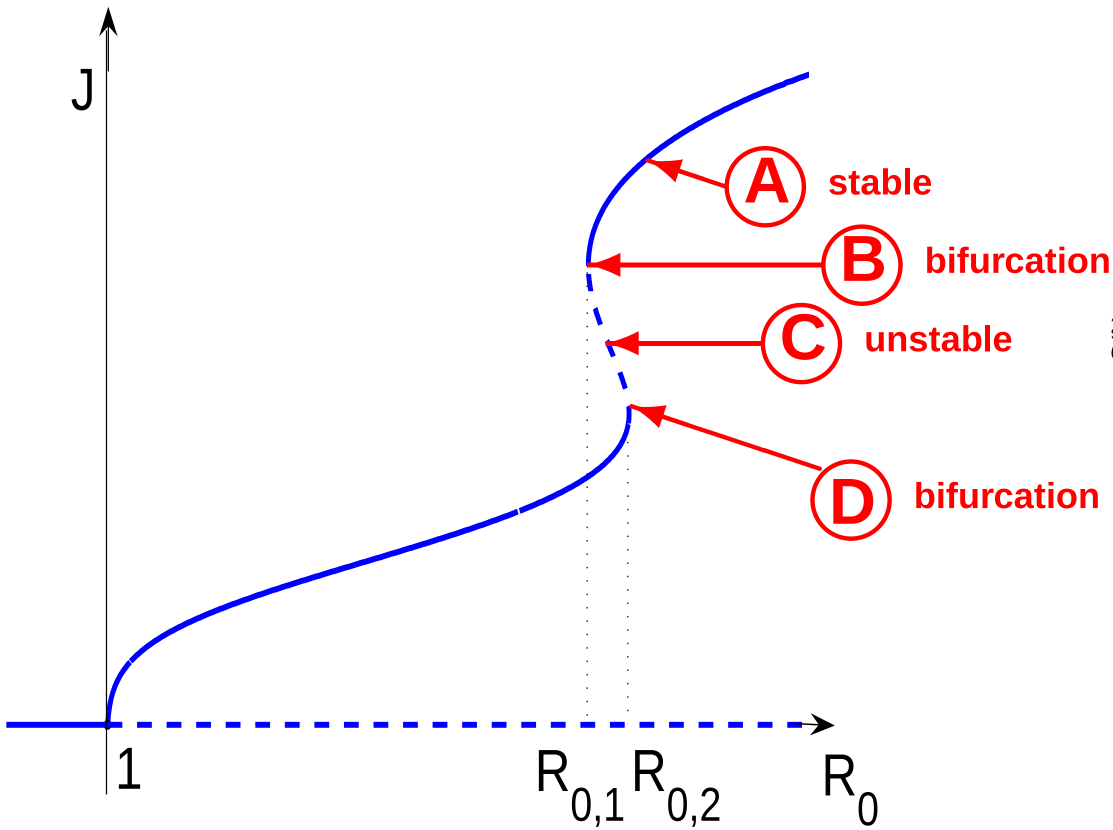

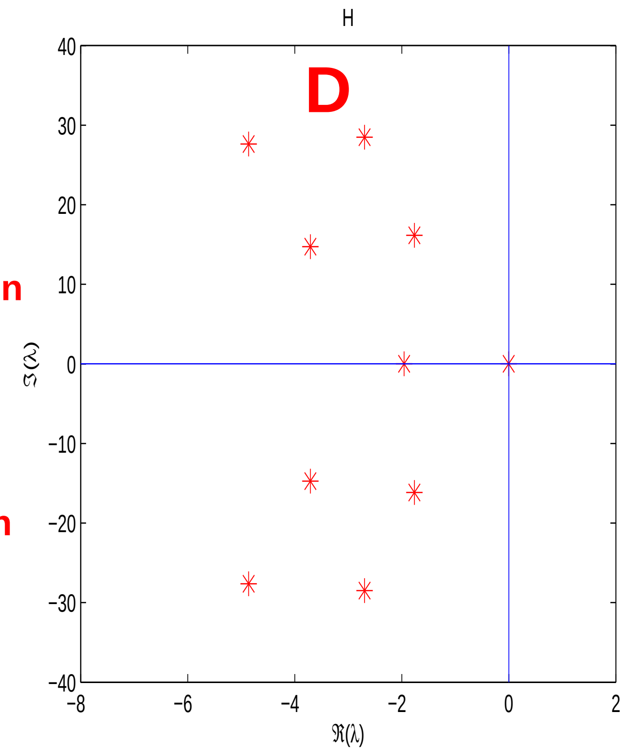

