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NEW MI-SOCP FORMULATIONS FOR THE
UNIT COMMITMENT PROBLEMS WITH
RAMPING CONSTRAINTS

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Abstract

The Unit Commitment (UC) problem in electrical power production requires to optimally operate a set of power generation units over a short time horizon (one day to a week). Operational constraints of each unit depend on its type (e.g., thermal, hydro, nuclear, . . .), and can be rather complex; typical ones concern minimum and maximum power output, minimum up- and down-time, start-up and shut-down limits, ramp-up and ramp-down limits. Also, the objective function is often nonlinear. Thus, even the Single-Unit Commitment (1UC) problem, in which only one unit is present, has a rich combinatorial structure. In this work we present the first MINLP formulation that describes the convex hull of the feasible solutions of (1UC) comprising all the above constraints, and Second-Order Cone representable power generation costs. The new formulation has a polynomial number of both variables and constraints, and it is based on the efficient Dynamic Programming algorithm proposed in [23] together with the *perspective reformulation* technique proposed in [22]. We then analyze the effect of using it to develop tight formulations for the more general (UC). Since the formulation, despite being polynomial-size, is rather large, we also propose two new formulations, based on partial aggregations of variables, with different trade-offs between quality of the obtained bound and cost of the solving the corresponding continuous relaxation. Our results show that navigating these trade-offs may lead to improved performances for the partial enumeration approach used to solve the problem.

Keywords Unit Commitment problem, Ramp Constraints, MIP Formulations, Dynamic Programming, Convex Costs

1. Introduction

The Unit Commitment (UC) problem is a basic problem arising in power industries to coordinate and manage power generation units. Although it was the typical problem to be solved in old monopolistic regimes, the need to solve UC problems has not disappeared. On the contrary, both generation companies and system operators need to routinely solve some UC variant even in the free market regime (e.g., see [1, 35, 9, 31, 41]), both before and after the price of energy (and other ancillary services) has been cleared in the relevant market. Due to the huge figures involved in real-world systems [48], even minor improvements on the quality of the obtained solutions can result in very significant economical (and, possibly, environmental) savings. Therefore, the efficient solution of UC problems is still very much relevant in practice. Besides, being UC a complex Mixed-Integer NonLinear Program (MINLP), its study is relevant from the methodological viewpoints. Indeed, some theoretical results (e.g., [22]) that have been originally motivated by UC have later found many more applications [33, 13, 19, 20, 15, 27].

The traditional UC problem requires finding the schedule of each power generation unit in order to minimize operational costs while satisfying both system-wide constraints and operational constraints associated with each unit. System-wide constraints usually comprise the satisfaction of the energy demand, the provision of different types of reserve, and the handling of the transmission network. Operational constraints depend on the type of generation units. Most power systems are mainly based on thermal units (comprised nuclear ones) and hydro units, but in recent years the contribution of renewable energy sources (wind, solar, . . .) has steadily increased. As these are characterized by uncertainty in the production output, uncertain (robust and/or stochastic) UC models are more and more necessary [53, 50]. Since uncertain variants of optimization problems are typically considerably more difficult to solve than deterministic ones, efficient solution methods for these problems are in high demand as much as ever. Moreover, there is a clear trend whereby production and consumption tend to become more geographically separated than they previously were (think offshore wind farms and large solar plants in semi-desertic areas), which is putting novel strain on transmission networks that were not originally planned for these scenarios. All this has a substantial impact both on costs and security/reliability [5, 11], providing further strong motivation for the development of new approaches capable of solving complex variants of UC problems in shorter and shorter computational times.

Traditionally, Lagrangian relaxation was the method of choice to solve UC (e.g., see [4, 56, 8], or [53, §3.3] for a complete survey), since it was capable of exploiting the spatial structure of the problem: the most complex constraints pertain to the behaviour of a single unit, and relatively fewer and simpler ones link the different units together. However, the advances in the solution of Mixed-Integer (linear and convex) Programming (MIP) problems that are now widely available in present commercial solver have made MIP approaches an attractive option. This is even more so as the two approaches can be fruitfully combined [51, 29].

The first MILP formulation for UC was described in [30] and used three sets of binary variables. Later on, formulations using only one set of binary variables (on/off state) became more popular [12, 28], although different ones continued to be used [2]. While a reduced number of variables may lessen the cost of computing relaxations, this is usually not the crucial factor; rather, the *tightness* (quality of the lower bound of the continuous relaxation) of a MIP formulation is key for the efficient solution of the problem.

As operational constraints of thermal units have a strong combinatorial structure, many efforts have been made to improve their MIP definition. While there are some different types of units, each with several different variants, thermal units (comprised nuclear ones) are still the bulk of most energy systems, and these most often have a common set of operational restrictions. In particular, constraints on minimum and maximum power output, minimum up- and down-time, and ramp-up/ramp-down in power are almost invariably imposed. Most units have a nonzero minimum power production (and, obviously, a maximum one), meaning that the produced power is a typical *semi-continuous* variable, which can either be 0 or live in a closed real interval. These are naturally modeled with the help of the (also natural) binary variables that dictate if the unit is on or off.

Minimum up- and down-time constraints establish a minimum number of consecutive time periods that a unit must be on, or off; they are typically imposed to limit technical stress of the thermal units due to frequent start-up and shut-down operations. Such constraints introduce a strong combinatorial structure. The first exact description by means of linear inequalities for minimum-up and -down time constraints has been given in [39] with an exponential number of inequalities and a polynomial time separation algorithm.

Afterwards, Rajan and Takriti [46] and independently Malkin and Wolsey [40] developed an extended linear description with a linear number of constraints.

Ramp-up and ramp-down constraints limit the maximum increase or decrease of the power production between two consecutive time periods. Often, these are described together with maximum limits on start-up and shut-down periods are also often imposed. In their simpler form, these establish minimum and/or maximum limits for the produced power on the time period following the start-up and on the period preceding the shut-down, which can be different from the previously mentioned minimum and maximum power output limits corresponding to "steady-state" operations of the unit (i.e., far from the start-up and shut-down operations). More in general, typically for large units, they can impose complex trajectories, spanning over several time periods, that start-up and shut-down operations must follow before the unit reaches the stable state in which it can be freely (subject to the other constraints) modulated, which is the state in which traditional UC formulations consider the unit "on" (e.g., see [2, 31]).

Further complex features of thermal units are related to the power production costs. The cost of producing energy is typically a nonlinear function of the produced energy already when the unit is in the stable state ("on"). Furthermore, usually start-up costs have to be paid in the period when the unit is started up, to account for the nontrivial start-up operations. In their simplest description start-up costs can be considered fixed, but in a more exact description they are dependent on how long the unit remained off before start-up. This is because, roughly speaking, the unit must reach a minimum temperature in order to be able to produce power, and the heating process requires energy that has to be paid for. The cost for reaching the required temperature depends on the temperature that the unit has when the start-up process begins, which in turn depends both on environmental factors (assumed known) and on the previous history of the unit, i.e., how many periods it has been off. Typically, cooling of a unit follows an exponential law, and therefore the start-up cost is a nonlinear and concave function of the time the unit is been off. However, if a unit is to remain off for a limited time (this being known in advance, it being precisely the result of the UC problem that is typically solved much before actual operations), an alternative is to keep burning some fuel to avoid the cooling. This can be more economical for short off periods, with the breakpoint easy to find with simple algebraic formulae. However, as the time is discretized the same happens with start-up costs, which therefore entail yet another combinatorial (as opposed to nonlinear) feature of UC models. Nowak and Römisch [43] gave a popular description of these constraints using only state variables. Recently, in [47] it was shown that a specific type of function for start-up costs can be exactly described by linear inequalities which are also computationally efficient in practice. Moreover in [10] a polyhedral description of the general start-up costs was proposed that uses an exponential number of inequalities that can be separated in polynomial time.

In this work we present the first MIP description of the convex hull of the solutions satisfying *all* the standard operational constraints for the thermal units: minimum up- and down-time constraints, minimum and maximum power output, ramp constraints (including start-up and shut-down limits), general start-up costs, and nonlinear power production costs belonging to the class of Second-Order Cone-representable functions. Our new formulation is derived by a Dynamic Programming algorithm [23] and contains a polynomial number of variables and constraints. A simpler version of this result, limited to linear power generation costs, was first presented in [25]. Analogous results, using different proof techniques and still limited to (piecewise-)linear production costs, where independently proposed in [37, 32]. Interestingly, the latter paper claims that the proposed formulation represents the convex hull for any possible convex power generation cost, but this is proven to be false (even for simple quadratic separable cost) in [3]. Indeed, nonlinearity of the cost function introduces a further complexity in the convex hull description, that none of the previous attempts addressed. Interestingly, this complexity can be tamed with the help of the *perspective reformulation* technique that, despite having very many other applications [33, 13, 19, 20, 15, 27], was originally developed precisely in the context of the solution of UC problems [23].

