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**A study on Symbolic Control Design with Quantized
State Measurements**

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Abstract

Most of the existing results available in the literature concerning symbolic control design of purely continuous or hybrid systems assume full information of the state which in concrete applications may be not available. This partial information to the controller translates into the necessity to revisit existing methods on symbolic control design. This paper aims at addressing this issue and deals with symbolic control design of discrete-time nonlinear control systems affected by disturbances with specifications expressed as regular languages and where controllers can access state information only through quantized measurements.

Key words: Formal methods, Symbolic Control, Quantized feedback

AMS subject classifications: 34A38, 93C30

1. Introduction

The use of discrete abstractions for the formal verification and control design of purely continuous or hybrid systems is becoming rather popular in the scientific community working on hybrid systems. The main advantages of this approach over conventional control design methods mostly consist in allowing to solve control problems with complex specifications as, for example, safety properties, obstacle avoidance, regular language and logic specifications, in the respect of nonidealities at the software/hardware implementation layer. The literature on discrete–abstractions is very rich. The interested reader can refer to e.g. [12, 7, 1] and the references therein for work on this research topic. However, to the best of our knowledge, current literature except [5, 10, 2] assumes full state information when designing controllers. This assumption appears limiting in concrete applications since state measurements are generally available through sensors which typically provide a quantized measurement of the state and also measurements errors show up. The present paper is along the work reported in [5, 10, 2] and addresses symbolic control design of discrete–time nonlinear control systems affected by disturbances with specifications expressed in terms of regular languages and with state quantized measurements. The approach we take is modular: starting from the control problem at the concrete (physical) domain (Problem 1), we define an auxiliary control problem at the systems’ domain (Problem 2), where the paradigm of systems [12] is used as a unified framework to describe both nonlinear systems and symbolic models, and an auxiliary control problem at the symbolic domain (Problem 3). From a solution to Problem 2, a solution to Problem 1 is derived. Further, from a solution to Problem 3, a solution to Problem 2 is derived. Symbolic models approximating the plant and solving Problem 3 are also proposed and based on the notion of strong alternating approximate simulation relation recently introduced in [2] and on an adaptation to discrete–time control systems of the notion of incremental forward complete (δ -FC) Lyapunov functions given in [13] for continuous–time control systems. The modular approach taken also allows to apply Problem 2 and Problem 3 to symbolic control problems involving any class of dynamical (and possibly hybrid) control system modeling the plant.

As already mentioned, symbolic control design with state quantized measurements has been also addressed in [5, 10, 2]. The present paper extends the results reported in [5] from continuous–time nonlinear switched systems and safety and reachability specifications to discrete–time nonlinear control systems affected by disturbances and regular languages specifications. The work in [10] addresses symbolic control design with specifications expressed in a behavioral framework. It introduces the notion of feedback refinement relations (FRR) and shows that if there exists a FRR from a symbolic model to the plant, described in the system paradigm, see e.g. [12], then the controller symbolic system solving a suitable auxiliary control problem can be used to enforce the original specification on the plant, by using a quantizer as the only interface between the controller and the plant. Symbolic models for continuous–time affine systems are also provided. As discussed in the paper, when the notion of FRR is properly extended to be compared with the notion of strong alternating simulation relations of [2], the former notion is stronger than the latter. Further, [10] addresses symbolic control design in the systems’ domain and controller symbolic systems obtained are not refined to obtain controllers that can be applied to the original plant. Our results instead provide in the solution to Problem 1 a controller which indeed is proven to correctly enforce the specification on the original plant given in the form of a nonlinear control system. The present paper can be viewed also an extension of the results reported in [2], addressing symbolic control design of networked control systems (NCS), from continuous–time NCS without disturbances and specifications expressed as nondeterministic transition systems

with neither labels nor marked states to discrete-time nonlinear systems with disturbances and regular language specifications. It is well known in the literature that regular languages allow handling a wider class of specifications of interest than those considered in [2].

This paper is organized as follows. In Section 2 we introduce the notation and recall basic notions on regular languages, notions of systems, approximate and strong alternating approximate simulations and approximate feedback composition. In Section 3 we introduce the class of control systems and the symbolic control problem set-up. Section 4 provides the solution to the symbolic control problem and Section 5 offers some concluding remarks.

