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**AN INTEGRAL LP RELAXATION FOR A
DRAYAGE PROBLEM**

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Abstract

This paper investigates a drayage problem, where a fleet of trucks must ship container loads from a port to importers and from exporters to the port, without separating trucks and containers during customer service. We present three reformulations for this problem that are valid when the capacity of trucks is equal to 1. For the third reformulation, we also assume that the arc costs are equal for all vehicles, and then we prove that its continuous relaxation admits integer optimal solutions, by checking that its constraint matrix is totally unimodular.

Key words: Logistics; Vehicle Routing Problem; Drayage Problem; Total Unimodularity

1. Introduction

The distribution of containers between customers and intermodal terminals is a critical part of the service provided by liner shipping companies. While their vessels carry thousands of containers in maritime networks, the supply of door-to-door services between intermodal terminals and customers must be performed by trucks carrying one or two containers. These transport services are typically called *drayage* and are much more resource-intensive and energy-consuming than the maritime counterpart [9]. Drayage problems were often faced by optimization methods from the *Vehicle Routing Problem (VRP)*, in order to plan how to serve the requests of import and export customers, who must receive and ship container loads, respectively. More precisely, drayage problems can be viewed as special variants of the *VRP with Backhauls* [10] and *VRP with split delivery* [1].

In the vast literature on drayage, two types of transportation requests are investigated: *drop&pick* and *stay-with*. In the first case, while containers are left at customer locations, drivers can move to other customers, thus bypassing packing and unpacking operations [12, 6, 13, 14, 2, 8]. Conversely, in *stay-with* operations, drivers wait for containers and trucks carry the same container(s) throughout their routes [9, 5, 3]. In this paper we focus on a drayage problem with *stay-with* operations to serve two type of transportation requests: the delivery of container loads from a port to importers and the shipment of container loads from exporters to the same port. Containers emptied at importers can be used to collect cargoes from the exporters.

In container drayage, mostly 20-foot and 40-foot standard containers are transported, and usually only up to two 20-foot containers or one single 40-foot container can be carried by a truck [11]. Therefore, it is worth adopting general mathematical formulations to model the transportation of one or two containers. A general model for drayage problems with *stay-with* and multiple container loads per truck was proposed in [7], together with a metaheuristic solution method.

When trucks carry only one container this model can be simplified. Here we propose three reformulations for this case. We also prove that the constraints matrix in the third reformulation is totally unimodular, then its continuous relaxation produce integer optimal solutions.

2. Modeling

We recall the general model proposed in [7]. We consider a port p , a set of Importers I , a set of Exporters E , a set of trucks K .

Given a graph $G(N, A)$, where $N = \{p\} \cup E \cup I$, $A = A_1 \cup A_2$, $A_1 = \{(i, j) | i \in I \cup \{p\}, j \in N, i \neq j\}$, $A_2 = \{(i, j) | i \in E, j \in E \cup \{p\}, i \neq j\}$.

The following decision variables are defined:

- x_{ij}^k link selection variable: it is equal to 1 if arc $(i, j) \in A$ is traversed by truck $k \in K$, 0 otherwise;
- y_{ij}^k number of loaded containers moved along arc $(i, j) \in A$ by truck $k \in K$;
- z_{ij}^k number of empty containers moved along arc $(i, j) \in A$ by truck $k \in K$;

Moreover c_{ij}^k is the routing cost of truck $k \in K$ on arc $(i, j) \in A$; h_{pj}^k the handling cost of a container put on truck $k \in K$ at the port p to serve customer $j \in N$; $d_i \geq 0$ the number of containers required by customer $i \in I \cup E$; u_k capacity for truck $k \in K$.

$$\min \sum_{k \in K} \left[\sum_{(i,j) \in A} c_{ij}^k x_{ij}^k + \sum_{j \in N} h_{pj}^k (y_{pj}^k + z_{pj}^k) \right] \quad (1)$$

$$\text{s.t.} \quad \sum_{k \in K} \sum_{l \in N} y_{il}^k = \sum_{k \in K} \sum_{j \in p \cup I} y_{ji}^k - d_i \quad \forall i \in I \quad (2)$$

$$\sum_{k \in K} \sum_{l \in N} z_{il}^k = \sum_{k \in K} \sum_{j \in p \cup I} z_{ji}^k + d_i \quad \forall i \in I \quad (3)$$

$$\sum_{l \in N} y_{il}^k \leq \sum_{j \in p \cup I} y_{ji}^k \quad \forall i \in I, \forall k \in K \quad (4)$$

$$\sum_{l \in N} z_{il}^k \geq \sum_{j \in p \cup I} z_{ji}^k \quad \forall i \in I, \forall k \in K \quad (5)$$