The structure of the paper is as follows. In Section 2 we recall the two main formulations of the UC problem. In Section 3 we give a survey of the main results concerning polyhedral results for UC formulations. In Section 4 we recall the Dynamic Programming algorithm described in [23]. In Section 5 we present the new formulation, we prove that it describes the convex hull of the solutions of the single-unit commitment problem. In Section 6 we propose two additional simplified formulations also based on the Dynamic Programming algorithm with a trade-off between tightness and compactness. In Section 7 we present some preliminary computational experiments aimed at gauging the practical effectiveness of the new formulations on a data set already used in the literature. Finally, in Section 8 we sum up the results, and draw some possible lines

for future research on the topic.

2. The Thermal Unit Commitment Problem

In this section we recall the two most popular MIP formulations of the thermal Unit Commitment problem; other kinds of units have entirely different constraints and therefore require specific study (e.g. [52]). Those formulations are usually named *1-bin* and *3-bin* formulations from the number of vectors of binary variables that are considered.

Let I be the set of (indices of) thermal generators, with $m = |I|$, and $T = \{1, \ldots, n\}$ be the set of (indices of) time periods in the planning horizon. Given two time instants t' and t'', we will denote by $T(t', t'')$ the set of all the time instants between t' and t'', extremes included (obviously, $T(t', t'') = \emptyset$ if $t' > t''$). For each $i \in I$ and $t \in T$, let p_{it} (the *power variables*) be the power level of unit i at time period t, and x_{it} (the *commitment variables*) be the binary variable denoting the on/off state of unit i at time period t. As previously recalled, "on" state means that the unit can be *modulated*, i.e., the power output of the unit can be increased or decreased at will, subject to the technical constraints. The "off" state does not necessarily mean that the unit is inactive, in that it could be performing a start-up or shut-down trajectory, or being "banking", i.e., burning fuel to keep the temperature of the unit in view of an imminent restart. All these details are largely transparent to our model: although complex start-up or shut-down trajectories may require some modifications to be completely accounted for [42], this does not impact the mathematical formulation of the individual units.

We start by describing the 1-bin formulation that uses only the above-mentioned variables. With l_i and u_i being the minimum and the maximum power output for unit $i \in I$, respectively, the minimum and maximum power output constraints are simply

$$l_i x_{it} \leq p_{it} \leq u_i x_{it} \qquad t \in T. \tag{1}$$

With τ_i^+ and τ_i^- being the minimum number of consecutive time periods that unit i has to be in on and off state, respectively, the minimum up- and down-time constraints can be expressed as follows:

$$x_{it} \geq x_{ir} - x_{i,r-1} \qquad\qquad t \in T(\tau_i^+ + 1, n) \ , \ r \in T(t - \tau_i^+, t - 1), \tag{2}$$

$$x_{it} \leq 1 - x_{i,r-1} + x_{ir} \qquad\qquad t \in T(\tau_i^- + 1, n) \ , \ r \in T(t - \tau_i^-, t - 1). \tag{3}$$

Further constraints are required to specify the initial conditions of the unit. Let τ_i^0 denote the initial state of unit i as follows: at the beginning of the planning horizon, if $\tau_i^0 > 0$ then unit i has been in on state for τ_i^0 time periods, thus one has to impose the condition

$$x_{it} = 1 \qquad t \in T(1, \tau_i^+ - \tau_i^0) \ . \tag{4}$$

Of course, this is only required if, besides $\tau_i^0 > 0$, one also has $\tau_i^0 < \tau_i^+$ (otherwise, $T(1, \tau_i^+ - \tau_i^0) = \emptyset$). Similarly, $\tau_i^0 < 0$ means that unit i has been in off state for $-\tau_i^0$ time periods, and one has to impose the condition

$$x_{it} = 0 \qquad t \in T(1, \tau_i^- + \tau_i^0) \tag{5}$$

(again, this is only significant if $\tau_i^0 < -\tau_i^-$).

Let now Δ_i^+ and Δ_i^- be the ramp-up and ramp-down limits for unit i, respectively, i.e., the maximum increase/decrease of power output w.r.t. the previous period. The corresponding ramp constraints would be quite simple to write, were it not for the special treatment required by the start-up and shut-down periods. These are clearly peculiar, since the power produced at the previous/subsequent period is not really significant. In particular, note that it could well be, for instance, that $\Delta_i^+ < l_i$; if the ramp constraint were written without considering start-ups (i.e., just taking $p_{i,t-1} = 0$ as the reference power value), then the whole system would be unfeasible. Since start-up and shut-down periods need be dealt with in a specific way, it is expedient to introduce the two specific values $\bar{l}_i$ and $\bar{u}_i$, known the start-up and shut-down limits for unit i. These are the maximum power value that the unit can have in a, respectively, start-up and shut-down period, and they can be different from l_i and u_i; for consistency, it must however be $l_i \leq \bar{l}_i \leq u_i$

and $l_i \leq \bar{u}_i \leq u_i$. Then, the ramp constraints can be formulated as follows:

$$p_{it} - p_{i,t-1} \leq \Delta_i^+ x_{i,t-1} + \bar{l}_i(1 - x_{i,t-1}) \qquad\qquad t \in T, \qquad\qquad (6)$$

$$p_{i,t-1} - p_{it} \leq \Delta_i^- x_{it} + \bar{u}_i(1 - x_{it}) \qquad\qquad t \in T. \qquad\qquad (7)$$

We can also assume that $0 \leq \Delta_i^+ \leq u_i - l_i$ and $0 \leq \Delta_i^- \leq u_i - l_i$, otherwise constraints (6)–(7) are either redundant or not feasible. Note that for $t = 1$ the constraints (6)–(7) refer to values p_{i0} and x_{i0}, which clearly are not variables but parameters to be set according to the initial conditions (cf. τ_i^0 above).

The objective function usually contains the minimization of the production costs, that depend on two main contributions: the generation costs and the start-up costs. The generation costs, for each unit i and time period t, are often expressed by a convex quadratic cost function of the type

$$f_i(p_{it}) = a_i p_{it}^2 + b_i p_{it} \ , \qquad\qquad (8)$$

possibly plus a fixed cost $c_i x_{it}$. This is an approximation of the true cost function, that does not take into account some technical characteristics of the units, such as the so-called "valve points". However, the approximation is generally deemed to be accurate enough for practical purposes. Indeed, in many cases the cost function is further approximated by a piecewise linear (or even downright linear) function in order to get good feasible solutions in short time [28].

The start-up costs should in general be expressed as a function $s_i(x_i)$ of the complete state vector x_i, as it depends on the time τ that unit i has been off. In its most accurate formulation, the start-up cost can be computed by means of two functions. One is a concave cost function of the type $\sigma_i(\tau) = \bar{\sigma}_i(1 - e^{-\beta_i \tau}) + \alpha_i$, corresponding to the fact that the cost of starting up the unit depends on the temperature, which, if the unit is left to cool, drops with an exponential law towards ambient temperature (e.g., see [47, 55, 49]). However, for shorter stops it might be preferable to spend some fuel just in order to keep the unit at the right temperature, which can be assumed to have a linear cost $\gamma_i \tau$ on the number of time periods. For each value of τ, then, the optimal choice between the two options (usually referred to as "cooling" and "banking") is just the one giving minimum start-up cost. For our purposes, this complex function only need to be known at the discrete set of values

$$\sigma_{i\tau} = \min(\bar{\sigma}_i(1 - e^{-\beta_i \tau}) + \alpha_i \ , \ \gamma_i \tau) \qquad \tau \in T(\tau_i^-, \bar{\tau}_i) \ , \qquad\qquad (9)$$

where $\bar{\tau}_i$ is the time such that $\bar{\sigma}_i(\bar{\tau}_i) \approx \bar{\sigma}_i(\bar{\tau}_i + 1) \approx \bar{\sigma}_i + \alpha_i$, i.e., the unit has reached ambient temperature and the start-up cost is maximum (in general, banking is only convenient for short stops, and cooling is preferable in the long run). Whatever the exact form of the function, the only relevant property needed for MIP formulations is that the values $\sigma_{i\tau}$ are non decreasing with respect to τ. Using this property, [43] suggested to express the start-up costs by means of a single extra new variable s_{it} and $\bar{\tau}_i - \tau_i^- + 1$ extra constraints (for each unit and time instant), as follows:

$$s_i(x_i) = \textstyle\sum_{t=1}^n s_{it} \qquad\qquad (10)$$

$$s_{it} \geq \sigma_{i\tau}\left(x_{it} - \textstyle\sum_{j=1}^\tau x_{i,t-j} \right) \qquad\qquad t \in T \ , \ \tau \in T(\tau_i^-, \bar{\tau}_i) \qquad\qquad (11)$$

$$s_{it} \geq 0 \qquad\qquad t \in T. \qquad\qquad (12)$$

Even though the number of extra variables and constraints in (10)–(12) is reasonably limited, using such a detailed representation of the start-up cost in a MIP model can have a substantial impact on the performances; this is why, most often the start-up costs are simply approximated with the fixed maximal cost $(\bar{\sigma}_i + \alpha_i)$. In general, since solution time is a crucial issue, the trade-off between an accurate representation of the physical behavior of generating units and the solution cost of the corresponding models is nontrivial. In practice, often simplified models are employed in order to quickly find and approximated solutions of good quality. We will refer to the parameters $\sigma_{i\tau}$ as *history-dependent start-up costs* if $\bar{\tau}_i > \tau_i^-$, while we will refer to *fixed start-up costs* when $\bar{\tau}_i = \tau_i^-$.

