Extended Formulations in Integer Programming (and Combinatorial Optimization)

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Projections

The *projection* of a set $\mathcal{S}$ of points $(x, z) \in \mathbb{R}^{n+p}$ onto the x-space is the set:

$$\operatorname{proj}_x(\mathcal{S}) = \{x \in \mathbb{R}^n : \exists z \in \mathbb{R}^p \text{ s.t. } (x, z) \in \mathcal{S}\}.$$

The following two programs are equivalent

$$\max\{f(x) : x \in \operatorname{proj}_x(\mathcal{S})\} \text{ and } \max\{f(x) + \mathbf{0}z : (x, z) \in \mathcal{S}\}$$

Extended formulations

We study sets of the type:

$$X = \{x \in \mathbb{R}^n : A'x \leq b', x_i \text{ integer}, i \in I \subseteq N\}$$

We wish to find a dimension p and a polyhedron

$$Q = \{(x, z) \in \mathbb{R}^{n+p} : Ax + Bz \leq b\}$$

such that:

- $\mathrm{proj}_x(Q) = \mathrm{conv}(X)$.

- The linear program $\max\{hx + gz : Ax + Bz \leq b\}$ is easy to solve.

This is an *extended formulation*.

Union of Polyhedra

Theorem 1 *(Balas) Given k polyhedra*

$$
\begin{array}{llllll}
P^1 = & A^1 x \leq b^1 & = \operatorname{conv}(v_1^1, \ldots, v_{p_1}^1) & + & \operatorname{cone}(r_1^1, \ldots, r_{q_1}^1) & A^1 x \leq 0 \\
P^2 = & A^2 x \leq b^2 & = \operatorname{conv}(\emptyset) & + & \operatorname{cone}(r_1^2, \ldots, r_{q_2}^2) & A^2 x \leq 0 \\
& \vdots & & & & \\
P^k = & A^k x \leq b^k & = \operatorname{conv}(v_1^k, \ldots, v_{p_1}^k) & + & \operatorname{cone}(r_1^k, \ldots, r_{q_k}^k) & A^k x \leq 0
\end{array}
$$

Consider the polyhedron Y in the space of variables (x, x^i, δ^i) defined by:

$$
Y := \begin{cases}
A^i x^i & \leq & \delta^i b^i \ i \in K \\
\sum_{i \in K} x^i & = & x \\
\sum_{i \in K} \delta^i & = & 1 \\
\delta^i & \geq & 0 \ i \in K.
\end{cases}
$$

Then

$$
\operatorname{proj}_x(Y) = \operatorname{conv}(\bigcup_{i \in K}(v_1^i, \ldots, v_{p_i}^i)) + \operatorname{cone}(\bigcup_{i \in K}(r_1^i, \ldots, r_{q_i}^i))
$$

Union of Polyhedra

Given $\mathrm{conv}(v_1, \ldots, v_p) + \mathrm{cone}(r_1, \ldots, r_q)$ find $v_1^i, \ldots, v_{p_i}^i$ and $r_1^i, \ldots, r_{q_i}^i$, $1 \leq i \leq k$ such that

- $\square$ $(v_1, \ldots, v_p) = \bigcup_{i \in K}(v_1^i, \ldots, v_{p_i}^i)$ and $(r_1, \ldots, r_q) = \bigcup_{i \in K}(r_1^i, \ldots, r_{q_i}^i)$

- $\square$ The polyhedra $\mathrm{conv}(v_1^i, \ldots, v_{p_i}^i) + \mathrm{cone}(r_1^i, \ldots, r_{q_i}^i)$ admit an easy description $A^i x \leq b^i$.

All-even subsets

We consider the set:

$$EVEN^n = \{x \in \{0,1\}^n : x \text{ has an even number of } 1^s\}$$

Jeroslow proves the following:

Theorem 2

$$\text{conv}(EVEN^n) = \{x \in \mathbb{R}^n : 0 \leq x_i \leq 1, \, i \in N,$$
$$\sum_{i \in S} x_i - \sum_{i \in N \setminus S} x_i \leq |S| - 1, \, |S| \text{ odd}\}.$$