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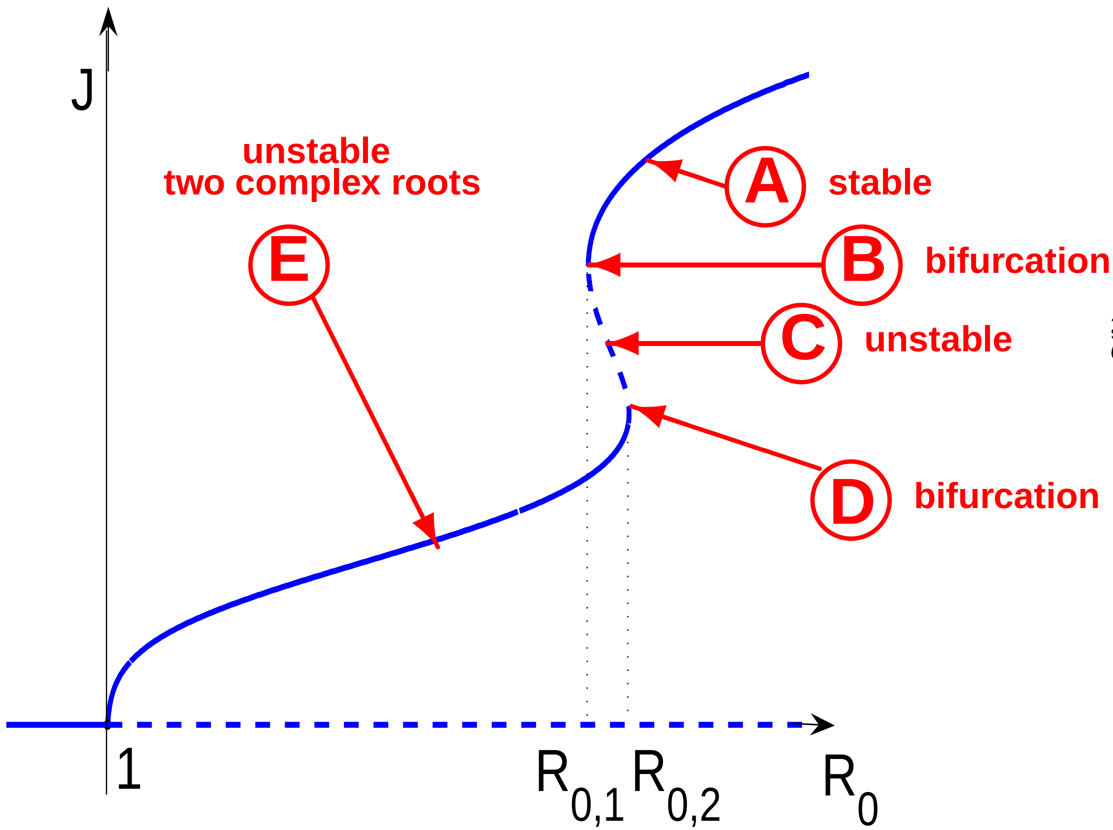
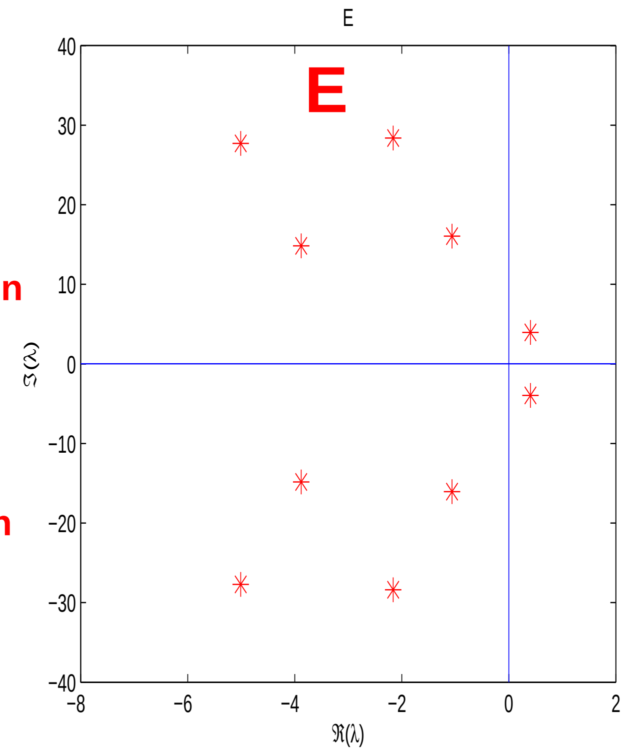