While most of the constraints of the standard UC problem concern the behavior of a specific unit $i \in I$, system-wide constraints that link the decisions of the different units are also present. The simplest and most common form of system-wide constraints is that of the demand constraints

$$\textstyle\sum_{i \in I} p_{it} = d_t \qquad t \in T, \qquad\qquad (13)$$

where d_t is the (forecasted) total energy demand at time period t. These constraints are valid for the so called *bus network*, i.e., the case in which the transmission network has ample capacity to accommodate energy transfer and therefore the physical location of generators and constraints is irrelevant. In some (but not all) applications the capacity of the transmission network may become a limiting factor impacting the production decisions, and more accurate representations of the network are needed. The simplest ones (DC model) boil down to just a set of linear constraints, while the most accurate ones (AC model) involve highly nonlinear terms that are much harder to deal with. This has recently motivated a quite active research stream where formulations (or tight relaxations) of AC constraints are proposed using Second-Order Cone or SemiDefinite constraints (e.g., [7, 18] and the references therein). Another source of complexity comes from the fact that the only (somewhat crude) way to influence how energy flows on an electrical network, for given production and consumption at the nodes, is to change the network itself; this in practice can typically only mean to disconnect lines, giving rise to the so-called Optimal Transmission Switching variants of UC where binary variables are used to model the corresponding disconnection decisions. Yet other system-wide constraints pertain to "reserve" (primary, secondary, or inertia) that are established to guarantee that the system will remain operational even if the actual conditions deviate (not too much) from the expected ones. Yet, all these variants typically do not impact of the formulation of the individual units, and therefore need not be discussed here in detail.

The 3-bin formulation has been introduced in [46], and independently in [40]. It starts from an exact formulation of the minimum up- and down-time constraints only, that is obtained by lifting the problem in an extended space defined by 3 vectors of binary variables: besides the commitment variables x_{it}, one introduces *start-up variables* v_{it} denoting if unit i has been started up at time period t (i.e., $x_{it} = 1$ and $x_{i,t-1} = 0$) and *shut-down variables* w_{it} denoting if i has been shut down t (i.e., $x_{it} = 0$ and $x_{i,t-1} = 1$). Using these, the minimum up- and down-time constraints (2)–(3) can be replaced by

$$\sum_{s \in T(t-\tau_+^i+1,t)} v_{is} \le x_{it} \qquad\qquad t \in T(\tau_i^+ + 1, n), r \qquad\qquad (14)$$

$$\sum_{s \in T(t-\tau_-^i+1,t)} w_{is} \le 1 - x_{it} \qquad\qquad t \in T(\tau_i^- + 1, n), \qquad\qquad (15)$$

$$x_{it} - x_{i,t-1} = v_{it} - w_{it} \qquad\qquad t \in T(2, n); \qquad\qquad (16)$$

(note how (16) are flow-conservation-type constraints). Consequently, the ramp constraints (6)–(7) can be reinforced as [44]

$$p_{it} - p_{i,t-1} \le \Delta_i^+ x_{i,t-1} + \bar{l}_i v_{it} \qquad\qquad t \in T, \qquad\qquad (17)$$

$$p_{i,t-1} - p_{it} \le \Delta_i^- x_{it} + \bar{u}_i w_{it} \qquad\qquad t \in T. \qquad\qquad (18)$$

In [44] it was also proposed to reinforce constraints (11) by using start-up and shut-down variables, as follows:

$$s_{it} \ge \sigma_{i\tau}(v_{it} - \sum_{j=2}^\tau w_{i,t-j+1}) \qquad t \in T \ , \ \tau \in T(\tau_i^-, \bar{\tau}_i) \ . \qquad\qquad (19)$$

Note that with fixed start-up costs s_i, the 3-bin formulation can be significantly simplified, as the start-up cost is then completely captured by adding the simple term

$$\sum_{t \in T} s_i v_{it} \qquad\qquad (20)$$

to the objective function, with no need of the extra variables s_{it} and the constraints (11) or (19).

As the 3-bin formulation is generally accepted as a better starting formulation, we adopt it as a benchmark for the new formulation that we will propose in the following. In particular, in the rest of the paper we will refer to the following model as the *3-bin* formulation:

$$
\begin{aligned}
\min \quad & \sum_{i \in I} \sum_{t \in T} (8) + \sum_{i \in I} \sum_{t \in T} c_i x_{it} + \sum_{i \in I} (20) \\
& (13) \\
& (14) \ , \ (15) \ , \ (16) \ , \ (17) \ , \ (18) \qquad\qquad i \in I \\
& x_{it}, v_{it}, w_{it} \in \{0, 1\} \qquad\qquad i \in I, t \in T,
\end{aligned}
\qquad (21)
$$

with the addition of proper initial conditions (4)-(5) for each unit.

3. Literature review of polyhedral descriptions

Here we revise the main polyhedral results proposed in the literature for UC. Due to the fact that the 3-bin formulation is usually stronger than the 1-bin formulation when ramp constraints are used, most attempts start from the former. One exception is [39], that gave a polyhedral description of the minimum up/down time constraints starting from the 1-bin formulation. This description has an exponential number of constraints that can be separated in polynomial time. Although this improve the lower bound provided by the 1-bin formulation, the same (or better) result(s) can be obtained with the 3-bin formulation, and therefore we will concentrate on that.

One of the first papers proposing new valid inequalities for the 3-bin formulation of UC was [12]. This paper was very influential, and several subsequent papers improved most of the results presented there. In particular, [44] presented several types of new constraints based on ramp limits. Then [16] proposed new inequalities based on the study of two distinct polytopes which include the UC polytope: the ramp-up and ramp-down polytopes; subsequently, [45] improved some of these result by considering the full UC polytope. In [31] the special case where only start-up and shut-down limits are imposed were characterized. On a different line of research, [47, 10] analyzed the start-up costs: the former paper presented the case where only "cooling" (i.e., no "banking") is allowed, while the latter gave a complete polyhedral charaterization for the definition of general history-dependent start-up costs. Finally, [38] presents a comprehensive review of the previous results and some new type of inequalities, studies the ways to combine the different types of inequalities in new models, and presents a large computational experience.

In the following we present some of the above cited results according to the classification in [38].

Strengthened generation limits. This type of inequalities limits the generation for each unit and time instant reinforcing upper bound inequalities in (1). In [44] (cf. Eq. (19)) the following constraints were proposed (here presented in the equivalent form of Eq. (37) described in [38]):

$$p_{it} \leq u_i x_{it} - \sum_{j=0}^{K_i^{SD}(t)} (u_i - (\bar{u}_i + j\Delta_i^-))w_{i,t+1+j} \ , \tag{22}$$

where $K_i^{SD}(t) = \min\{T_i^{RD}, \tau_i^+ - 1, T - t - 1\}$ and $T_i^{RD} = \lfloor \frac{u_i - \bar{u}_i}{\Delta_i^-} \rfloor$. These constraints reduce the upper bound on p_{it} if a shut-down occurs in the period from $t+1$ to $t+1+K_i^{SD}(t)$.

The approach was furthered in [45], where new variable upper bounds on the generation were introduced that include information of both the ramp-up and ramp-down trajectories:

$$p_{it} \leq u_i x_{it} - (u_i - \bar{u}_i)w_{i,t+1} - \sum_{j=0}^{\min\{\tau_i^+ - 2, T_i^{RU}\}} (u_i - \bar{l}_i - j\Delta_i^+)v_{i,t-i}, \tag{23}$$

where $T_i^{RU} = \lfloor \frac{u_i - \bar{l}_i}{\Delta_i^+} \rfloor$. Combining the results of [44] and [45], new inequalities of this class were found in [38] which can strengthen the formulation.