2. Notation and preliminary definitions

2.1. Notation

Given a pair of sets X and Y and a relation $\mathcal{R} \subseteq X \times Y$, the symbol \mathcal{R}^{-1} denotes the inverse relation of \mathcal{R} , i.e. $\mathcal{R}^{-1} = \{(y, x) \in Y \times X : (x, y) \in \mathcal{R}\}$. Given $X' \subseteq X$ and $Y' \subseteq Y$, we denote $\mathcal{R}(X') = \{y \in Y \mid \exists x \in X' \text{ s.t. } (x, y) \in \mathcal{R}\}$ and $\mathcal{R}^{-1}(Y') = \{x \in X \mid \exists y \in Y' \text{ s.t. } (x, y) \in \mathcal{R}\}$. Given sets A , B and C and relations $\mathcal{R}_{a,b} \subseteq A \times B$ and $\mathcal{R}_{b,c} \subseteq B \times C$ we recall that the composition relation $\mathcal{R} = \mathcal{R}_{a,b} \circ \mathcal{R}_{b,c} \subseteq A \times C$ is defined as $\mathcal{R}_{a,b} \circ \mathcal{R}_{b,c} = \{(a, c) \in A \times C \mid \exists b \in B \text{ s.t. } (a, b) \in \mathcal{R}_{a,b} \wedge (b, c) \in \mathcal{R}_{b,c}\}$. Note that, for any $A' \subseteq A$, $\mathcal{R}(A') = \mathcal{R}_{b,c}(\mathcal{R}_{a,b}(A'))$ and for any $C' \subseteq C$, $\mathcal{R}^{-1}(C') = \mathcal{R}_{a,b}^{-1}(\mathcal{R}_{b,c}^{-1}(C'))$. The symbols \mathbb{N}_0 , \mathbb{Z} , \mathbb{R} , \mathbb{R}^+ and \mathbb{R}_0^+ denote the set of nonnegative integer, integer, real, positive real, and nonnegative real numbers, respectively. Given $n \in \mathbb{N}_0$ and $n > 0$ we denote by $[1; n]$ the set $\{1, 2, \dots, n\}$. Given a vector $x \in \mathbb{R}^n$ we denote by $x(i)$ the i -th element of x and by $\|x\|$ the infinity norm of x . Given $a \in \mathbb{R}$ and $X \subseteq \mathbb{R}^n$, the symbol aX denotes the set $\{y \in \mathbb{R}^n \mid \exists x \in X \text{ s.t. } y = ax\}$. Given $\theta \in \mathbb{R}^+$ and $x \in \mathbb{R}^n$, the symbol $\mathcal{B}_{[-\theta, \theta]}(x)$ denotes the set $\{y \in \mathbb{R}^n \mid y(i) \in [-\theta + x(i), \theta + x(i)], i \in [1; n]\}$. Note that for any $\theta \in \mathbb{R}^+$, the collection of $\mathcal{B}_{[-\theta, \theta]}(x)$ with x ranging in $2\theta\mathbb{Z}^n$ is a partition of \mathbb{R}^n . Given $z \in \mathbb{R}^n$ the symbol $[z]_\theta$ denotes the unique vector in $\theta\mathbb{Z}^n$ such that $z \in \mathcal{B}_{[-\theta/2, \theta/2]}([z]_\theta)$. Given $X \subseteq \mathbb{R}^n$, we set $[X]_\theta = (\theta\mathbb{Z}^n) \cap X$; if X is convex and with interior there always exists $\theta_X \in \mathbb{R}^+$ such that for any $x \in X$ there exists $y \in [X]_{\theta_X}$ such that $[x]_{\theta_X} \in X$. A continuous function $\gamma : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is said to belong to class \mathcal{K} if it is strictly increasing and $\gamma(0) = 0$; function γ is said to belong to class \mathcal{K}_∞ if $\gamma \in \mathcal{K}$ and $\gamma(r) \rightarrow \infty$ as $r \rightarrow \infty$.

2.2. Systems, Regular Languages and Approximate Bisimulation

We start by recalling from e.g. [3] some notions on formal language theory. Let Y be a finite set representing the alphabet. A word over Y is a finite sequence $y_1 y_2 \dots y_l$ of symbols in Y . The empty word is denoted by ϵ . The symbol Y^* denotes the Kleene closure of Y , that is the collection of all words over Y including ϵ . A language L over Y is a subset of Y^* .

We proceed a step further by recalling the notion of systems:

Definition 2.1. *A system is a tuple $S = (X, X_0, U, \longrightarrow, X_m, Y, H)$, consisting of a set of states X , a set of initial states $X_0 \subseteq X$, a set of inputs U , a transition relation $\longrightarrow \subseteq X \times U \times X$, a set of marked states $X_m \subseteq X$, a set of outputs Y and an output function $H : X \rightarrow Y$.*

The above definition slightly extends the one of [12] to systems with marked states. A system enters a marked state whenever it completes some operation or task, see e.g. [3]. Marked states are also instrumental in defining hereafter regular languages. A transition $(x, u, x') \in \longrightarrow$

of S is denoted by $x \xrightarrow{u} x'$. For later purposes, given $x \in X$ define $U(x)$ as the collection of $u \in U$ for which there exists a transition $x \xrightarrow{u} x'$ for some $x' \in X$. The evolution of systems is captured by the notions of state, input and output runs. Given a sequence of transitions of S

$$x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} \dots \xrightarrow{u_{l-1}} x_l \quad (1)$$

with $x_0 \in X_0$, the sequences

$$r_X : x_0 x_1 \dots x_l, \quad (2)$$

$$r_U : u_0 u_1 \dots u_{l-1}, \quad (2)$$

$$r_Y : H(x_0) H(x_1) \dots H(x_l), \quad (3)$$

are called a *state run*, an *input run* and an *output run* of S , respectively. System S is said to be *symbolic* if X and U are finite sets, *metric* if Y is equipped with a metric $\mathbf{d} : Y \times Y \rightarrow \mathbb{R}_0^+$, *deterministic* if for any $x \in X$ and $u \in U$ there exists at most one transition $x \xrightarrow{u} x^+$ and *nondeterministic* otherwise. System S is said *nonblocking* if for any transitions sequence (1) of S with $x_0 \in X_0$ either $x_l \in X_m$ or there exists a continuation

$$x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} \dots \xrightarrow{u_{l-1}} x_l \xrightarrow{u_l} \dots \xrightarrow{u_{l'-1}} x_{l'}$$

of (1) such that $x_{l'} \in X_m$, and *blocking*, otherwise.

The *input language* (resp. *output language*) generated by system S is the collection of all its input runs (resp. output runs) and is denoted as $\mathcal{L}^u(S)$ (resp. $\mathcal{L}^y(S)$). The *marked input language* (resp. *marked output language*) of S , denoted as $\mathcal{L}_m^u(S)$ (resp. $\mathcal{L}_m^y(S)$), is the collection of all input runs r_U in (2) (resp. output runs r_Y in (3)) such that the corresponding transitions sequence in (1) is with ending state $x_l \in X_m$. Following e.g. [3], a language L over a finite set U is said *regular* if there exists a symbolic system S with input set U such that $L = \mathcal{L}_m^u(S)$.

We now recall from [6] the notion of approximate simulation relations and from [2] the recently introduced notion of strong alternating simulation relations.