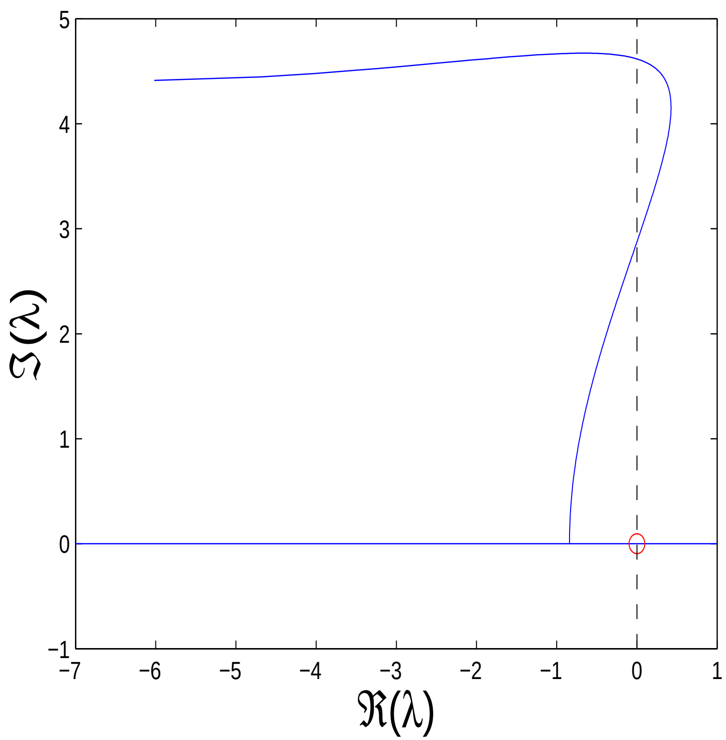
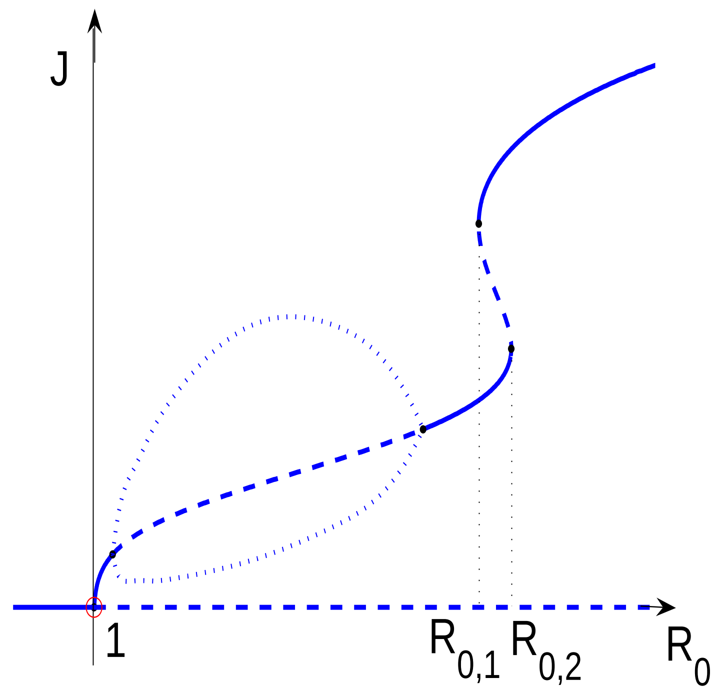