$$\sum_{k \in K} \sum_{l \in p \cup E} y_{il}^k = \sum_{k \in K} \sum_{j \in N} y_{ji}^k + d_i \quad \forall i \in E \quad (6)$$

$$\sum_{k \in K} \sum_{l \in p \cup E} z_{il}^k = \sum_{k \in K} \sum_{j \in N} z_{ji}^k - d_i \quad \forall i \in E \quad (7)$$

$$\sum_{l \in p \cup E} y_{il}^k \geq \sum_{j \in N} y_{ji}^k \quad \forall i \in E, \forall k \in K \quad (8)$$

$$\sum_{l \in p \cup E} z_{il}^k \leq \sum_{j \in N} z_{ji}^k \quad \forall i \in E, \forall k \in K \quad (9)$$

$$\sum_{(ji) \in A} (y_{ji}^k + z_{ji}^k) = \sum_{(il) \in A} (y_{il}^k + z_{il}^k) \quad \forall i \in I \cup E, \forall k \in K \quad (10)$$

$$y_{ij}^k + z_{ij}^k \leq u_k x_{ij}^k \quad \forall (i,j) \in A, \forall k \in K \quad (11)$$

$$\sum_{j \in N} x_{ji}^k - \sum_{l \in N} x_{il}^k = 0 \quad \forall i \in N, \forall k \in K \quad (12)$$

$$\sum_{j \in N} x_{pj}^k \leq 1 \quad \forall k \in K \quad (13)$$

$$\sum_{k \in K} \sum_{i \in I \cup E} z_{ip}^k - \sum_{k \in K} \sum_{i \in I \cup E} z_{pi}^k = \sum_{i \in I} d_i - \sum_{i \in E} d_i \quad (14)$$

$$x_{ij}^k \in \{0, 1\} \quad \forall (i,j) \in A, \forall k \in K \quad (15)$$

$$y_{ij}^k \in \mathbb{Z}^+ \quad \forall (i,j) \in A, \forall k \in K \quad (16)$$

$$z_{ij}^k \in \mathbb{Z}^+ \quad \forall (i,j) \in A, \forall k \in K \quad (17)$$

Constraints (2) and (3) are the flow conservation constraints of loaded and empty containers at each importer node respectively; constraints (4) and (5) guarantee that the number of loaded containers cannot be increased after a service at each importer and the number of empty containers cannot be reduced; constraints (6) and (7) are the flow conservation constraints of loaded and empty containers at each exporter node respectively; constraints (8) and constraints (9) guarantee that the number of loaded containers cannot be reduced after a service at each exporter and the number of empty containers cannot be increased; constraints (10) guarantee that the number of containers carried by each truck does not change after visiting a customer; constraints (11) impose that the number of containers carried by each truck is not larger than its

transportation capacity u_k , constraints (12) represent flow conservation constraints for trucks at each node; constraints (13) impose that trucks are not used more than once; constraint (14) is the flow conservation of empty containers at port p .

2.1. First reformulation

When the capacity of the trucks is one, the model can be simplified. The number of variables can be reduced and associated with one-to-one correspondence to possible routes.

The variable y_{ij}^k is equal to x_{ij}^k if $i = p, j \in I$, or $i \in E, j = p$, and 0 otherwise; z_{ij}^k is equal to x_{ij}^k if $i \in \{p\} \cup I, j \in \{p\} \cup E, i \neq j$, and 0 otherwise.

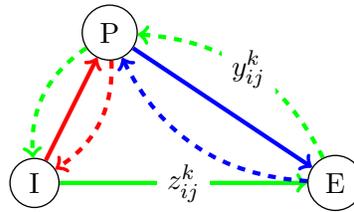


Figure 1: Possible routes

In Fig. 1 we show the allowed routes:

- from the port to an importer and from the importer to the port (red lines);
- from the port to an importer, from the importer to an exporter, from the exporter to the port (green lines);
- from the port to an exporter and from the exporter to the port (blue lines);

dashed and solid lines represent loaded containers and empty containers respectively.

$$\min \sum_{k \in K} \left[\sum_{i \in I; l \in E} f_{il}^k x_{il}^k + \sum_{i \in I} f_{ip}^k x_{ip}^k + \sum_{l \in E} f_{pl}^k x_{pl}^k \right] \quad (18)$$

$$x_{pi}^k - \sum_{l \in \{p\} \cup E} x_{il}^k = 0 \quad \forall i \in I, k \in K \quad (19)$$

$$\sum_{j \in \{p\} \cup I} x_{jl}^k - x_{lp}^k = 0 \quad \forall l \in E, k \in K \quad (20)$$

$$\sum_{j \in I \cup E} x_{jp}^k - \sum_{i \in I \cup E} x_{pi}^k = 0 \quad \forall k \in K \quad (21)$$

$$\sum_{i \in I \cup E} x_{pi}^k \leq 1 \quad \forall k \in K \quad (22)$$

$$\sum_{k \in K} x_{pi}^k = d_i \quad \forall i \in I \quad (23)$$

$$\sum_{k \in K} x_{lp}^k = d_l \quad \forall l \in E \quad (24)$$

$$x_{pj}^k, x_{ip}^k, x_{ij}^k \in \{0, 1\} \quad \forall i \in I, j \in E, k \in K \quad (25)$$