Definition 2.2. Let $S_i = (X_i, X_{0,i}, U_i, \xrightarrow{i}, X_{m,i}, Y_i, H_i)$ ($i = 1, 2$) be metric systems with the same output sets $Y_1 = Y_2$ and metric \mathbf{d} , and let $\varepsilon \in \mathbb{R}_0^+$ be a given precision. Consider a relation

$$\mathcal{R} \subseteq X_1 \times X_2 \quad (4)$$

satisfying the following conditions:

$$(i) \quad \forall x_1 \in X_{0,1} \exists x_2 \in X_{0,2} \text{ such that } (x_1, x_2) \in \mathcal{R};$$

$$(ii) \quad \forall x_1 \in X_{m,1} \exists x_2 \in X_{m,2} \text{ such that } (x_1, x_2) \in \mathcal{R};$$

$$(iii) \quad \forall (x_1, x_2) \in \mathcal{R}, \mathbf{d}(H_1(x_1), H_2(x_2)) \leq \varepsilon.$$

Relation \mathcal{R} is an ε -approximate simulation relation from S_1 to S_2 if it enjoys conditions (i)–(iii) and the following one:

$$(iv) \quad \forall (x_1, x_2) \in \mathcal{R} \text{ if } x_1 \xrightarrow[1]{u_1} x'_1 \text{ then } \exists x_2 \xrightarrow[2]{u_2} x'_2 \text{ such that } (x'_1, x'_2) \in \mathcal{R}.$$

System S_1 is ε -simulated by S_2 or S_2 ε -simulates S_1 , denoted $S_1 \preceq_\varepsilon S_2$, if there exists an ε -approximate simulation relation from S_1 to S_2 .

Definition 2.3. [2] Relation \mathcal{R} as in (4) is a strong alternating ε -approximate (strong $A\varepsilon A$) simulation relation from S_1 to S_2 if it enjoys conditions (i)–(iii) of Definition 2.2 and the following one:

$$(iv') \quad \forall (x_1, x_2) \in \mathcal{R} \quad \forall u_1 \in U_1(x_1), u_2 = u_1 \in U_2(x_2) \text{ and } \forall x_2 \xrightarrow{u_2}_2 x'_2 \quad \exists x_1 \xrightarrow{u_1}_1 x'_1 \text{ such that } (x'_1, x'_2) \in \mathcal{R}.$$

System S_1 is strongly alternatingly ε -simulated by S_2 , denoted $S_1 \preceq_\varepsilon^{s, \text{alt}} S_2$, if there exists a strong $A\varepsilon A$ simulation relation from S_1 to S_2 .

The notions in Definitions 2.2 and 2.3 slightly extend those reported in [6, 12] and [2] to systems with marked states.

Interaction between plants and controllers in the systems' domain is formalized by the following notion of approximate feedback composition adapted from [12] to systems equipped with marked states and to the notion of strong $A\varepsilon A$ simulation relations.

Definition 2.4. [12] Consider a pair of metric systems $S_i = (X_i, X_{0,i}, U_i, \xrightarrow{i}, X_{m,i}, Y_i, H_i)$ ($i = 1, 2$) with the same output sets $Y_1 = Y_2$, and let $\varepsilon \in \mathbb{R}_0^+$ be a given precision. Let \mathcal{R} be a strong $A\varepsilon A$ simulation relation from S_2 to S_1 . The ε -approximate feedback composition of S_1 and S_2 , with composition relation \mathcal{R} , is the system

$$S_1 \times_\varepsilon^{\mathcal{R}} S_2 = (X, X_0, U, \longrightarrow, X_m, Y, H)$$

where $X = \mathcal{R}^{-1}$, $X_0 = X \cap (X_{0,1} \times X_{0,2})$, $U = U_1$, $(x_1, x_2) \xrightarrow{u} (x'_1, x'_2)$ if $x_1 \xrightarrow{u}_1 x'_1$ and $x_2 \xrightarrow{u}_2 x'_2$, $X_m = X_{m,1} \times X_{m,2}$, $Y = Y_1$ and $H(x_1, x_2) = H_1(x_1)$ for any $(x_1, x_2) \in X$.

The notion in Definition 2.3 has been inspired by the notion of feedback refinement relations (FRR) recently introduced in [10]. A brief discussion on connections with FRR follows. A FRR is a relation \mathcal{R} satisfying conditions (i) and

$$(iv'') \quad \forall (x_1, x_2) \in \mathcal{R} \quad \forall u_1 \in U_1(x_1), u_2 = u_1 \in U_2(x_2) \text{ and } \forall x_2 \xrightarrow{u_2}_2 x'_2 \quad \forall x_1 \xrightarrow{u_1}_1 x'_1, (x'_1, x'_2) \in \mathcal{R}.$$

When FRR is extended to metric systems and additional conditions (ii) and (iii) of Definition 2.2 are imposed, a straightforward comparison of conditions (iv') and (iv'') reveals that the notion of FRR is stronger than the one in Definition 2.3 which in turn is stronger than the notion of alternating approximate simulation of [9, 12] when extended with condition (ii) of Definition 2.2.

3. Control Systems and Problem Formulation

The class of control systems that we consider in this paper is described by

$$P : \begin{cases} x(t+1) = f(x(t), u(t), d(t)), \\ x(t) \in \mathcal{X}, u(t) \in \mathcal{U}, d(t) \in \mathcal{D}, \end{cases} \quad (5)$$

with $t \in \mathbb{N}_0$. As usual, symbol $x(t)$ denotes the state, $u(t)$ is the control input and $d(t)$ is the disturbance input, at time $t \in \mathbb{N}_0$. We assume that $d(t)$ is not measurable, from which P is nondeterministic. We further assume that $\mathcal{X} \subset \mathbb{R}^n$ is convex, bounded and with interior,

$\mathcal{U} \subset \mathbb{R}^m$ is finite, $\mathcal{D} \subset \mathbb{R}^l$ is convex, bounded and with interior, and $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \rightarrow \mathbb{R}^n$ is locally Lipschitz in its first and third arguments, uniformly with respect to the second argument. Finiteness of \mathcal{U} is placed because in the sequel we assume to control P through a digital controller which in practical applications is in fact characterized by a finite collection of control inputs. A trajectory of P is a function

$$x : [0; t_f] \rightarrow \mathcal{X} \quad (6)$$

satisfying (5) for all times $t \in [0; t_f[$, for some time $t_f \in \mathbb{N}_0$. Trajectory

$$x' : [0; t'_f] \rightarrow \mathcal{X} \quad (7)$$

is said to be a continuation of trajectory $x(\cdot)$ as in (6) if $t_f < t'_f$ and $x'(t) = x(t)$ for all $t \in [0; t_f]$. We now proceed with a further step by formalizing the class of specifications that we focus on in this paper. Let Y_Q be a finite subset of the state space \mathcal{X} of P . The specification is expressed as a regular language

$$L_Q \subset Y_Q^*, \quad (8)$$

where Y_Q^* is the Kleene closure of Y_Q . This class of specifications is rather rich and comprises, as also argued in [11], reachability and motion planning specifications, periodic orbits, state-based switching specifications, specifications involving sequences of smaller tasks that need to be performed according to a given order (e.g. in manufacturing systems, autopilots etc.).