Exploration of juveniles-adults dinamics

A complete pattern





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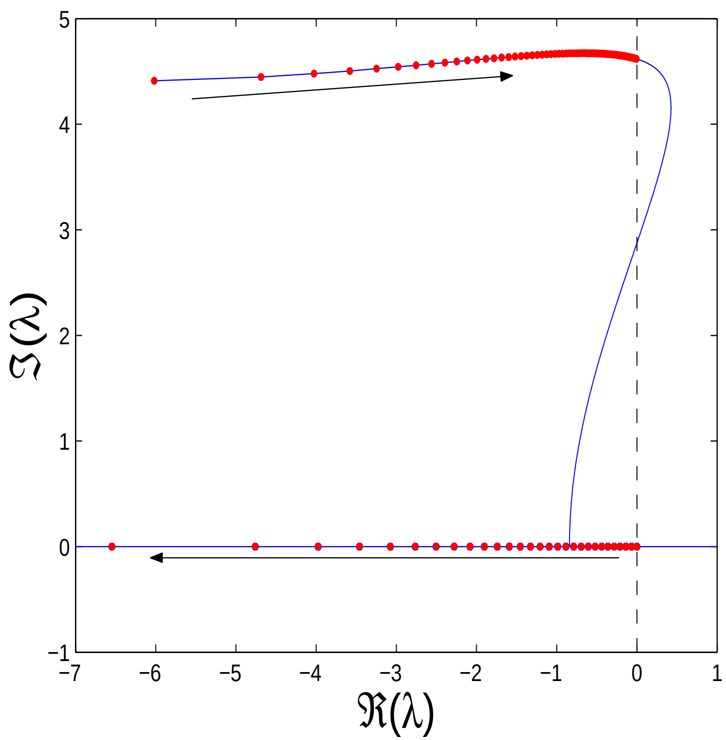