Strengthened ramp-up and ramp-down constraints. Under appropriate conditions, one can replace (17)–(18) with stronger inequalities [44]. In particular, if $\Delta_i^+ > \bar{u}_i - l_i$ and $\tau_i^+ \geq 2$ then the following inequalities are valid:

$$p_{it} - p_{i,t-1} \leq \Delta_i^+ x_{it} - l_i w_{it} - (\Delta_i^+ - \bar{u}_i + l_i)w_{i,t+1} + (\bar{l}_i - \Delta_i^+)v_{it} \qquad t \in T(1, n-1) \ . \tag{24}$$

Indeed, at most one among w_{it}, $w_{i,t+1}$, and v_{it} can be equal to one. If $w_{it} = 1$, then $x_{it} = p_{it} = 0$ and (24) implies to $p_{i,t-1} \geq l$. If $w_{i,t+1} = 1$ then $x_{it} = 1$ and the constraint reduces to $p_{it} - p_{i,t-1} \leq \bar{u}_i - l_i$, that is valid because in this case $p_{it} \leq \bar{u}_i$ and $p_{i,t-1} \geq l$. If $v_{it} = 1$, then (24) reduces to $p_{it} \leq \bar{l}_i$. Finally, if $w_{it} = w_{i,t+1} = v_{it} = 0$, then constraint (24) reduces to $p_{it} - p_{i,t-1} \leq \Delta_i^+$ if $x_{it} = 1$ or to $p_{it} = p_{i,t-1} = 0$ if $x_{it} = 0$. A symmetric version of (24) is also presented, and other inequalities can be derived under some special conditions. Moreover, in [44, 16, 45] different cases of *two-periods ramping contraints*, *three-periods ramping constraints*, and further generalizations that results in an exponential number of constraints are presented.

Strengthened description of start-up costs. In [47] a model for the start-up costs based on temperatures is proposed that works when banking is not allowed. It requires to add new continuous variables:

$temp_{it}$, the temperature of unit i at time t, and h_{it}, the heating needed to restart at time t. The start-up cost is then expressed by the cost of the heating $\bar{\sigma}_i$ plus the fixed cost α_i:

$$s_{it} = \bar{\sigma}_i h_{i,t-1} + \alpha_i v_{it}. \tag{25}$$

The temperature is normalized to be equal to 1 when the unit is on and decreases exponentially when the unit is off. The heating and the temperatures are linked by the following constraints:

$$x_{it} \le temp_{it} \le 1 \qquad\qquad t \in T \tag{26}$$
$$temp_{i1} = e^{-\beta_i \max(-\tau_0, 0)} + h_{i0} \tag{27}$$
$$temp_{it} = e^{-\beta_i} temp_{i,t-1} + (1 - e^{-\beta_i}) x_{i,t-1} + h_{i,t-1} \qquad t \in T(2, n) \ . \tag{28}$$

The constraints (26) force the temperature to be equal to 1 when the unit is on. Due to the cost paid in the objective function for (25), $h_{i,t-1} > 0$ only when $x_{it} = 1$ and the temperature $temp_{it} = 1$. Therefore, from (28) we get that $h_{i,t-1} = temp_{it} - e^{-\beta_i} temp_{i,t-1} = 1 - e^{-\beta_i} e^{-\beta_i} temp_{i,t-2} = \ldots = 1 - e^{-\beta_i \tau}$ where τ is equal to the number of time instants that the unit has been off. The term $1 - e^{-\beta} x_{i,t-1}$ in (28) is need to neutralize the heating when the unit is kept on; i.e., if $x_{i,t-1} = x_{it} = 1$, then $temp_{it} = temp_{i,t-1} = 1$ and (28) is satisfied with $h_{i,t-1} = 0$. If, rather, $x_{i,t-1} = 1$ and $x_{it} = 0$, then $temp_{it} \in [0, 1]$ and (28) is satisfied with $temp_{it} = e^{-\beta_i} + (1 - e^{-\beta_i}) + h_{i,t-1} \Rightarrow temp_{it} = 1$ and $h_{i,t-1} = 0$. The result is extended in [10], where an exponential class of valid inequalities (which can be separated in polynomial time) is presented that can be used to express the general history-dependent start-up costs.

Strengthened convex generation costs. While (8) is already convex, p_{it} is a semi-continuous, hence nonconvex, function. In other words, the "true" objective function should rather be expressed as

$$f_i(p_{it}, x_{it}) = \begin{cases} a_i p_{it}^2 + b_i p_{it} & \text{if } l_i \le p_{it} \le u_i \text{ and } x_{it} = 1 \\ 0 & \text{if } p_{it} = x_{it} = 0 \\ \infty & \text{otherwise} \end{cases} . \tag{29}$$

While (29) is nonconvex, its *convex envelope* (best possible convex approximation) turns out to be easily computed [22]:

$$h_i(p_{it}, x_{it}) = \begin{cases} a_i \frac{p_{it}^2}{x_{it}} + b_i p_{it} & \text{if } x_{it} > 0 \\ 0 & \text{if } x_{it} = 0. \end{cases} \tag{30}$$

This is called the *perspective function* of f_i; note that $h_i(p_{it}, x_{it}) = f_i(p_{it})$ if $x_{it} \in \{0, 1\}$, but $h_i(p_{it}, x_{it}) > f_i(p_{it})$ if $0 < x_{it} < 1$. Thus, substituting (30) to (8)—a technique known as *perspective reformulation*—has the potential to significantly increase the lower bound, as confirmed in several studies [22, 33, 24, 28, 29, 26, 13, 19, 20, 36, 27, 15]. Different special methods [22, 33, 26, 19, 20] have been studied to efficiently deal with this "very nonlinear" term in the continuous relaxation without increasing too much its computational cost w.r.t. the case of the "simple" (8). One of these (that it is not even always the best choice [20]) is through the generation of the so-called *perspective cuts*, that can be efficiently used in a branch-and-cut framework,

$$z_{it} \ge a_i \bar{p} p_{it} - a_i \bar{p}^2 x_{it} \quad \bar{p} \in [0, u_i] \ , \tag{31}$$

where the variables z_{it} substitutes $f_i(p_{it})$ in the objective function. Even choosing a fixed limited number of perspective cuts can be an effective computational strategy, much more so than standard piecewise-approximation of (8) alone [28].

All the above results show that describing the convex hull of (1UC) solutions, when all the technical constraints are considered, is highly nontrivial. Hence, formulations used in practice usually have to resort to carefully picking only some of the above ideas. In [38] a large computational experience is presented to compare several different models that mix in many different types the formulation tricks described in this section.

4. The dynamic programming algorithm

While (1UC) is nontrivial to describe in the variable spaces proposed in the previous sections, it is actually relatively easy to solve. Indeed, in [23] a Dynamic Programming (DP) algorithm was proposed that can solve (1UC) with all the constraints—minimum up- and down-time, ramp and generation limits—in $O(n^3)$ with the standard quadratic separable cost function (8) (and that can be generalized to more complex objectives). We now present an improvement of that DP algorithm and recall the basic ingredients of the approach that are necessary to present the MIP formulation.

In this paragraph, since the unit index $i \in I$ is fixed we will drop it for notational simplicity. The DP algorithm is based on defining a state-space graph $G = (N, A)$. The nodes in N are of two types: ON_t and OFF_t for each $t \in T$, plus two special nodes, the source s and the sink d. The arcs in A are of two types: arcs (OFF_h, ON_k), denoting that the unit is turned ON at the beginning of time period h and that the unit remains ON until the end of time period k (indicated as *ON arcs*); arcs (ON_k, OFF_r), denoting that the unit is OFF from time periods $k + 1$ to time period $r - 1$ (indicated as *OFF arcs*).

The OFF arcs satisfy minimum-down time constraints, that is $(ON_k, OFF_r) \in A$ if and only if $r \geq k + \tau^- + 1$, and are labeled with the start-up cost that depends on the length $r - k - 1$ of the off period. Note that the most general time-dependent start-up costs (9) are easily handled within this framework, since the computation is done entirely offline.