In the objective function (18) $f_{il}^k = (c_{pi}^k + c_{il}^k + c_{lp}^k + h_{pi}^k)$, $f_{ip}^k = (c_{pi}^k + c_{ip}^k + h_{pi}^k)$, and $f_{pl}^k = (c_{pl}^k + c_{lp}^k + h_{pl}^k)$ for each $i \in I, l \in E$. Constraints (19), (20), and (21) are the flow conservation constraints at each importer node, at each exporter node, and at the port node, respectively; constraints (22) guarantee that trucks are not used more than once; constraints (23) and (24) represent the demand constraints at each importer node and at each exporter node respectively.

2.2. Second reformulation

We exploit the constraints (19) and (20) to replace variables x_{pi}^k and $x_{lp}^k \forall i \in I, \forall l \in E, \forall k \in K$ into constraints (21), (22), (23), (24).

Constraint (21) results be an identity. Constraints (22), (23), (24) reduced to (26), (27), (28) respectively as follows:

$$\sum_{k \in K} (x_{ip}^k + \sum_{l \in E} x_{il}^k) = d_i \quad \forall i \in I \quad (26)$$

$$\sum_{k \in K} (x_{pl}^k + \sum_{j \in I} x_{jl}^k) = d_l \quad \forall l \in E \quad (27)$$

$$\sum_{i \in I} \sum_{l \in E} x_{il}^k + \sum_{i \in I} x_{ip}^k + \sum_{i \in E} x_{pi}^k \leq 1 \quad \forall k \in K \quad (28)$$

$$x_{ij}^k \in \{0, 1\} \quad (29)$$

Constraints (26) and (27) represent demand constraints at each importer node and at each exporter node respectively; constraints (28) guarantee that trucks are not used more than once. The objective function is equal to (18).

The constraints matrix of the model (26)-(29) can be written with the following blockstructure

:

$$M = \left[\begin{array}{c|c|c|c|c} B & B & B & \dots & B \\ \hline D_1 & D_2 & D_3 & \dots & D_K \end{array} \right]$$

where $B \in \mathbb{R}^{(n+m) \times (n \times m + m + n)}$ and $D_1, \dots, D_K \in \mathbb{R}^{K \times (n \times m + m + n)}$.

The first and the second part of the matrix in Fig. 2 represent the block B and the third part is the block D_1 of the previous matrix. In general, the matrix D_k has all rows equal to zero, except the row k that has all entries equal to 1.

	$i_1 j_1$	$i_1 j_2$...	$i_2 j_1$	$i_2 j_2$...	$i_n j_m$	$i_1 p$	$i_2 p$...	$i_n p$	$p j_1$	$p j_2$...	$p j_m$	$k_2 \dots$
i_1	1	1	...	0	0	...	0	1	0	...	0	0	...	0
i_2	0	0	...	1	1	...	0	0	1	...	0	0	...	0
\vdots																
i_n	0	0	...	0	0	...	1	0	0	...	1	0	0	...	0	...
j_1	1	0	...	1	0	0	0	...	0	1	0	...	0	...
j_2	0	1	...	0	1	0	0	...	0	0	1	...	0	...
\vdots																
j_m	0	0	...	0	0	...	1	0	0	...	0	0	0	...	1	...
k_1	1	1	...	1	1	...	1	1	1	...	1	1	...	1
k_2	0	0	...	0	0	...	0	0	0	...	0	0	...	0
\vdots				\vdots				\vdots							\vdots	...
K	0	0	...	0	0	...	0	0	0	...	0	0	0	...	0	...

Figure 2: Constraint matrix M

Consider the submatrix of the constraints matrix M associated with row and column indices as indicated in Fig. 3: its determinant is -2 , then M is not totally unimodular.