We now proceed to define the class of controllers we focus on. Controller C for P is specified by partial map:

$$C : [\mathcal{X}]_\mu \rightarrow 2^{\mathcal{U}}, \quad (9)$$

associating a set $C([x]_\mu)$ of admissible control inputs to the quantization $[x]_\mu$ of the state $x \in \mathcal{X}$. From the formulation of the controller it is clear that here we do not assume full state information of the plant but only quantized state information. The system P in closed-loop with controller C is denoted as P^C . Since map C is partial, closed-loop system P^C may exhibit blocking behaviors. In fact, existence of a trajectory $x(\cdot)$ of P^C in the form of (6) implicitly requires that for all times $t \in [0; t_f[$:

$$C([x(t)]_\mu) \neq \emptyset. \quad (10)$$

We can now formalize the control problem we consider:

Problem 1. *Given the control system P in (5), the regular language specification L_Q in (8) and a desired accuracy $\theta \in \mathbb{R}^+$ find a set of initial states $\mathcal{X}_0 \subseteq \mathcal{X}$, a set of final states $\mathcal{X}_f \subseteq \mathcal{X}$, an accuracy $\mu \in \mathbb{R}^+$ of the quantizer and a controller C as in (9) such that:*

- (i) *for any trajectory $x(\cdot)$ of P^C as in (6) with $x(0) \in \mathcal{X}_0$ either $x(t_f) \in \mathcal{X}_f$ or there exists a continuation $x'(\cdot)$ as in (7) of $x(\cdot)$ such that $x'(t'_f) \in \mathcal{X}_f$;*
- (ii) *for all trajectories $x(\cdot)$ of P^C as in (6) with $x(0) \in \mathcal{X}_0$ and $x(t_f) \in \mathcal{X}_f$, there exists a word $q_0q_1\dots q_{t_f} \in L_Q$ such that for all times $t \in [0; t_f]$:*

$$\|x(t) - q_t\| \leq \theta. \quad (11)$$

Since condition (11) relaxes condition $x(t) = q_t$, the above control problem can be viewed as an approximate version of classical supervisory control problems traditionally given for DESs (see e.g. [3]) and here studied for nonlinear control (infinite-state) systems with state quantized measurements.

4. Main results

In this section we provide the solution to Problem 1. To this purpose we provide:

- in Section 4.1, the auxiliary Problem 2 in the systems' domain, whose solution is proven to give the solution of Problem 1;
- in Section 4.2, the auxiliary Problem 3 in the symbolic domain, whose solution is proven to give the solution of Problem 2;
- in Section 4.3, symbolic models approximating the plant P and the specification L_Q (in Problem 1), which give the solution to Problem 3.

4.1. From the systems' domain to the concrete domain

Starting from the plant P , the specification L_Q and Problem 1, in this section we derive a systems' representation $S_{\mathcal{X}_0, \mathcal{X}_f}(P)$ of P , a systems' representation S_q of L_Q and the auxiliary Problem 2. We then show that if a controller system S_c enforces the specification S_q on $S_{\mathcal{X}_0, \mathcal{X}_f}(P)$, then S_c can be properly refined to obtain a controller C in the form of (9) that enforces the specification L_Q on P .

We start by providing a systems' representation of P :

Definition 4.1. *Given control system P in (5) and the sets $\mathcal{X}_0, \mathcal{X}_f \subseteq \mathcal{X}$ in Problem 1, define system*

$$S_{\mathcal{X}_0, \mathcal{X}_f}(P) = (X_p, X_{0,p}, U_p, \xrightarrow{p}, X_{p,m}, Y_p, H_p) \quad (12)$$

where $X_p = \mathcal{X}$, $X_{0,p} = \mathcal{X}_0$, $U_p = \mathcal{U}$, $x \xrightarrow{p} x^+$ if $x^+ = f(x, u, d)$ for some $d \in \mathcal{D}$, $X_{m,p} = \mathcal{X}_f$, $Y_p = \mathcal{X}$ and $H_p(x) = x$ for any $x \in \mathcal{X}$.

System $S_{\mathcal{X}_0, \mathcal{X}_f}(P)$ is nondeterministic and metric when we regard Y_p as being equipped with the metric

$$\mathbf{d}(x, x') = \|x - x'\|, \forall x, x' \in \mathcal{X}. \quad (13)$$

System $S_{\mathcal{X}_0, \mathcal{X}_f}(P)$ is not symbolic because the cardinality of sets X_p and U_p is infinite. Let the system

$$S'_q = (X'_q, X'_{0,q}, Y_Q, \xrightarrow{t,q}, X'_{m,q}, Y'_q, H'_q) \quad (14)$$

be symbolic, deterministic and nonblocking and such that its input marked language coincides with the language specification, i.e.

$$\mathcal{L}_m^u(S'_q) = L_Q.$$

Automatic tools for constructing S'_q are well known in the literature, see e.g. [4]. Given S'_q , it is useful to define a symbolic system S_q whose states are transitions of S'_q and vice versa. More formally,

Definition 4.2. *Given symbolic system S'_q in (14) define*

$$S_q = (X_q, X_{0,q}, U_q, \xrightarrow{q}, X_{m,q}, Y_q, H_q) \quad (15)$$

where:

- $X_q = \xrightarrow{t,q}$;

- $X_{0,q}$ is the collection of states $x'_q \xrightarrow[\iota,q]{u'_q} x'_{q,+}$ of X_q such that $x'_q \in X'_{0,q}$;
- $U_q = \{u_q\}$ where u_q is a dummy input;
- $\xrightarrow[q]{}$ is such that

$$\left(x'_q \xrightarrow[\iota,q]{u'_q} x''_q \right) \xrightarrow[q]{u_q} \left(x'_q \xrightarrow[\iota,q]{u'_q} x'_q \right)$$

if $x''_q = x'_q$;

- $X_{m,q}$ is the collection of states $x'_q \xrightarrow[\iota,q]{u'_q} x'_{q,+}$ of X_q such that $x'_{q,+} \in X'_{m,q}$;
- $Y_q = \mathcal{X}$;
- $H_q(x'_q \xrightarrow[\iota,q]{u'_q} x'_{q,+}) = u'_q$.