The ON arcs satisfy minimum-up time constraints, that is $(OFF_h, ON_k) \in A$ if and only if $k \geq h + \tau^+ - 1$, and are labeled with the cost of the optimal dispatch in the associated period. This is composed of two parts: fixed cost, and variable cost. The first is just the sum of c_i for all periods from h to k (extremes included), since the unit will be committed in that interval. The variable cost, that depends on the p_{it} variables, is just the optimal value $z^{hk} = z(ED^{hk})$ of the following *Economic Dispatch problem with Ramping Constraints*

$$\min \sum_{t \in T(h,k)} f(p_t) \tag{32}$$
$$l \leq p_t \leq u \qquad\qquad t \in T(h, k) \tag{33}$$
$$p_h \leq \bar{l} \tag{34}$$
$$p_{t+1} \leq p_t + \Delta^+ \qquad\qquad t \in T(h, k - 1) \tag{35}$$
$$p_t \leq p_{t+1} + \Delta^- \qquad\qquad t \in T(h, k - 1) \tag{36}$$
$$p_k \leq \bar{u} \tag{37}$$

which we will denote as (ED^{hk}). Since all the relevant binary variables are fixed, (ED^{hk}) is an optimization problem with convex objective function and linear constraints. Hence, its optimal objective function value $z^{hk} = z(ED^{hk})$ can be computed in polynomial time.

Moreover, there are the connections between the source node s and the ON and OFF nodes defined according to the initial state of the unit. That is, if the unit is committed since τ^0 time periods, then there is an arc from s to each node ON_k such that $k + \tau^0 \geq \tau^+$. If, instead, the unit is uncommitted since $-\tau^0$ time periods, then there is an arc from s to each node OFF_h such that $h - \tau^0 - 1 \geq \tau^-$; the latter arcs are labeled with the corresponding start-up cost. The ending node defines the type of the arcs starting from s. All nodes are then connected to the sink node d: arcs (ON_t, d) are of type OFF, arcs (OFF_t, d) are of type ON and their costs is computed for the period $[t, T]$ only. The arc (s, d) means that the unit remains with the same status as at the beginning of the period and it is an ON arc if the unit was ON at time 0, and an OFF arc with zero cost if the unit was OFF at time 0.

Summing up, the state-space graph G has $2n + 2$ nodes and $O(n^2)$ arcs; every s–d path on G represents a feasible schedule for the unit. Hence, (1UC) is reduced to a shortest path problem on an acyclic graph with $O(n)$ nodes and $O(n^2)$ arcs. Thus, the problem can be solved in $O(n^2)$ once that all the data has been computed. We remark that a larger, more complex graph with $O(n^2)$ nodes was proposed in [23], but the one described in this paragraph is clearly preferable. Yet, in [23] it is proved that all $O(n^2)$ Economic Dispatch problems with Ramping Constraints can be solved in $O(n^3)$ by means of another Dynamic Programming algorithm, which is therefore the cost of the overall procedure, as it was with the original graph. The new graph state-space graph G will be the starting point for developing our MIP formulation in next paragraph.

5. The convex hull for the thermal single-unit polytope

In this section we introduce a new formulation for (1UC) that is inspired by the DP algorithm presented in Section 4. This new formulation is composed of two parts:

- the shortest path formulation based on the state-space graph G of the DP algorithm;

- new power variables, their related cost, and the linking constraints with the previous part.

As in the previous section, the unit index $i \in I$ is fixed and therefore we drop it.

The shortest path formulation is straightforward: one just introduces the node-arcs incidence matrix of the graph and writes the obvious system of inequalities. Then we can then simply write this part of the formulation as

$$Ey = b \ , \ \ y \geq 0 \ , \tag{38}$$

where E is the node-arcs incidence matrix of G, y is the vector of arc flow variables, and b is the vector with all zero entries except $b_s = -1$ and $b_d = 1$ for the source node s and the sink node d, respectively.

We now add variables p_t^{hk} associated with each ON arc $(OFF_h, ON_k) \in A$ and with $t \in T(h,k)$ to compute the power level for each time instant and the related costs. With these variables we define the sub-problem

$$\min \quad c^{hk} y^{hk} + \sum_{t \in T(h,k)} f(p_t^{hk}) \tag{39}$$

$$\left. \begin{array}{ll} ly^{hk} \leq p_h^{hk} \leq \bar{l}y^{hk} \\ ly^{hk} \leq p_t^{hk} \leq uy^{hk} & t \in T(h+1, k-1) \\ ly^{hk} \leq p_t^{hk} \leq \bar{u}y^{hk} \\ p_{t+1}^{hk} \leq p_t^{hk} + y^{hk}\Delta^+ & t \in T(h, k-1) \\ p_t^{hk} \leq p_{t+1}^{hk} + y^{hk}\Delta^- & t \in T(h, k-1) \end{array} \right\} \tag{40}$$

$$y^{hk} \in \{0,1\} \tag{41}$$

which, basically, computes the cost of traversing the arc $(OFF_h, ON_k) \in A$. Hereafter, we denote with $A_{ON} = \{(h,k)|h,k \in T, (OFF_h, ON_k) \in A\}$ the set of pairs (h,k) such that (OFF_h, ON_k) is an ON arc. We can easily describe the convex hull of (39)–(41), since the p_t^{hk} for all $t \in T(h,k)$ are semi-continuous variables governed by the *same* y^{hk}. Indeed, the already recalled results about the perspective reformulation show that all that is needed for this is to replace (39) with

$$\min \ c^{hk} y^{hk} + \left(h(p^{hk}, y^{hk}) = \sum_{t \in T(h,k)} y^{hk} f(p_t^{hk}/y^{hk}) \right) \ . \tag{42}$$

A function f is *Second-Order Cone-representable* (SOC-representable) if its epigraph $Epi(f) = \{ (x,z) : z \geq f(x) \}$ can be represented with a finite set of Second-Order Cone (SOC) constraints of the form

$$\|\, b_j - A_j x - B_j y \,\|_2 \leq a_j - q_j x - r_j y \quad j \in J \ . \tag{43}$$

It is well-known [6] that for a SOC-representable function f, the associated perspective function h is "almost always" SOC-representable. Because our results hinge on conic representability and duality, we will require that (43) is available for h such that the *strict feasibility* condition holds: for some $(\bar{x}, \bar{y})$, $\|\, b_j - A_j\bar{x} - B_j\bar{y} \,\|_2 < a_j - q_j\bar{x} - r_j\bar{y}$ for all $j \in J$, so that strong duality holds. For instance, for $f(p) = p^2$, $h(p,y) = p^2/y$, and

$$Epi(h) = \left\{ (p, y, z) \ : \ \sqrt{4p^2 + (z-y)^2} \leq z + y \ , \ \ z \geq 0 \right\}$$

Hence, the (SOCP) problem

$$\begin{aligned} \min \quad & c^{hk} y^{hk} + \sum_{t \in T(h,k)} z_t^{hk} \\ & (40) \\ & \sqrt{4 \sum_{t \in T(h,k)} (p_t^{hk})^2 + (z^{hk} - y^{hk})^2} \leq z^{hk} + y^{hk} \\ & y^{hk} \in [0,1] \ , \ \ z^{hk} \geq 0 \end{aligned} \tag{44}$$

is equivalent to (39)–(41), i.e., it describes the convex hull of its feasible solutions. Separately, (38) has the integrality property, and therefore it would represent the convex hull of (1UC) were the objective function linear. We will show that the combination of the two preserves the property, i.e., defines the convex hull of (1UC), using the nonlinear analogous of the well-known "Approach no. 4" of [54] (used by Edmonds in [17] and by others). To do that, we need the following characterization of the convex hull of a Mixed-Integer SOCP set:

Proposition 5.1. *Consider the (SOCP) set*

$$C = \left\{ (x,y) \in \mathbb{R}^{n+1} \ : \ \| b_j - A_j x - B_j y \|_2 \leq a_j - q_j x - r_j y \quad j \in J \ , \ x \geq 0 \ , \ y \geq 0 \right\} \ , \tag{45}$$

where $A_j \in \mathbb{R}^{m_j \times n}$, $B_j \in \mathbb{R}^{m_j}$, $b_j \in \mathbb{R}^{m_j}$, $q_j \in \mathbb{R}^n$, and $r_j, a_j \in \mathbb{R}$, and its mixed-integer restriction

$$S = \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{Z} \ : \ (45) \ , \ x_k \in \mathbb{Z} \quad k \in K \subseteq \{1, \ldots, n\} \right\} \ . \tag{46}$$

Assume that for each objective function $(c,d) \in \mathbb{R}^{n+1}$ strong duality holds for C (say, strict feasibility holds), and furthermore there exists $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^{|J|}$, where $m = \sum_{j \in J} m_j$, such that

$$\max \{ cx + dy \ : \ (x,y) \in S \} = \min \quad \begin{array}{ll} \sum_{j \in J} \lambda_j b_j + \mu_j a_j & \\ \sum_{j \in J} \lambda_j A_j + \mu_j q_j \geq c & \\ \sum_{j \in J} \lambda_j B_j + \mu_j r_j \geq d & : \\ \|\lambda_j\|_2 \leq \mu_j & j \in J \end{array} \tag{47}$$

then $C = conv(S)$.