All-even subsets

Let $S_k = \{x \in \{0,1\}^n : x \text{ has } k \ 1^s\}$. Then $EVEN^n = \bigcup_{k \text{ even}} S_k$.
So we can apply Balas theorem. Since

$$\text{conv}(S_k) = \{x \in \mathbb{R}^n : \mathbf{0} \le x \le \mathbf{1}, \sum_{i \in N} x_i = k\},$$

consider the polytope Y defined by the following system:

$$
\begin{aligned}
x_i - \sum_{k \text{ even}} x_i^k &= 0 & i \in N \\
\sum_{i \in N} x_i^k &= k\lambda^k & k \text{ even} \\
\sum_{k \text{ even}} \lambda^k &= 1 \\
x_i^k &\le \lambda^k & i \in N,\ k \text{ even} \\
x_i^k &\ge 0 & i \in N,\ k \text{ even} \\
\lambda^k &\ge 0 & k \text{ even}
\end{aligned}
$$

Then $\text{conv}(EVEN^n) = \text{proj}_x(Y)$.

Mixed Integer Programming

$$X = \{(s, y, x) \in \mathbb{R}_+^1 \times \mathbb{R}_+^n \times \mathbb{Z}_+^n : s + y_t + x_t \geq b_t, \ 1 \leq t \leq n\}.$$

Lemma 1 *Let (s^*, y^*, x^*) be a vertex of $\mathrm{conv}(X^{CMIX})$. Define $b_0 = 0$.*
Then

1. *$s^* \equiv b_t \bmod 1$ for some $0 \leq t \leq n$.*

2. *For $1 \leq t \leq n$, either $y_t^* = 0$ or $y_t^* \equiv b_t - s \bmod 1$.*

For $0 \leq i \leq n$, define

$$X_i = \{(s, y, x) \in X^{CMIX} : s \equiv b_i \bmod 1\}.$$

Then

$$\mathrm{Vert}(\mathrm{conv}(X)) = \bigcup \mathrm{Vert}(\mathrm{conv}(X_i)) \quad \mathrm{Rays}(\mathrm{conv}(X)) = \mathrm{Rays}(\mathrm{conv}(X_i)).$$

Mixed Integer Programming

Let $f_{ti} = (b_t - f_i) - \lfloor b_t - f_i \rfloor$.

Theorem 3 *The following set of inequalities gives a formulation for* $\operatorname{conv}(X_i)$:

$$s \geq f_i$$
$$s + y_t + x_t \geq b_t, \; 1 \leq t \leq n$$
$$y_t + f_{ti}(x_t + s) \geq f_{ti}(\lceil b_t - f_i \rceil + f_i), \; 1 \leq t \leq n$$
$$s \in \mathbb{R}^1_+, y \in R^n_+, x \in R^n_+.$$

The size of an extended formulation

Given polytope

$$P = \{x \in \mathbb{R}^n : A^= x = b^=, A^< x \le b^<\} = \text{conv}\{v^1, \ldots, v^p\}$$

where $v^1, \ldots, v^p$ are the vertices of P, and the inequalities in $A^< x \le b^<$ correspond to the facets of P. Let f be the number of rows of $A^<$. We define the *slack matrix* of P to be the $f \times p$ matrix SM whose ij-entry is $b_i^< - a_i^< v^j$, that is, the slack taken by vertex v^j in the inequality $a^< x \le b^<$.

Theorem 4 *(Yannakakis) Let m be the smallest number such that SM can be written as the product of two nonnegative matrices of size $f \times m$ and $m \times p$.*
The minimum number of variables and constraints that defines a polyhedron Q such that $\text{proj}_x(Q) = P$ is of order $\Theta(m + n)$.

The spanning tree polytope

$SP(G)$ is the convex hull of the incidence vectors of spanning trees of G.

$$SP(G) = \left\{ x \in \mathbb{R}^m : \begin{array}{ll} \sum_{e \in E} x_e = n - 1, & \\ \sum_{e \in E(S)} x_e \leq |S| - 1, & S \subset V \\ x_e \geq 0, & e \in E \end{array} \right\}$$