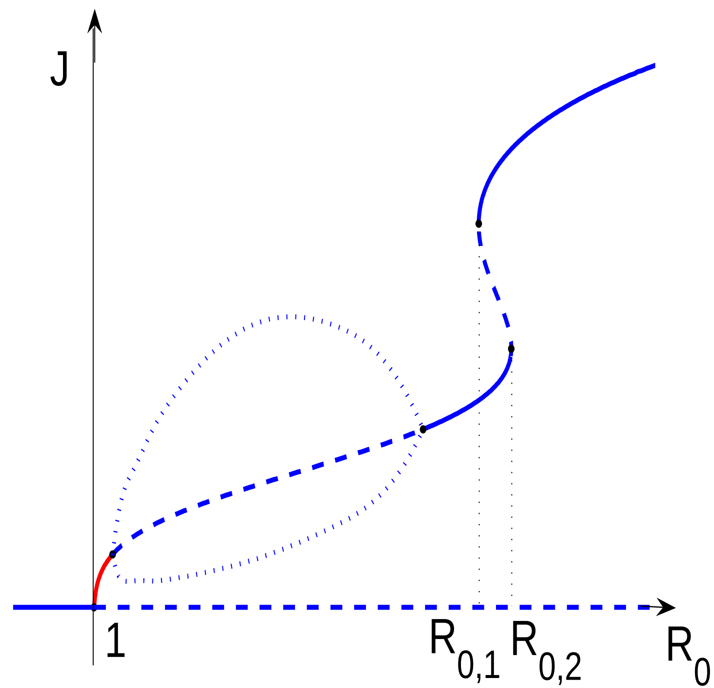

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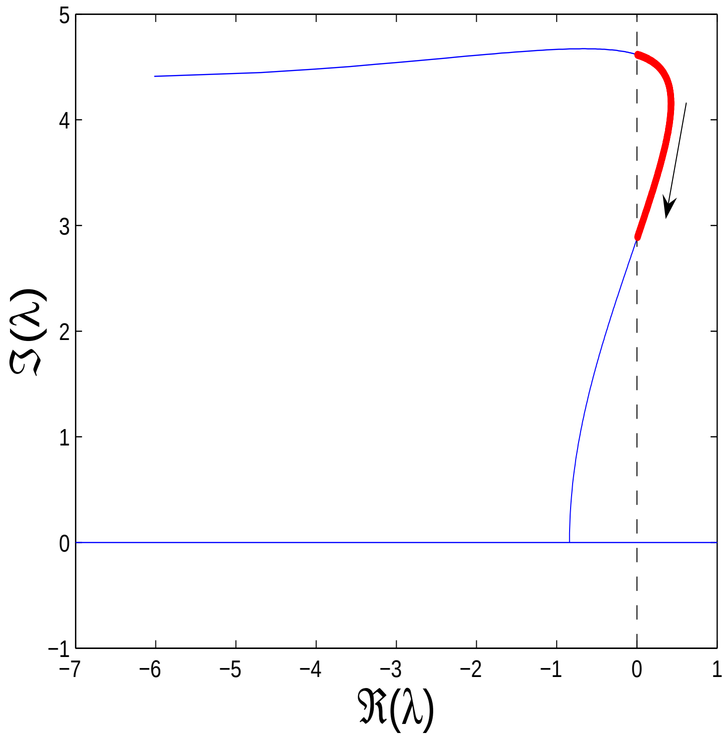

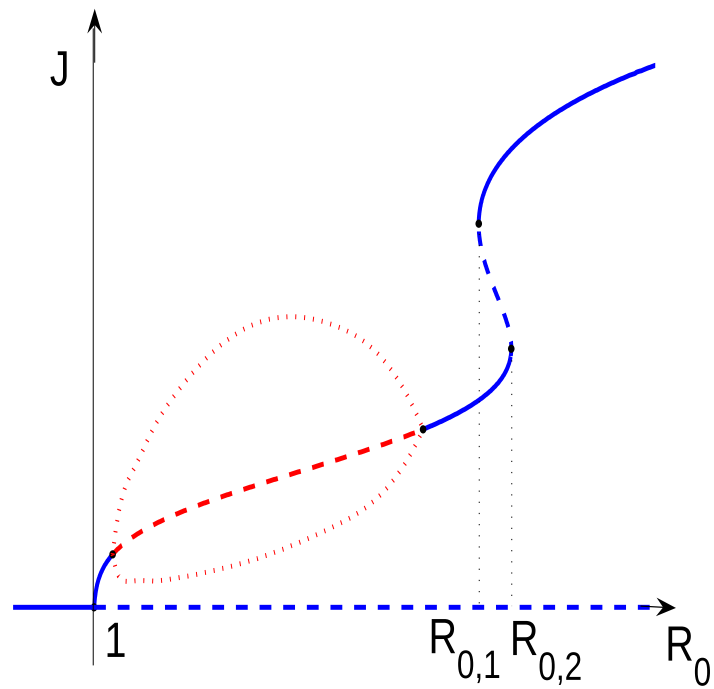

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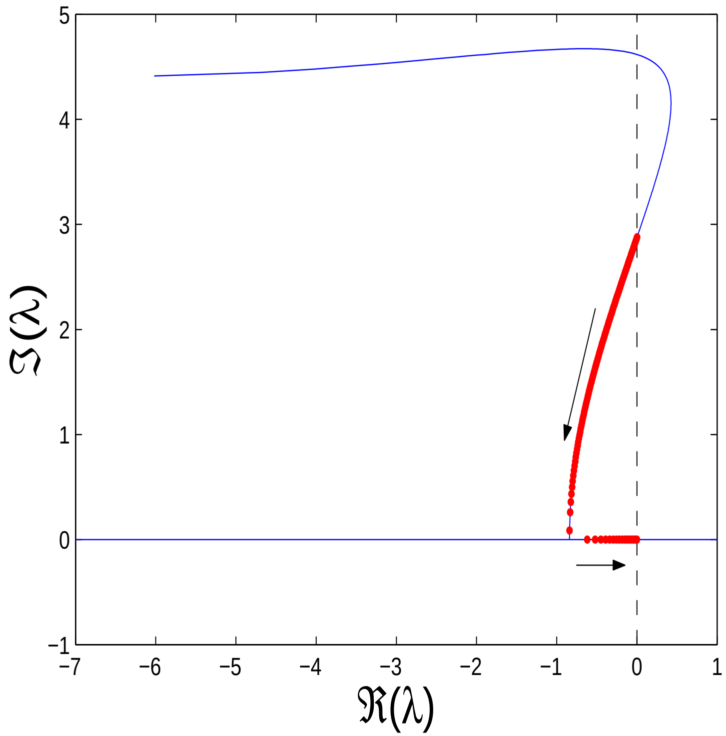