Proof. The rightmost optimization problem in (47) is the conic dual of $\max \{ cx + dy \ : \ (x,y) \in C \}$; since strong duality holds, (47) states that $\sigma_C = \sigma_S$, where $\sigma_X(v) = \max \{ vx \ : \ x \in X \}$ is the *support function* of the set X. Standard convex analysis results [34, §V.2] show that for any set P, $\sigma_P = \sigma_{conv(P)}$ ("the support function does not distinguish a set from its closed convex hull"), and that a closed convex set is completely determined by its support function [34, Theorem V.2.2.2]; since $\sigma_C = \sigma_S = \sigma_{conv(S)}$, then $C = conv(S)$. ∎

The required result is now that, for an appropriate definition of *composition* of (SOCP) sets, the description of the convex hull of the composed set can be obtained from the descriptions of the convex hulls of the composing ones.

Definition 5.2. *For $h = 1, 2$, let $S^h \subset \mathbb{R}^{n_h} \times \mathbb{R}$ be two sets; their 1-sum composition is*

$$S^1 \oplus S^2 = \left\{ (x^1, x^2, y) \in \mathbb{R}^{n_1 + n_2 + 1} \ : \ (x^h, y) \in S^h \quad h = 1, 2 \right\} \ .$$

Lemma 5.3. *For $h = 1, 2$, let $S^h \subset \mathbb{R}^{n_h} \times \mathbb{R}$ be two sets. Assume that*

$$C^h = \left\{ (x^h, y) \in \mathbb{R}^{n_h + 1} \ : \ \| b_j^h - A_j^h x^h - B_j^h y \|_2 \leq a_j^h - q_j^h x^h - r_j^h y \quad j \in J(h) \ , \ x^h \geq 0 \ , \ y \geq 0 \right\}$$

(where all matrices/vectors have the appropriate dimension) describes the convex hull of S^h, that strong duality holds, and that $(x^h, y) \in S^h$ implies that $y \in \{0,1\}$: then, the system

$$\begin{array}{ll} \|b_j^h - A_j^h x^h - B_j^h y\|_2 \leq a_j^h - q_j^h x^h - r_j^h y & h \in \{1,2\}, \ j \in J(h) \\ x^h \geq 0, y \geq 0 & h \in \{1,2\} \end{array} \tag{48}$$

describes $conv(S^1 \oplus S^2)$ and strong duality holds for each linear objective function (c^1, c^2, d).

Proof. We prove that for every $(c^1, c^2, d) \in \mathbb{R}^{n_1 + n_2 + 1}$ there exists a dual optimal solution for (48) whose objective function value is equal to $z = \max\{ c^1 x^1 + c^2 x^2 + dy \ : \ (x^h, y) \in S^h \quad h = 1, 2 \}$, according to Proposition 5.1. This proof follows similar steps as the proof by Chvátal [14] for composition of stable set polyhedra by clique-cutsets. The (conic) dual of $\max\{ c^1 x^1 + c^2 x^2 + dy \ : \ (48) \}$ is

$$\min \sum_{h \in H} \sum_{j \in J(h)} \lambda_j^h b_j^h + \mu_j^h a_j^h \tag{49}$$

$$\sum_{j \in J(h)} \lambda_j^h A_j^h + \mu_j^h q_j^h \geq c^h \qquad\qquad h = 1, 2 \tag{50}$$

$$\sum_{h=1}^2 \sum_{j \in J(h)} \lambda_j^h B_j^h + \mu_j^h r_j^h \geq d \tag{51}$$

$$\|\lambda_j^h\|_2 \leq \mu_j^h \qquad\qquad h = 1, 2 \ , \ j \in J(h) \tag{52}$$

For $y \in \{0,1\}$, we define $z_y^h = \max\{c^h x^h + dy \ : \ (x^h, y) \in S^h\}$. Consider the following problems for $h = 1, 2$:

$$\sigma^h = \max\{c^h x^h + (d + z_0^h - z_1^h)y \ : \ (x^h, y) \in S^h\} \ . \tag{53}$$

Note that $\sigma^h = z_0^h$ for $h = 1, 2$. Therefore, by applying Proposition 5.1 to problems (53) for $h = 1, 2$, there exists (λ^h, μ^h) such that (50) and (52) hold together with

$$\sum_{j \in J(h)} \lambda_j^h B_j^h + \mu_j^h r_j^h \geq d + z_0^h - z_1^h \quad , \quad \lambda^h b^h + \mu^h a^h = z_0^h \ .$$

Let $z_0 = z_0^1 + z_0^2$, $z_1 = z_1^1 + z_1^2 - d$, and $z = \max\{z_0, z_1\}$. Clearly $z - z_0 \geq 0$; then by defining

$$\sigma = \max\{(z - z_0)y \ : \ (x^1, y) \in S^1\}, \tag{54}$$

we have that $\sigma = z - z_0$ (indeed, $y \in \{0,1\}$ for all $(x^1, y) \in S^1$, and the maximum is attained for $y = 1$). By applying Proposition 5.1 to problem (54) there exists $(\tilde{\lambda}, \tilde{\mu})$ such that

$$\begin{aligned} &\sum_{j \in J(1)} \tilde{\lambda}_j A_j^1 + \tilde{\mu}_j q_j^1 \geq 0 \ , \ \sum_{j \in J(1)} \tilde{\lambda}_j B_j^1 + \tilde{\mu}_j r_j^1 \geq z - z_0 \ , \ \tilde{\lambda} b^1 + \tilde{\mu} a^1 = z - z_0 \\ &\|\tilde{\lambda}_j\|_2 \leq \tilde{\mu}_j \hspace{7cm} j \in J(1) \end{aligned} \quad .$$

Now, for $\hat{\lambda}^1 = \lambda^1 + \tilde{\lambda}$, $\hat{\lambda}^2 = \lambda^2$, $\hat{\mu}^1 = \mu^1 + \tilde{\mu}$, $\hat{\mu}^2 = \mu^2$ the following relationships hold:

$$\sum_{j \in J(1)} \hat{\lambda}_j^1 A_j^1 + \hat{\mu}_j^1 q_j^1 = \sum_{j \in J(1)} (\lambda_j^1 + \tilde{\lambda}_j) A_j^1 + (\mu_j^1 + \tilde{\mu}_j) q_j^1 =$$

$$= \sum_{j \in J(1)} \lambda_j^1 A_j^1 + \mu_j^1 q_j^1 + \sum_{j \in J(1)} \tilde{\lambda}_j A_j^1 + \tilde{\mu}_j q_j^1 \geq c^1 + 0 = c^1 \ ;$$

$$\sum_{j \in J(2)} \hat{\lambda}_j^2 A_j^2 + \hat{\mu}_j^2 q_j^2 = \sum_{j \in J(2)} \lambda_j^2 A_j^2 + \mu_j^2 q_j^2 \geq c^2 \ ;$$

$$\sum_{h \in H} \sum_{j \in J(h)} \hat{\lambda}_j^h B_j^h + \hat{\mu}_j^h r_j^h = \sum_{j \in J(1)} (\lambda_j^1 + \tilde{\lambda}_j) B_j^1 + (\mu_j^1 + \tilde{\mu}_j) r_j^1 + \sum_{j \in J(2)} \lambda_j^2 B_j^2 + \mu_j^2 r_j^2 =$$

$$= \sum_{j \in J(1)} \lambda_j^1 B_j^1 + \mu_j^1 r_j^1 + \sum_{j \in J(1)} \tilde{\lambda}_j B_j^1 + \tilde{\mu}_j r_j^1 + \sum_{j \in J(2)} \lambda_j^2 B_j^2 + \mu_j^2 r_j^2 \geq$$

$$\geq (d + z_0^1 - z_1^1) + (z - z_0) + (d + z_0^2 - z_1^2) =$$

$$= d + z + (z_0^1 + z_0^2 - z_0) - (z_1^1 + z_1^2 - d) = d + z - z_1 \geq d \ ;$$

$$\|\hat{\lambda}_j^1\|_2 = \|\lambda_j^1 + \tilde{\lambda}_j\|_2 \leq \|\lambda_j^1\|_2 + \|\tilde{\lambda}_j\|_2 \leq \mu_j^1 + \tilde{\mu}_j = \hat{\mu}_j^1 \ ;$$

$$\|\hat{\lambda}_j^2\|_2 = \|\lambda_j^2\|_2 \leq \mu_j^2 = \hat{\mu}_j^2 \ ;$$

$$\sum_{h \in H} \sum_{j \in J(h)} \hat{\lambda}_j^h b_j^h + \hat{\mu}_j^h a_j^h = \sum_{j \in J(1)} (\lambda_j^1 + \tilde{\lambda}_j) b_j^1 + (\mu_j^1 + \tilde{\mu}_j) a_j^1 + \sum_{j \in J(2)} \hat{\lambda}_j^2 b_j^2 + \hat{\mu}_j^2 a_j^2 =$$

$$= \sum_{h \in H} \sum_{j \in J(h)} \lambda_j^h b_j^h + \mu_j^h a_j^h + \sum_{j \in J(1)} \tilde{\lambda}_j b_j^1 + \tilde{\mu}_j a_j^1 = \sum_{h \in H} z_0^h + (z - z_0) = z \ .$$