The slack matrix corresponding to the inequalities $\sum_{e \in E(S)} x_e \leq |S| - 1$ is the following. Given a node $k \in V$, let A_k be the arborescence obtained by rooting T at k. Then one can readily verify that, if $k \in S$, the slack is is the number of nodes of S whose father in A_k is not in S. Let

$$\lambda_{kij} = 1 \text{ if } j \text{ is the father of } i \text{ in } A_k, 0 \text{ otherwise, } i, j, k \in V$$

then

$$SM_{S,T} = \sum_{i \in S, j \in V \setminus S} \lambda_{kij}.$$

The spanning tree polytope

This yields a factorization of the slack matrix SM of $SP(G)$ and the following extended formulation:

$$\sum_{i,j \in V} x_{ij} = n - 1,$$
$$x_{ij} \geq 0, \qquad\qquad\qquad ij \in E,$$
$$x_{ij} = \lambda_{kij} + \lambda_{kji}, \qquad\quad 1 \leq i, j, k \leq n$$
$$\sum_j \lambda_{kij} = 1, \qquad\qquad 1 \leq i, k \leq n,\ i \neq k$$
$$\lambda_{kij} \geq 0,\ \lambda_{kkj} = 0,\ \lambda_{kii} = 0, \quad 1 \leq i, j, k \leq n.$$

A general idea

$$S = \{(x,y) \in \mathbb{R}^{n+q} \,|\, Ax + Gy \geq b, \, y \geq 0, \, x \in \mathbb{Z}^n\}$$

☐ Δ : least common multiple of the subdeterminants of G;

☐ k : smallest number such that kb is integral.

then, given a vertex $(\bar{x}, \bar{y})$ of $\text{conv}(S)$, $(k\Delta)\bar{y}$ is integral.

Proof. $\bar{y}$ must be a vertex of $\{y \geq 0 \,|\, Gy \geq b - A\bar{x}\}$ ...

If G is totally unimodular, i.e. $\Delta{=}1$, we can write y_i as the average of k integer variables

$$y_i \;\; = \;\; \frac{1}{k}(\mu_i^0 + \cdots + \mu_i^{k-1})$$
$$\mu_i^0, \ldots, \mu_i^{k-1} \geq 0$$
$$\mu_i^0, \ldots, \mu_i^{k-1} \in \mathbb{Z}$$

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y_i \ &= \ \frac{1}{k}(\mu_i^0 + \cdots + \mu_i^{k-1}) \\
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An example

$$x + y \geq \frac{h}{k}$$
$$y \geq 0$$
$$x \in \mathbb{Z}$$

w.l.o.g. $0 \leq h < k$

we get

$$y - \frac{1}{k}(\mu^0 + \cdots + \mu^{k-1}) = 0$$
$$x + \frac{1}{k}(\mu^0 + \cdots + \mu^{k-1}) \geq \frac{h}{k}$$
$$\mu^0, \ldots, \mu^{k-1} \geq 0$$
$$x, \mu^0, \ldots, \mu^{k-1} \in \mathbb{Z}$$

$$x + \mu^0 \geq \lfloor \frac{h}{k} \rfloor$$
$$\vdots$$
$$x + \mu^{h-k-1} \geq \lfloor \frac{h}{k} \rfloor$$
$$x + \mu^{h-k} \geq \lceil \frac{h}{k} \rceil$$
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An example

$$\begin{aligned} x + y &\geq \frac{h}{k} \\ y &\geq 0 \\ x &\in \mathbb{Z} \end{aligned}$$

w.l.o.g. $0 \leq h < k$

$$\begin{aligned} x + \mu^0 &\geq \lfloor \frac{h}{k} \rfloor \\ &\vdots \\ x + \mu^{h-k-1} &\geq \lfloor \frac{h}{k} \rfloor \\ x + \mu^{h-k} &\geq \lceil \frac{h}{k} \rceil \\ &\vdots \\ x + \mu^{k-1} &\geq \lceil \frac{h}{k} \rceil \end{aligned}$$

we get

$$\begin{aligned} y - \frac{1}{k}(\mu^0 + \cdots + \mu^{k-1}) &= 0 \\ x + \frac{1}{k}(\mu^0 + \cdots + \mu^{k-1}) &\geq \frac{h}{k} \\ \mu^0, \ldots, \mu^{k-1} &\geq 0 \\ x, \mu^0, \ldots, \mu^{k-1} &\in \mathbb{Z} \end{aligned}$$

Projections

$$P = \{(z, \mu) \mid Cz + D\mu \geq f, \ \mu \geq 0\}$$

Given $\lambda \geq 0$ s.t. $\lambda D \leq 0$, then

$$(\lambda C)z \geq \lambda f$$

is valid for P and for $\mathrm{proj}_z(P)$.