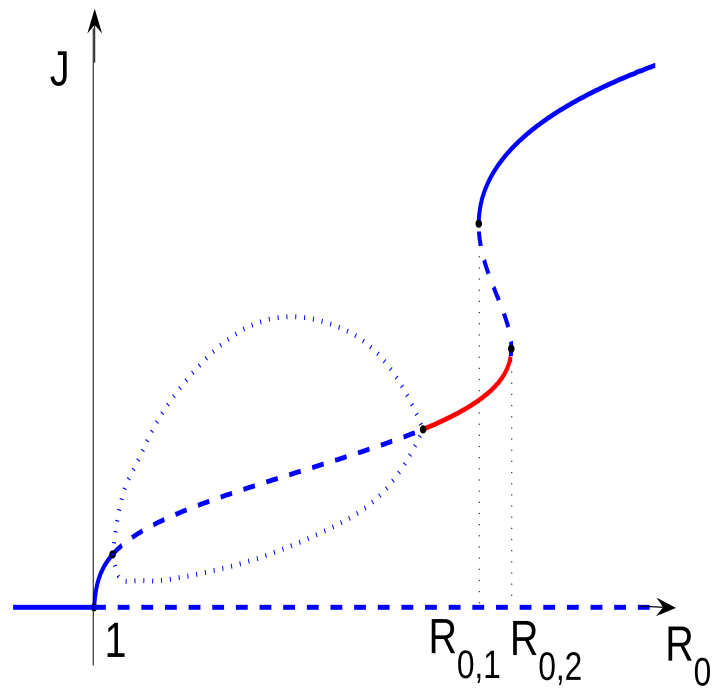

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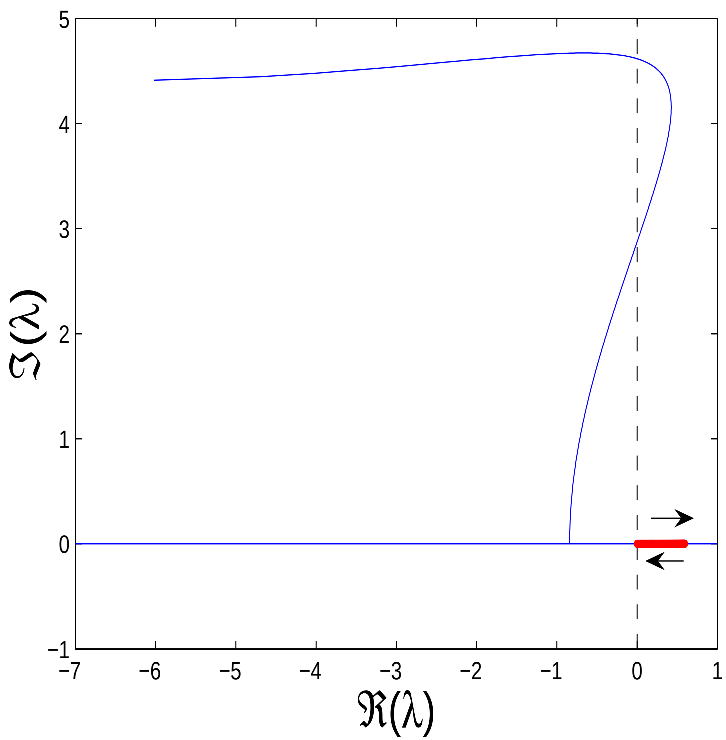