That is, $\{(\hat{\lambda}^h, \hat{\mu}^h)\}_{h=1,2}$ is feasible for the dual problem (49)–(52) and has objective function value equal to z; applying Proposition 5.1 the lemma follows. ∎

We are now ready for the announced result: the formulation

$$\begin{aligned} \min \quad & cy + \sum_{(h,k) \in A_{ON}} z^{hk} \\ & (38) \\ & z^{hk} \geq \sum_{t \in T(h,k)} y^{hk} f(p_t^{hk}/y^{hk}) \quad (h,k) \in A_{ON} \ , \\ & (40) \ , \ y^{hk} \in [0,1] \hspace{2cm} (h,k) \in A_{ON} \end{aligned} \tag{55}$$

describes the convex hull of the feasible solutions for (1UC).

Theorem 5.4. *Formulation (55) describes the convex hull of the feasible solutions for (1UC).*

Proof. Define S_0 the set of feasible solutions of the network flow problem (38) associated with the DP graph G, and S^{hk} the set of feasible solutions of (44). We can build the set of solutions for the complete problem by iteratively composing the solutions of S_0 with the sets S^{hk}, e.g., in lexicographic order of the pairs $(h, k) \in A_{ON}$. At each step $j \geq 1$ of the process, we are combining a set S_{j-1} and a set S^{hk} (for some fixed pair (h, k)) that only share the single binary variable y^{hk} to obtain the set $S_j = S_{j-1} \oplus S^{hk}$. It is immediate to prove by induction that the SOCP formulation obtained by adding to the inequalities of the system (38) all the inequalities of (44) for all the pairs (h, k) used describes the convex hull of S_j, and satisfies strong duality for each objective function. Indeed, at the first iteration we are combining S_0 with one S^{hk}; both sets satisfy the hypotheses of Lemma 5.3 (having only linear constraints, S_0 does not need any strict feasibility assumption for strong duality to hold), and therefore also the corresponding formulation satisfies them for $S_1 = S_0 \oplus S^{hk}$. Repeating the process, at each step the corresponding SOCP describes $conv(S_j)$ and satisfies strong duality. At the end of the composition process we have obtained all the constraints in (55), which therefore define a SOCP formulation for the convex hull of the overall set of solutions for (1UC). ∎

Formulation (55) is the first formulation for (1UC) that describe the convex hull of feasible solutions considering all the constraints described above and a convex (actually, SOC-representable) objective function. As previously mentioned, the formulation presented in [32] was claimed to have this property, but the claim is proven false in [3] with a counterexample.

Finally, we present a formulation for the UC problem based on the exact single-unit formulation can then be summarized as follows:

$$
\begin{aligned}
\min \quad & \sum_{i \in I} c^i y_i + \sum_{i \in I} \sum_{hk \in A_{ON}} z_i^{hk} \\
& \sum_{i \in I} \sum_{hk:t \in T(h,k)} p_{it}^{hk} = d_t && t \in T \\
& z_i^{hk} \geq \sum_{t \in T(h,k)} y_i^{hk} f^i (p_{it}^{hk}/y_i^{hk}) && (h, k) \in A_{ON}^i \ , i \in I \\
& (38) \ , \ (40) && i \in I.
\end{aligned}
\tag{56}
$$

In the rest of the paper, we will refer to (56) as the *DP formulation*. Note that the number of variables is $O(n^2)$ (n being the number of time instants) for the network flow system and $O(n)$ for each of the $O(n^2)$ subproblems (44) associated with each pair (h, k); hence, the total number of variables in the proposed formulation is $O(n^3)$.

6. Additional DP based formulations

In this Section, we introduce other two formulations which are based on the DP algorithm introduced in Section 4. They represent two different levels of complexity and tightness. When restricted to (1UC), both are less tight than the exact formulation (55) and keep the network constraints (38). The first one uses the original power variables p_{it}, while the second one presents a new type of variables whose cardinality is intermediate between 3-bin and DP formulations.

Given a unit i, consider the commitment variable x_{it}, the start-up/shut down variables v_{it}/w_{it} and the set of variables y_i^{hk} associated with ON arcs (OFF_h, ON_k) such that $t \in T(h, k)$. It is easy to see that, by definition, these variables are related by the following equations:

$$
x_{it} = \sum_{(h,k):t \in T(h,k)} y_i^{hk} \ , \quad v_{it} = \sum_{k \geq t} y_i^{tk} \ , \quad w_{it+1} = \sum_{h \leq t} y_i^{ht} \ .
\tag{57}
$$

Consequently, the ramp-up/ramp-down constraints assume, respectively, the following form:

$$
p_{it} - p_{it-1} \leq \bar{l} + (\Delta^+ - \bar{l}) \sum_{(h,k):t-1 \in T(h,k-1)} y_i^{hk} \qquad i \in I, t \in T(2, T)
\tag{58}
$$

$$
p_{it-1} - p_{it} \leq \bar{u} + (\Delta^- - \bar{u}) \sum_{(h,k):t-1 \in T(h,k-1)} y_i^{hk} \qquad i \in I, t \in T(2, T)
\tag{59}
$$

Note that, in case the unit is on at the beginning of time horizon ($\tau_i^0 > 0$), the initial ramp-up/ramp-down conditions have to be set by

$$
p_{i1} \leq (\Delta_i^+ + p_{i0}) \sum_{k:t \leq k} y_i^{0k} \qquad i \in I : \tau_i^0 > 0
\tag{60}
$$

$$
-p_{i1} \leq (\Delta_i^- - p_{i0}) \sum_{k:t \leq k} y_i^{0k} \qquad i \in I : \tau_i^0 > 0
\tag{61}
$$

Then, the minimum and maximum power output constraints can be re-written as follows:

$$l \sum_{(h,k):t \in T(h,k)} y_i^{hk} \leq p_{it} \leq u \sum_{(h,k):t \in T(h,k)} y_i^{hk} \qquad i \in I, t \in T \tag{62}$$

Note that, if $\tau_i^+ \geq 2$, the right-hand side of constraints (62) can be renforced in this way:

$$p_{it} \leq \bar{l} \sum_{k:t<k} y_i^{tk} + \bar{u} \sum_{h:t>h} y_i^{ht} + u \sum_{(h,k):h<t<k} y_i^{hk} \qquad i \in I : \tau_i^+ \geq 2, t \in T \tag{63}$$

Otherwise, if $\tau_i^+ = 1$, this can be done as follows:

$$p_{it} \leq \bar{l} \sum_{k:t<k} y_i^{tk} + \bar{u} \sum_{h:t>h} y_i^{ht} + u \sum_{(h,k):h<t<k} y_i^{hk} + \min\{\bar{l}, \bar{u}\} y_i^{tt} \qquad i \in I : \tau_i^+ \geq 2, t \in T \tag{64}$$

Finally, by assuming for each unit $i \in I$ fixed start-up costs s_i, the objective funtion is the following, where variables y_{i-}^{hk} are associated with OFF arcs (ON_h, OFF_k):

$$\min \quad \begin{array}{l} \sum_{i \in I} \sum_{t \in T} \sum_{(h,k):t \in T(h,k)} (c_i^{hk} y_i^{hk} + s_i y_{i-}^{hk}) + \\ \sum_{i \in I} \sum_{t \in T} (\sum_{(h,k):t \in T(h,k)} y_i^{hk}) f(p_{it}/(\sum_{(h,k):t \in T(h,k)} y_i^{hk})) \end{array} \quad . \tag{65}$$

We will denote as p_t *model* the following formulation:

$$\min \quad \begin{array}{l} (65) \\ (13) \;, \; (38) \;, \\ (58)\text{--}(61) \;, \; (62)\text{--}(64) \end{array} \tag{66}$$