Theorem 5

$$\mathrm{proj}_z(P) = \{z \mid (\lambda C)z \geq \lambda f, \ \forall \lambda \in \mathrm{extr}Q\}$$

where Q is the projection cone

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Example: projecting out the μs

$$
\begin{array}{rllll}
(\bar{\lambda}) & y - \frac{1}{k}(\mu^0 + \cdots + \mu^{k-1}) & = & 0 & \\
(\lambda^i) & x + \mu^t & \geq & \lfloor \frac{h}{k} \rfloor & t = 0, \ldots, k - h - 1 \\
(\lambda^i) & x + \mu^t & \geq & \lceil \frac{h}{k} \rceil & t = k - h, \ldots, k - 1 \\
& \mu^0, \ldots, \mu^{k-1} & \geq & 0 &
\end{array}
$$

Projection cone:

$$
\begin{array}{rll}
\lambda^t - \frac{1}{k}\bar{\lambda} & \leq & 0 \qquad t = 0, \ldots, k - 1; \\
\lambda^t & \geq & 0 \qquad t = 0, \ldots, k - 1;
\end{array}
$$

Choosing $\bar{\lambda} = k$, $\lambda^{k-h} = 1$, $\lambda^t = 0$, $t \neq k - h$ we get

$$
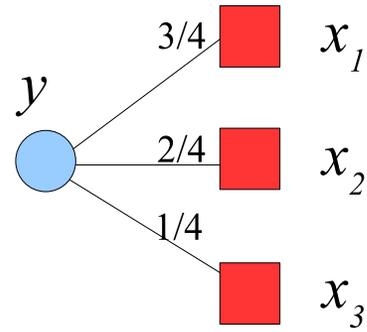x + ky \geq \lceil \frac{h}{k} \rceil \qquad \textit{(MIR inequality)}
$$

The mixing set

$$x_i + y \geq \frac{h_i}{k} \qquad i = 1, \ldots, n$$

$$y \geq 0$$

$$x \in \mathbb{Z}^n$$

we get

$$y - \frac{1}{k}(\mu^0 + \cdots + \mu^{k-1}) = 0$$

$$\mu^0, \ldots, \mu^{k-1} \geq 0$$

$$x_i + \mu_i^t \geq \lfloor \frac{h_i}{k} \rfloor, \quad i = 1, \ldots, n; \ t = 0, \ldots, k - h_i - 1$$

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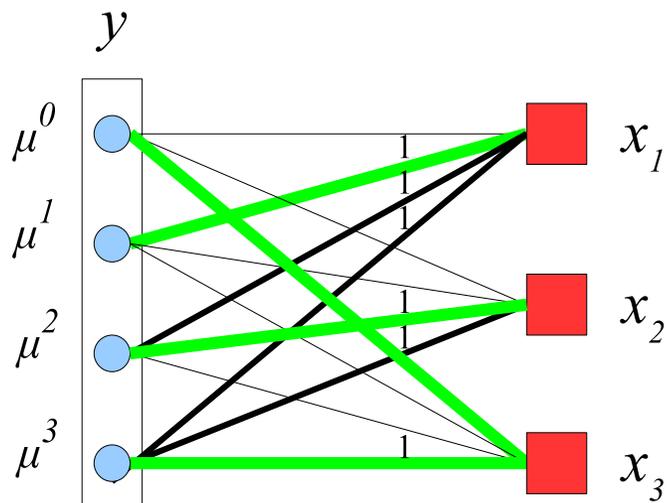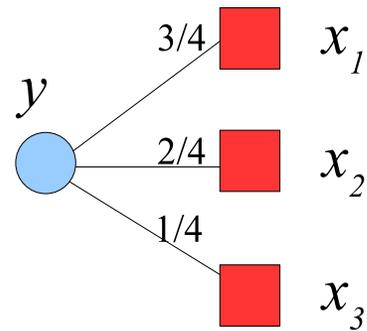
$$x, \mu^0, \ldots, \mu^{k-1} \in \mathbb{Z}$$

The mixing set

$$
\begin{array}{rcll}
y & +x_1 & \geq & 3/4 \\
y & +x_2 & \geq & 2/4 \\
y & +x_3 & \geq & 1/4 \\
y & & \geq & 0 \\
& x_1, \quad x_2, \quad x_3 & \in & \mathbb{Z}
\end{array}
$$