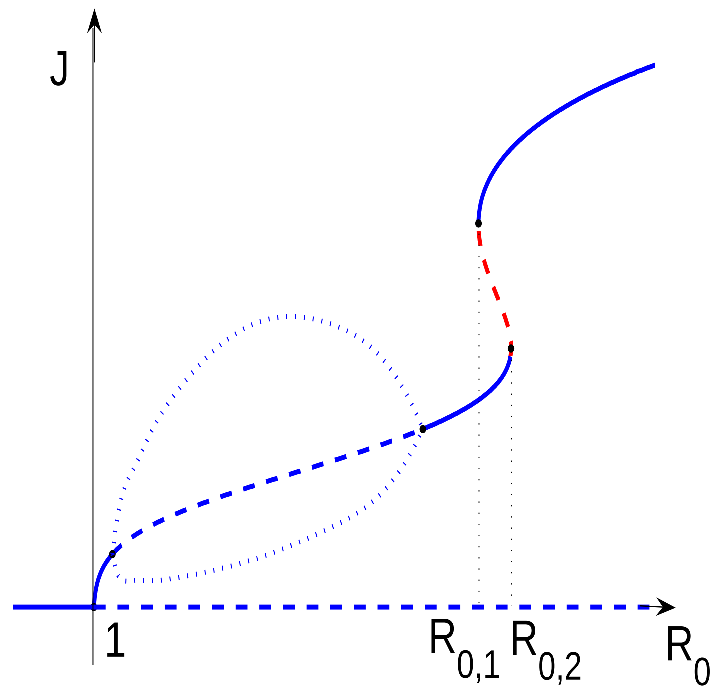

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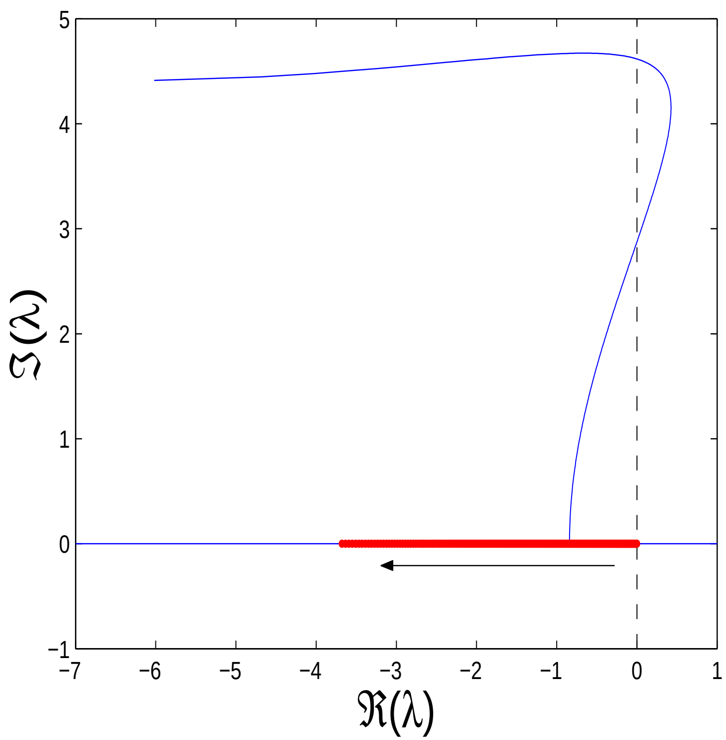
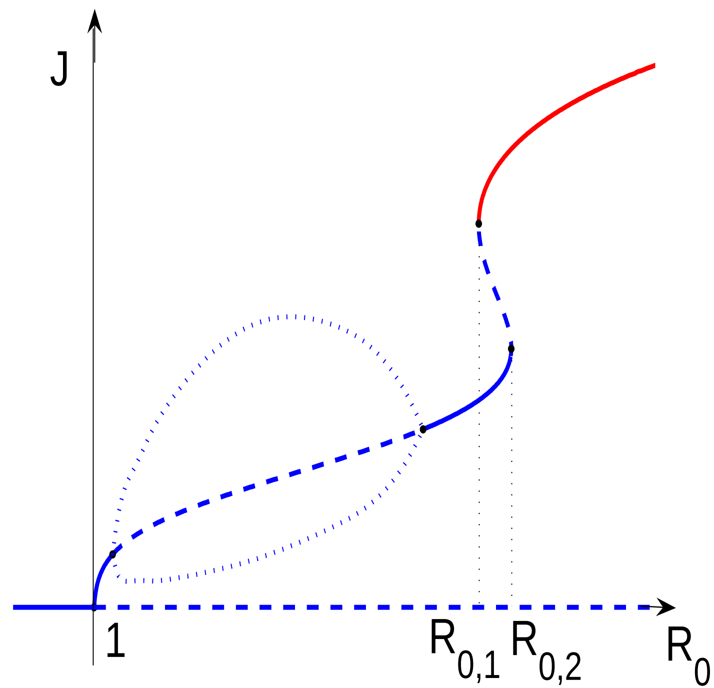