	$x_{i_1 p}^{k_1}$	$x_{p j_1}^{k_1}$	$x_{i_1 j_1}^{k_2}$
i_1	1	0	1
j_1	0	1	1
k_1	1	1	0

Figure 3: A submatrix of M

2.3. Third reformulation

If the cost of the arc is not depending on the choice of the truck, that is $f_{il} = f_{il}^k$, $f_{pi} = f_{pi}^k$, and $f_{lp} = f_{lp}^k$ for any $k \in K$, then we can introduce a new set of integer variables:

$$t_{ij} = \sum_{k \in K} x_{ij}^k : \text{number of trucks moved along arc } (i, j) \in A$$

Formulation (26)-(29) can be rewritten as follows:

$$\min \sum_{i \in I; l \in E} f_{il} t_{il} + \sum_{i \in I} f_{ip} t_{ip} + \sum_{l \in E} f_{pl} t_{pl} \quad (30)$$

$$t_{ip} + \sum_{l \in E} t_{il} = d_i \quad \forall i \in I \quad (31)$$

$$t_{pl} + \sum_{j \in I} t_{jl} = d_l \quad \forall l \in E \quad (32)$$

$$\sum_{i \in I} \sum_{l \in E} t_{il} + \sum_{i \in I} t_{ip} + \sum_{i \in E} t_{pi} \leq |K| \quad (33)$$

$$t_{ij} \in \mathbb{Z} \quad (34)$$

Equations (31) and (32) represent demand constraints at each importer node and exporter node respectively; constraint (33) guarantees that trucks are not used more than once.

We now report a result by Ghouila-Houri [4] to prove that the continuous relaxation of the model (31)-(34) always admits an integer optimal solution.

Theorem 1. *Let $A \in \mathbb{Z}^{m \times n}$. A is totally unimodular if and only if for every $C \subseteq \{1, \dots, m\}$, there exists a partition of C into C_1 and C_2 such that*

$$\left| \sum_{c_1 \in C_1} a_{c_1 j} - \sum_{c_2 \in C_2} a_{c_2 j} \right| \leq 1 \quad \forall j. \quad (35)$$

Theorem 2. *The constraints matrix of reformulation (31)-(34) is totally unimodular.*

Proof.

Let us denote with A the constraint matrix of (31)-(34). This matrix is depicted in Fig. 2 as the restriction of matrix M limited to rows indexed by $I \cup E \cup \{k_1\}$, where $I = \{i_1, \dots, i_n\}$ and $E = \{l_1, \dots, l_m\}$, and the columns indexed by set $A_1 \cup A_2 \cup A_3$, where $A_1 = \{(i, l, k_1) | i \in I, l \in E\}$, $A_2 = \{(i, p, k_1) | i \in I\}$, $A_3 = \{(p, l, k_1) | l \in E\}$. Let I' and E' be two subsets of I and E respectively.

We will use Theorem 1 to prove that A is totally unimodular. We distinguish two cases.

- Case 1 : Let C be a subset of rows of A such that $k_1 \in C$. Then we can write $C = I' \cup E' \cup \{k_1\}$ and choose the partition: $C_1 = I' \cup E'$, $C_2 = \{k_1\}$. We indicate in the following table the sum of the elements of each column:

		A_1	A_2	A_3
	$\sum_{c_1 \in I'} a_{c_1 j}$	$1 \vee 0$	$1 \vee 0$	0
+	$\sum_{c_1 \in E'} a_{c_1 j}$	$1 \vee 0$	0	$1 \vee 0$
-	$\sum_{c_2 \in C_2} a_{c_2 j}$	1	1	1
=		$1 \vee 0 \vee -1$	$0 \vee -1$	$0 \vee -1$

For every column indexed by $(i, l, k) \in A_1 \cup A_2 \cup A_3$ the sum of the elements in $I' \cup E'$ may be 2, 1, or 0. Subtracting the row indexed by k_1 , we obtain 1, 0, -1 . Then condition (35) holds and Theorem 1 applies.

- Case 2: Let C be a subset of rows of A such that $k_1 \notin C$. Then we can write $C = I' \cup E'$ and choose the partition: $C_1 = I', C_2 = E'$.

		A_1	A_2	A_3
	$\sum_{c_1 \in C_1} a_{c_1 j}$	$1 \vee 0$	$1 \vee 0$	0
-	$\sum_{c_2 \in C_2} a_{c_2 j}$	$1 \vee 0$	0	$1 \vee 0$
=		$1 \vee 0 \vee -1$	$0 \vee -1$	$0 \vee -1$

For every column indexed by $(i, l, k) \in A_1 \cup A_2 \cup A_3$ the sum of the elements in I' may be 1 or 0. Subtracting the row indexed by E' , we obtain 1, 0, -1 . Then condition (35) holds and Theorem 1 applies.

Summing up Case 1 and Case 2, we can conclude that A is totally unimodular. We can conclude that formulation (30)-(34) has a continuous relaxation that always admits an integer optimal solution.

3. Conclusion

We proposed three new reformulations of the problem in the case with capacity equal to 1. We proved that the constraint matrix is not totally unimodular in the first and second reformulations. Finally, we proved total unimodularity of constraint matrix of the third formulation, that is valid when the vehicles have the same arc cost. Thus the continuous relaxation of the last reformulation always provides an integer optimal solution.

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