To introduce the last formulation, we define a variable p_{it}^h denoting the unit i is committed at time t and it has been turned on at time instant h. Differently from the variable p_{it}^{hk}, with p_{it}^h the time when the unit will be turned off is not explained. The relation between variables p_{it} and p_{it}^h is explicated by the following equation:

$$p_{it} = \sum_{h:t \geq h} p_t^h \tag{67}$$

In what follows, we present a new formulation based on the DP algorithm and by considering equation (67). In particular, the ramp-up/down constraints are the following ones:

$$p_{it}^h - p_{it-1}^h \leq -l y_i^{ht-1} + \Delta^+ \sum_{k:t-1<k} y_i^{hk} \qquad i \in I, t \in (2,T), h : t \geq h+1 \tag{68}$$

$$p_{it-1}^h - p_{it}^h \leq \bar{u} y_i^{ht-1} + \Delta^- \sum_{k:t-1<k} y_i^{hk} \qquad i \in I, t \in (2,T), h : t \geq h+1 \tag{69}$$

Again, if $\tau_i^0 > 0$, the initial ramp-up/ramp-down conditions has to be imposed:

$$p_{i1}^0 \leq (\Delta^+ + p_0) \sum_{k:t \leq k} y_i^{0k} \qquad i \in I \tag{70}$$

$$-p_{i1}^0 \leq (\Delta^- - p_0) \sum_{k:t \leq k} y_i^{0k} \qquad i \in I \tag{71}$$

Then, the minimum/maximum power output constraints have the following form:

$$l \sum_{k:t \leq k} y_i^{hk} \leq p_{it}^h \leq u \sum_{k:t \leq k} y_i^{hk} \qquad i \in I, t \in T, h : t \geq h \tag{72}$$

Note that when $t = h$, we can improve the right hand side of constraints (72) with

$$p_{ih}^h \leq \bar{l} \sum_{k:t \leq k} y_i^{hk} \qquad i \in I, t \in T, h : t = h \tag{73}$$

In conclusion, we derive the objective function and demand constrains

$$\min \quad \sum_{i \in I} \sum_{t \in T} \sum_{(h,k):t \in T(h,k)} (c_i^{hk} y_i^{hk} + s_i y_{i-}^{hk}) +$$

$$\sum_{i \in I} \sum_{t \in T} \sum_{h:t \geq h} (\sum_{k:t \leq k} y_i^{hk}) f(p_{it}^h/(\sum_{k:t \leq k} y_i^{hk})) \tag{74}$$

$$\sum_{i \in I} \sum_{h:t \geq h} p_{it}^h = d_t \qquad t \in T \tag{75}$$

In the rest of the paper, with p_t^h *model* we will refer to the following model:

$$\min \quad \begin{array}{l} (74) \\ (75) \;, \; (38) \;, \\ (68)\text{--}(71) \;, \; (72) \;, \; (73) \end{array} \tag{76}$$

	3-bin		DP		p_t		p_t^h	
units	time^{lp}	gap^{lp}%	time^{lp}	gap^{lp}%	time^{lp}	gap^{lp}%	time^{lp}	gap^{lp}%
10	0.12	1.52	16.40	0.67	0.56	1.28	2.85	1.17
20	0.28	1.44	59.07	0.51	1.23	1.17	8.48	1.04
50	0.96	0.88	300.53	0.08	3.84	0.63	25.26	0.55

Table 1: Root node gaps of the DP, p_t, p_t^h and 3bin formulations

7. Computational tests

In this section we test the computational performances of the new formulation DP presented in Section 4 and formulations p_t and p_t^h introduced in Section 6. The main issue, of course, is that of the trade-off between the bound improvement w.r.t. less tight formulations and the cost increase due to the larger size. Indeed, the *3-bin* formulation has $O(n)$ variables only and $O(n)$ constraints for each unit, while the new ones proposed have $O(n^2)$ binary variables, at least $O(n)$ continuous variables, and at least $O(n)$ constraints, for each unit. As already described, the new formulations based on the DP algorithm differ on the type of continuous variables and associated constraints. The first formulation contains $O(n^3)$ variables p_{it}^{hk} for each unit, the second formulation contains $O(n)$ variables p_{it} for each unit, and the third formulation represents an intermediate case and contains $O(n^2)$ variables p_{it}^h for each unit. We denote the new formulations *DP-model*, p_t-*model*, p_t^h-*model*, respectively.

The experiments have been carried out with CPLEX 12.9 on a PC with 2.2 GHz Intel Xeon Gold 5120 CPUs and 64 GB of RAM, under a GNU/Linux Ubuntu 18.04.3 LTS operating system. We used the set of instances published at

<div align="center">http://www.di.unipi.it/optimize/Data/UC.html</div>

considering pure thermal instances ranging from 10 to 50 units and $n = 24$ time periods. For each instance size we performed 5 tests, and we present the average of the results thus obtained. In Table 1 we compare the running times in seconds to solve the continuous relaxation (time^{lp}) of the new formulations presented in this paper and the standard *3-bin* one, with the corresponding gap in percentage (gap^{lp}%) w.r.t. the minimum solution among all formulations. Note that the gap is the one of the "pure" formulation, i.e., before any cut added by CPLEX.

The results in Table 1 show that the root node gaps of each model actually decreases when the size of the instances increases. Furthermore, the DP formulation always provides the best gaps, while the 3-bin the worst ones. However, the required average running time by the former model is considerably larger than the latter. On the other hand, formulations p_t and p_t^h seem to provide (especially the second one) good gaps w.r.t. reasonable computing times.

We have performed some computational results for solving the UC problem at optimality, with 0.01% (Table 2) and with 0.1% (Table 3) gap. In Table 2 and Table 3, for each model, column *time(opt)* reports the average computing time in seconds and the number of instances, over five, solved to optimality (in parentheses) within a time limit of 10000 seconds, columns *nodes* and *gap*% denote the average number of nodes explored and the final gap, respectively.

Overall, the results in Table 2 show that the number of solved instances is the same regardless of the formulation used, except for the model p_t^h which seems to be less efficient. On the other hand, the average computing times are lower for formulation p_t and then for formulation 3-bin. Note that, on instances with 10 units, the DP model takes much longer time than any other model. Obviously, by setting a lower gap (Table 3) more instances are solved; in this case, formulation p_t clearly has the best performances, and can solve all the instances with 50 units.

8. Conclusions

We have presented the first exact formulation for the (1UC) problem with convex (actually, SOC-representable) cost function and all the main operational constraints proposed in the literature. The formulation is based

units	3-bin time(opt)	nodes	gap%	DP time(opt)	nodes	gap%	p_t time(opt)	nodes	gap%	p_t^h time(opt)	nodes	gap%
10	28(5)	275	0.01	832(5)	599	0.01	17(5)	145	0.01	311(5)	865	0.01
20	7036(2)	3561	0.08	7902(2)	1961	0.05	6081(2)	1944	0.05	8590(1)	3933	0.14
50	10000(0)	1619	0.12	10000(0)	695	0.14	10000(0)	1817	0.06	10000(0)	1949	0.21

Table 2: Computational results with gap 10^{-4}

units	3-bin time(opt)	nodes	gap%	DP time(opt)	nodes	gap%	p_t time(opt)	nodes	gap%	p_t^h time(opt)	nodes	gap%
10	21(5)	163	0.09	500(5)	444	0.10	8(5)	42	0.09	218(5)	690	0.10
20	6002(2)	1980	0.11	5490(4)	1237	0.11	4369(3)	1580	0.10	7644(2)	3588	0.17
50	6052(2)	1042	0.14	6927(3)	504	0.11	921(5)	274	0.09	10000(0)	1939	0.21

Table 3: Computational results with gap 10^{-3}

on the combination of the flow formulation of the DP approach, which "has the integrality property for the constraints", and of the perspective reformulation for the objective, which "has the integrality property for the objective". We believe that this combined approach is interesting in its own right, and it could have other applications.

While the proposed (1UC) formulation indeed produces the strongest lower bounds when used as the basis of a (UC) formulation, its large size makes it not completely practical. We have therefore proposed and analyzed two alternative formulations, based on partial variable aggregation, that offer different trade-offs between size and potential bound quality. The experimental results show that the DP and (especially) p_t formulations have good performances w.r.t. the 3-bin one, while the p_t^h model does not seem to be very effective.

We believe that the new formulations presented in this paper show promise, and warrant further investigation in at least two directions. The first is to help in the definition of heuristic algorithms that exploit the much smaller gap and use the better continuous solution to quickly produce feasible solutions with the quality required by practical applications (say, a gap smaller than 0.5%). The second is the fact that, like with all "large" formulations, the number of variables and constraints that are actually required to characterize at the optimal solution is a small fraction of the total number. Thus, generation of variables and constraints, such as the Structured Dantzig-Wolfe Decomposition [21], could very considerably speed-up the overall performances of the algorithm, thereby overcoming the disadvantage related to the larger size of the proposed formulations and making them even more competitive with the *3-bin* one.

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