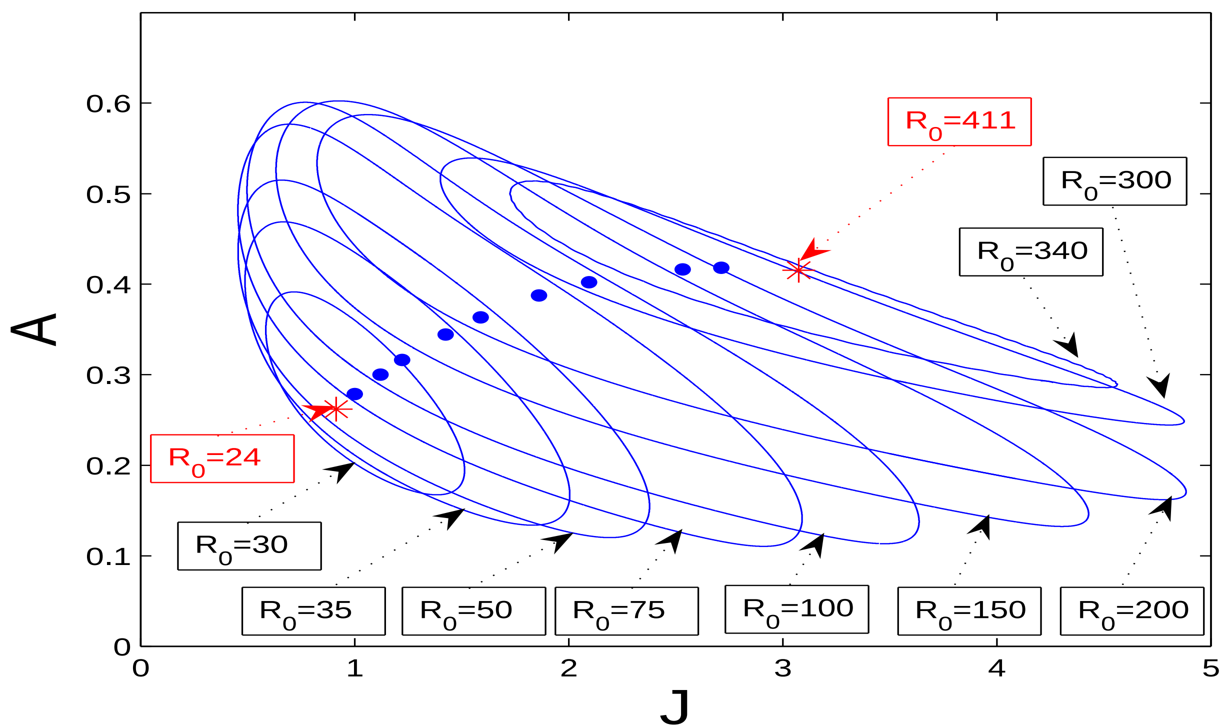


Exploration of juveniles-adults dinamics

Orbits by numerical computation of the solution

Exploration of juveniles-adults dinamics

Orbits by numerical computation of the solution



Future work

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- extension to epidemic models
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  - numerical methods for the computation of the solution
  - computation of steady states
  - stability analysis via numerical computation of characteristic roots

THANK YOU FOR YOUR ATTENTION