The above construction resembles translation of Mealy machines into Moore machines (see e.g. [8]) with technical differences which we do not discuss here for lack of space. System S_q is metric with metric \mathbf{d} in (13). From Definition 4.2, it is readily seen that:

$$\mathcal{L}^y(S_q) = \mathcal{L}^u(S'_q), \quad \mathcal{L}_m^y(S_q) = \mathcal{L}_m^u(S'_q) = L_Q.$$

Moreover since S'_q is nonblocking, S_q is nonblocking as well. In the sequel, for the sake of an easier notation, we denote a state $x'_q \xrightarrow[\iota,q]{u'_q} x'_{q,+}$ of X_q by x_q .

We can now introduce the following auxiliary problem:

Problem 2. Consider system $S_{\mathcal{X}_0, \mathcal{X}_f}(P)$ and a specification S_q . Given a desired accuracy $\theta' \in \mathbb{R}^+$, find

- an accuracy $\mu \in \mathbb{R}^+$ of the quantizer,
- a nonblocking metric controller system

$$S_c = (X_c, X_{0,c}, U_c, \xrightarrow[c]{}, X_{m,c}, Y_c, H_c) \quad (16)$$

with $Y_c = \mathcal{X}$ and metric \mathbf{d} in (13),

- an accuracy $\varepsilon \in \mathbb{R}^+$ and a strong $A\varepsilon A$ simulation relation $\mathcal{R}_{c,p}$ from S_c to $S_{\mathcal{X}_0, \mathcal{X}_f}(P)$ implying that:

$$\mathcal{X}_0 = \mathcal{R}_{c,p}(X_{0,c}), \quad (17)$$

$$\mathcal{X}_f = \mathcal{R}_{c,p}(X_{m,c}), \quad (18)$$

such that:

$$(c.1) \quad S_{\mathcal{X}_0, \mathcal{X}_f}(P) \times_{\varepsilon}^{\mathcal{R}_{c,p}} S_c \preceq_{\theta'} S_q;$$

$$(c.2) \quad \text{the system } S_{\mathcal{X}_0, \mathcal{X}_f}(P) \times_{\varepsilon}^{\mathcal{R}_{c,p}} S_c \text{ is nonblocking};$$

$$(c.3) \quad \text{for any pair of states } x \text{ and } x' \text{ of } S_{\mathcal{X}_0, \mathcal{X}_f}(P) \text{ if } [x]_{\mu} = [x']_{\mu}, \text{ then } \mathcal{R}_{c,p}^{-1}(\{x\}) = \mathcal{R}_{c,p}^{-1}(\{x'\}).$$

The above control problem can be viewed as an extension of approximate similarity games as in e.g. [12], to control problems where information on the state of the plant system $S_{\mathcal{X}_0, \mathcal{X}_f}(P)$ is available only through its quantization. We take care of this partial information on the plant state by requiring relation $\mathcal{R}_{c,p}$ in the above problem to be a strong $A\varepsilon A$ simulation relation (whereas in e.g. [12] $\mathcal{R}_{c,p}$ is required to be (only) an alternating approximate simulation relation [9, 12]) and by adding condition (c.3) (which is not present in e.g. [12]). Condition (c.3) requires that states of $S_{\mathcal{X}_0, \mathcal{X}_f}(P)$ with the same quantization are indistinguishable to the controller. In the remainder of this section, we assume that a solution to Problem 2 exists and derive a solution to Problem 1. The following result holds.

Theorem 4.3. *Given a desired accuracy $\theta \in \mathbb{R}^+$ as requested in Problem 1, select accuracy $\theta' \in \mathbb{R}^+$ in Problem 2 such that:*

$$\theta' = \theta. \quad (19)$$

Suppose there exist an accuracy of the quantizer $\mu \in \mathbb{R}^+$, a nonblocking controller system S_c and a strong $A\varepsilon A$ simulation relation $\mathcal{R}_{c,p}$ from S_c to $S_{\mathcal{X}_0, \mathcal{X}_f}(P)$ for some accuracy $\varepsilon \in \mathbb{R}^+$, implying that (17) and (18) hold, which solve Problem 2. Then, the controller C in (9) defined for any $x \in \mathcal{R}_{c,p}(X_c)$ by

$$C([x]_\mu) = \bigcup_{(x_c \xrightarrow[c]{u} x') \wedge ((x_c, x) \in \mathcal{R}_{c,p})} \{u\} \quad (20)$$

solve Problem 1.