The mixing set

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y & +x_1 & & & \geq & 3/4 \\
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y & & & +x_3 & \geq & 1/4 \\
y & & & & \geq & 0 \\
& x_1, & x_2, & x_3 & \in & \mathbb{Z}
\end{array}
$$

$$
\begin{array}{rcl}
4y - \mu^0 - \mu^1 - \mu^2 - \mu^3 & = & 0 \\
\mu^1 + x_1 & \geq & 1 \\
\mu^2 + x_2 & \geq & 1 \\
\mu^3 + x_3 & \geq & 1 \\
\mu^0 + x_3 & \geq & 0
\end{array}
$$

$\Rightarrow \qquad 4y + x_1 + x_2 + 2x_3 \geq 3 \qquad$ *(Mixing inequality)*

Mixed-integer vertex covers

□ $G = (V, E)$: bipartite graph;

□ $I \subseteq V$: set of integer variables;

$X(G, b, I)$ set of **mixed-integer vertex covers** $x \in \mathbb{R}^V$ i.e.

$$\begin{aligned}
x_i + x_j &\geq b_{ij} & ij \in E \\
x_i &\geq 0 & i \in V \\
x_j &\text{ integer} & j \in I
\end{aligned}$$

There is an extended formulation whose size is polynomial on $|V|$, $|E|$ and k (where k is the smallest number s.t. kb is integral).
[Conforti, Di Summa, Eisenbrand, Wolsey]

Extended Formulation (for $k = 2$)

Since any vertex $\tilde{x}$ of $\mathrm{conv}\,X(G, b, I)$ is half-integral, we can write

$$x_i = \frac{1}{2}(\mu_i^0 + \mu_i^1).$$

where $\mu_i^0, \mu_i^1 \geq$ and integer. We will express $x_i \in \mathbb{Z}$ by $\mu_i^0 = \mu_i^1$.

$$\frac{1}{2}(\mu_i^0 + \mu_i^1) + \frac{1}{2}(\mu_j^0 + \mu_j^1) \geq b_{ij}$$

For integral μ's this is equivalent to

$$\left. \begin{array}{ccc} \mu_i^0 & + \mu_j^0 & \geq \quad \lfloor b_{ij} \rfloor \\ \mu_i^1 & + \mu_j^1 & \geq \quad \lceil b_{ij} \rceil \end{array} \right\} \text{ if } \{b_{ij}\} = \frac{1}{2}$$

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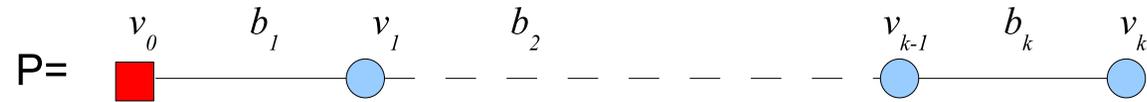where $\mu_i^0, \mu_i^1 \geq$ and integer. We will express $x_i \in \mathbb{Z}$ by $\mu_i^0 = \mu_i^1$.

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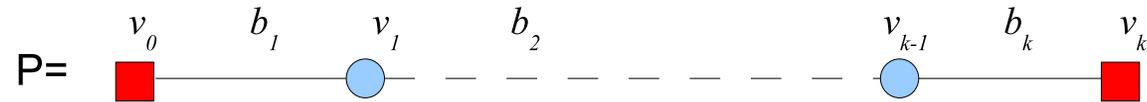
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$$\left.\begin{array}{rcl} \mu_i^0 \qquad + \mu_j^1 & \geq & b_{ij} \\ \mu_i^1 + \mu_j^0 & \geq & b_{ij} \end{array}\right\} \text{ if } \{b_{ij}\} = 0$$

Projection: I-path inequalities

$$x_{v0} + 2(x_{v_1} + \cdots x_{v_{k-1}}) \geq \lceil b_1 + b_2 + \cdots + b_k \rceil$$

$$x_{v0} + 2(x_{v_1} + \cdots x_{v_{k-1}}) + x_{v_k} \geq \lceil b_1 + b_2 + \cdots + b_k \rceil$$

[Conforti, Gerards, Zambelli]

A conjecture

□ The complexity status of solving mixed integer vertex cover problems on bipartite graphs is open.

□ The problem is polynomial on acyclic graphs.

Conjecture 2

$$\mathrm{conv}(X(G,b,I)) = \bigcap_T \mathrm{conv}(X(T,b,I))$$

taken over all subtrees of G where all nodes of $V(T) \cap I$ are leaves of T.

This would imply that the problem is in $NP \cap coNP$.