Proof. Let $S_{\mathcal{X}_0, \mathcal{X}_f}(P) \times_{\varepsilon}^{\mathcal{R}_{c,p}} S_c = (X_{p,c}, X_{0,p,c}, U_{p,c}, \xrightarrow[p,c]{\quad}, X_{m,p,c}, Y_{p,c}, H_{p,c})$, where each entity is as in Definition 2.4. Let $\mathcal{R}_{p,c,q}$ be a θ' -approximate simulation relation from $S_{\mathcal{X}_0, \mathcal{X}_f}(P) \times_{\varepsilon}^{\mathcal{R}_{c,p}} S_c$ to S_q , which exists by condition (c.1) of Problem 2. Consider any trajectory $x(\cdot)$ of PC as in (6) with initial condition $x(0) \in \mathcal{X}_0$. By definition of the set \mathcal{X}_0 in (17) and of $\mathcal{R}_{c,p}$ there exists $x_c(0) \in X_c$ such that

$$(x_c(0), x(0)) \in \mathcal{R}_{c,p} \quad (21)$$

from which, by recalling Definition 2.4, $(x(0), x_c(0)) \in X_{p,c}$. By definition of $\mathcal{R}_{p,c,q}$ there exists $x_q(0) \in X_q$ such that

$$((x(0), x_c(0)), x_q(0)) \in \mathcal{R}_{p,c,q}. \quad (22)$$

Define $q_0 = H_q(x_q(0))$. By Eqns. (22) and (19), definition of $\mathcal{R}_{p,c,q}$ and Definitions 2.4 and 4.1, we get:

$$\begin{aligned} \|x(0) - q_0\| &= \|H_p(x(0)) - H_q(x_q(0))\| \\ &= \|H_{p,c}(x(0), x_c(0)) - H_q(x_q(0))\| \\ &\leq \theta' = \theta. \end{aligned} \quad (23)$$

We now use induction and show that if the following condition holds:

(H) Given $\tau \in \mathbb{N}_0$, suppose that there exist state runs $\{x_c(t)\}_{t \in [0; \tau]}$, $\{x_q(t)\}_{t \in [0; \tau]}$ of S_c and S_q , respectively, such that:

(H1) $((x(t), x_c(t)), x_q(t)) \in \mathcal{R}_{p,c,q}$ for all $t \in [0; \tau]$,

(H2) by defining $q_t = H_q(x_q(t))$ for all $t \in [0; \tau]$, sequence $q_0 q_1 \dots q_\tau \in \mathcal{L}^y(S_q)$ and satisfies condition (11) for all $t \in [0; \tau]$,

then, one of the following conditions (T1) and (T2) hold:

(T1) $x(\tau) \in \mathcal{X}_f$, the word $q_0 q_1 \dots q_\tau \in L_Q$ and satisfies condition (11) for all $t \in [0; t_f]$ with $t_f = \tau$;

(T2) $C([x(\tau)]_\mu) \neq \emptyset$ and for any given $u(\tau) \in C([x(\tau)]_\mu)$ and any state

$$x(\tau + 1) = f(x(\tau), u(\tau), d(\tau)) \quad (24)$$

reached by P for any possible disturbance $d(\tau) \in \mathcal{D}$, there exist a transition

$$x_c(\tau) \xrightarrow[c]{u(\tau)} x_c(\tau + 1) \quad (25)$$

and a transition

$$x_q(\tau) \xrightarrow[q]{u_q} x_q(\tau + 1) \quad (26)$$

such that conditions in (H) hold with time τ replaced by time $\tau + 1$.

Suppose that conditions in (H) hold. Condition (H1) implies:

$$(x_c(\tau), x(\tau)) \in \mathcal{R}_{c,p}. \quad (27)$$

Since S_c is nonblocking then, either (case 1) $x_c(\tau) \in X_{m,c}$ or (case 2) there exists a transition

$$x_c(\tau) \xrightarrow[c]{u'} x'_c. \quad (28)$$

In case 1, by Eqns. (27) and (18) and definition of $\mathcal{R}_{c,p}$ we get $x(\tau) \in \mathcal{X}_f$. Hence, condition (i) of Problem 1 holds for $t_f = \tau$. Since by Definition 2.4, $(x(\tau), x_c(\tau)) \in X_{m,p,c}$ then, by condition (ii) of Definition 2.2, we get that $x_q(\tau) \in X_{m,q}$ from which, $q_0 q_1 \dots q_\tau \in L_Q$. Hence, also in view of (H2), condition (ii) of Problem 1 holds for $t_f = \tau$.

We now address case 2. By Eqns. (27), (28) and definition of the controller $C(\cdot)$ in (20), $u' \in C([x(\tau)]_\mu)$ from which $C([x(\tau)]_\mu) \neq \emptyset$. Pick any $u(\tau) \in C([x(\tau)]_\mu)$, any $d(\tau) \in \mathcal{D}$ and consider $x(\tau + 1)$ defined in (24). By Definition 4.1 we get:

$$x(\tau) \xrightarrow[p]{u(\tau)} x(\tau + 1). \quad (29)$$

By Eqns. (27), (29) and definition of $\mathcal{R}_{c,p}$, there exists transition in (25) such that

$$(x_c(\tau + 1), x(\tau + 1)) \in \mathcal{R}_{c,p}. \quad (30)$$

By Eqns. (27), (29), (25), (30) and Definition 2.4, we get

$$(x(\tau), x_c(\tau)) \xrightarrow[p,c]{u(\tau)} (x(\tau + 1), x_c(\tau + 1)).$$

Then, by definition of $\mathcal{R}_{p,c,q}$, there exists transition (26) such that (H1) holds with time τ replaced by time $\tau + 1$. Define $q_{\tau+1} = H_q(x_q(\tau + 1))$. By (26) we get that $q_0 q_1 \dots q_{\tau+1} \in \mathcal{L}^y(S_q)$. Moreover, by (H2), Eqn. (19), definition of $\mathcal{R}_{p,c,q}$ and Definition 2.4, we get:

$$\begin{aligned} & \|x(\tau + 1) - q_{\tau+1}\| = \\ & \|H_p(x(\tau + 1)) - H_q(x_q(\tau + 1))\| = \\ & \|H_{p,c}(x(\tau + 1), x_c(\tau + 1)) - H_q(x_q(\tau + 1))\| \leq \\ & \theta' = \theta. \end{aligned} \quad (31)$$

By Eqn. (31) and (H2) at time τ , we get (H2) at time $\tau + 1$. Hence, conditions in (T2) are proven.

In order to conclude the proof we first point out that the first part of the proof shows that (H) holds for $\tau = 0$. Second, we now show that there exists a time $t_f \in \mathbb{N}$ such that $x(t_f) \in \mathcal{X}_f$. Consider the sequence of transitions in (25) constructed in the induction iterations. Since S_c is nonblocking there exists a time $t_f \in \mathbb{N}$ such that $x_c(t_f) \in X_{m,c}$ which implies, by Eqns. (18) and (27) and definition of $\mathcal{R}_{c,p}$, that $x(t_f) \in \mathcal{X}_f$. ■

The above result relies upon the existence of a solution to Problem 2. In the next section we derive a possible approach to solve Problem 2 which is based on the use of discrete abstractions.

4.2. From the symbolic domain to the systems' domain

Starting from the plant system $S_{\mathcal{X}_0, \mathcal{X}_f}(P)$, the specification system S_q and Problem 2, in this section we assume the existence of a symbolic system S_a properly approximating $S_{\mathcal{X}_0, \mathcal{X}_f}(P)$ and of a symbolic system $S_{a,q}$ properly approximating S_q such that, if there exists a controller symbolic system $S_{c,a}$ enforcing the specification $S_{q,a}$ on S_a , then the symbolic controller $S_c = S_{a,c}$ correctly enforces the specification S_q on $S_{\mathcal{X}_0, \mathcal{X}_f}(P)$ in Problem 2.

We start by introducing the following auxiliary problem:

Problem 3. *Given a plant symbolic system*

$$S_a = (X_a, X_{0,a}, U_a, \xrightarrow{a}, X_{m,a}, Y_a, H_a), \quad (32)$$

and the specification symbolic system

$$S_{q,a} = (X_{q,a}, X_{0,q,a}, U_q, \xrightarrow{q,a}, X_{m,q,a}, Y_{q,a}, H_{q,a}) \quad (33)$$

with $Y_{q,a} = Y_a$, find

- a nonblocking controller symbolic system

$$S_{c,a} = (X_{c,a}, X_{0,c,a}, U_{c,a}, \xrightarrow{c,a}, X_{m,c,a}, Y_{c,a}, H_{c,a}) \quad (34)$$

with $U_{c,a} = U_a$ and $Y_{c,a} = Y_a$,

- a strong A0A simulation relation $\mathcal{R}_{c,a}$ from $S_{c,a}$ to S_a

such that:

(c.4) $S_a \times_0^{\mathcal{R}_{c,a}} S_{c,a} \preceq_0 S_{q,a}$;

(c.5) the system $S_a \times_0^{\mathcal{R}_{c,a}} S_{c,a}$ is nonblocking.

Classical similarity games, see e.g. [12], are generally given in the form of the above problem but with relation $\mathcal{R}_{c,a}$ that is an alternating simulation relation [9, 12] and not a strong A0A simulation relation, as instead required in Problem 3. Solutions provided in e.g. [12] can be easily extended to consider the case of strong A0A simulation relations. In fact, requiring $\mathcal{R}_{c,a}$ to be a strong A0A simulation relation is not needed to solve Problem 3 itself but it plays a role when using the solution of Problem 3 to solve Problem 2. The following result shows that a solution to Problem 2 can be derived from a solution to Problem 3.

Theorem 4.4. Consider plant system $S_{X_0, X_f}(P)$ and specification system S_q . Given a desired accuracy $\theta' \in \mathbb{R}^+$, select accuracy of the quantizer $\mu \in \mathbb{R}^+$ and accuracy $\varepsilon \in \mathbb{R}^+$ in Problem 2 such that:

$$\varepsilon + \mu \leq \theta'. \quad (35)$$

Suppose there exist

(c.6) a symbolic metric system S_a as in (32) with $Y_a = \mathcal{X}$ and metric \mathbf{d} in (13),

(c.7) a strong $A\varepsilon A$ simulation relation $\mathcal{R}_{a,p}$ from S_a to $S_{X_0, X_f}(P)$, with $\varepsilon \in \mathbb{R}^+$ satisfying (35), and satisfying

$$[x]_\mu = [x']_\mu \Rightarrow \mathcal{R}_{a,p}^{-1}(x) = \mathcal{R}_{a,p}^{-1}(x'), \quad (36)$$

(c.8) a symbolic metric system $S_{q,a}$ as in (33) with $Y_{q,a} = \mathcal{X}$ and metric \mathbf{d} in (13),

(c.9) a μ -approximate simulation relation \mathcal{R}_q from $S_{q,a}$ to S_q with $\mu \in \mathbb{R}^+$ satisfying (35),

(c.10) a nonblocking symbolic metric system $S_{c,a}$ as in (34),

(c.11) a strong $A0A$ simulation relation $\mathcal{R}_{c,a}$ from $S_{c,a}$ to S_a satisfying (c.4) and (c.5),

solving Problem 3. Then, controller symbolic system $S_c = S_{c,a}$ and relation $\mathcal{R}_{c,p} = \mathcal{R}_{c,a} \circ \mathcal{R}_{a,p}$ solve Problem 2.

Proof. Let $S_{X_0, X_f}(P) \times_{\varepsilon}^{\mathcal{R}_{c,p}} S_c = (X_{p,c}, X_{0,p,c}, U_{p,c}, \xrightarrow{p,c}, X_{m,p,c}, Y_{p,c}, H_{p,c})$ and $S_a \times_0^{\mathcal{R}_{c,a}} S_c = (X_{a,c}, X_{0,a,c}, U_{a,c}, \xrightarrow{a,c}, X_{m,a,c}, Y_{a,c}, H_{a,c})$ where each entity is as in Definition 2.4.

We first show that $\mathcal{R}_{c,p} = \mathcal{R}_{c,a} \circ \mathcal{R}_{a,p}$ is a strong $A\varepsilon A$ simulation relation from S_c to $S_{X_0, X_f}(P)$. Consider any $(x_c, x_p) \in \mathcal{R}_{c,p}$. There exists $x_a \in X_a$ such that $(x_c, x_a) \in \mathcal{R}_{c,a}$ and $(x_a, x_p) \in \mathcal{R}_{a,p}$. We first show condition (i) of Definition 2.3. Suppose $x_c \in X_{0,c}$. By condition (i) of Definition 2.3 applied to $\mathcal{R}_{c,a}$, since $(x_c, x_a) \in \mathcal{R}_{c,a}$ then $x_a \in X_{0,a}$; similarly, since $(x_a, x_p) \in \mathcal{R}_{a,p}$ then $x_p \in X_{0,p}$. Condition (ii) of Definition 2.3 can be shown similarly to condition (i). Regarding condition (iii) we get:

$$\begin{aligned} \mathbf{d}(H_c(x_c), H_p(x_p)) &\leq \\ \mathbf{d}(H_c(x_c), H_a(x_a)) + \mathbf{d}(H_a(x_a), H_p(x_p)) &\leq \\ 0 + \varepsilon &= \varepsilon. \end{aligned}$$

We now show condition (iv') of Definition 2.3. Consider any $u \in U_c = U_a$ and pick any transition $x_p \xrightarrow{u} x_p^+$. Since $(x_a, x_p) \in \mathcal{R}_{a,p}$ there exists a transition $x_a \xrightarrow{u} x_a^+$ such that $(x_a^+, x_p^+) \in \mathcal{R}_{a,p}$. Consequently, since $(x_c, x_a) \in \mathcal{R}_{c,a}$, there exists a transition $x_c \xrightarrow{u} x_c^+$ such that $(x_c^+, x_a^+) \in \mathcal{R}_{c,a}$. Hence, $(x_c^+, x_p^+) \in \mathcal{R}_{c,p}$. Thus, $\mathcal{R}_{c,p}$ is a strong $A\varepsilon A$ simulation relation from S_c to $S_{X_0, X_f}(P)$.

We now show condition (c.1) of Problem 2. Let $\mathcal{R}_{a,c,q}$ be a 0-approximate simulation relation from $S_a \times_0^{\mathcal{R}_{c,a}} S_{c,a}$ to $S_{q,a}$ which exists because of (c.4). Define relation $\mathcal{R}_{p,c,q} \subseteq \mathcal{R}_{c,p}^{-1} \times X_q$ such that $((x_p, x_c), x_q) \in \mathcal{R}_{p,c,q}$ if and only if there exist $x_a \in X_a$ and $x_{q,a} \in X_{q,a}$ such that:

$$(x_c, x_a) \in \mathcal{R}_{c,a}, \quad (37)$$

$$((x_a, x_c), x_{q,a}) \in \mathcal{R}_{a,c,q}, \quad (38)$$

$$(x_a, x_p) \in \mathcal{R}_{a,p}, \quad (39)$$

$$(x_{q,a}, x_q) \in \mathcal{R}_q. \quad (40)$$

We first note that existence of $x_a \in X_a$ satisfying Eqns. (37) and (39) is guaranteed by definition of $\mathcal{R}_{c,p}$. We now show that $\mathcal{R}_{p,c,q}$ is a θ' -approximate simulation relation from $S_{\mathcal{X}_0, \mathcal{X}_f}(P) \times_{\varepsilon}^{\mathcal{R}_{c,p}} S_c$ to S_q , thus showing (c.1). We start with conditions (i) of Definition 2.2. Consider any $((x_p, x_c), x_q) \in \mathcal{R}_{p,c,q}$ with $(x_p, x_c) \in X_{0,p,c}$. By Definition 2.4 we get $x_p \in X_{0,p}$ and $x_c \in X_{0,c}$. There exists $x_a \in X_a$ such that Eqns. (37) and (39) hold. Since $(x_c, x_a) \in \mathcal{R}_{c,a}$ and $x_c \in X_{0,c}$ by condition (i) of Definition 2.2 applied to $\mathcal{R}_{c,a}$ we get $x_a \in X_{0,a}$ from which, by Definition 2.4, we get $(x_a, x_c) \in X_{0,a,c}$. Since $(x_a, x_c) \in X_{0,a,c}$, by condition (i) of Definition 2.2 applied to $\mathcal{R}_{c,a,q}$, we get $x_{q,a} \in X_{0,q,a}$. Since $x_{q,a} \in X_{0,q,a}$, by condition (i) of Definition 2.2 applied to \mathcal{R}_q , we get $x_q \in X_{0,q}$. Condition (ii) of Definition 2.2 can be shown by using the same reasoning as in condition (i). We now show condition (iii) of Definition 2.2. We first note that $Y_{p,c} = Y_p = Y_a = Y_{a,c} = Y_q = Y_{q,a} = \mathcal{X}$. Moreover, by Definition 2.4, definition of $\mathcal{R}_{p,c,q}$ and Eqn. (35), we get:

$$\begin{aligned} & \mathbf{d}(H_{p,c}(x_p, x_c), H_q(x_q)) = \\ & \mathbf{d}(H_p(x_p), H_q(x_q)) \leq \\ & \mathbf{d}(H_p(x_p), H_a(x_a)) + \mathbf{d}(H_a(x_a), H_{q,a}(x_{q,a})) + \\ & \quad \mathbf{d}(H_{q,a}(x_{q,a}), H_q(x_q)) = \\ & \mathbf{d}(H_p(x_p), H_a(x_a)) + \mathbf{d}(H_{a,c}(x_a, x_c), H_{q,a}(x_{q,a})) + \\ & \quad \mathbf{d}(H_{q,a}(x_{q,a}), H_q(x_q)) \leq \varepsilon + 0 + \mu \leq \theta'. \end{aligned}$$

We now show condition (iv) of Definition 2.2. Consider any transition

$$(x_p, x_c) \xrightarrow[u]{p,c} (x_p^+, x_c^+). \quad (41)$$

By Eqn. (41) and Definition 2.4 applied to $S_{\mathcal{X}_0, \mathcal{X}_f}(P) \times_{\varepsilon}^{\mathcal{R}_{c,p}} S_c$ we get

$$x_p \xrightarrow[u]{p} x_p^+. \quad (42)$$

By definition of $\mathcal{R}_{c,p}$ there exists $x_a \in X_a$ such that Eqns. (37) and (39) hold. By Eqns. (39) and (42) and by condition (iv') of Definition 2.3 applied to $\mathcal{R}_{a,p}$, there exists $x_a^+ \in X_a$ such that

$$x_a \xrightarrow[u]{a} x_a^+, \quad (43)$$

$$(x_a^+, x_p^+) \in \mathcal{R}_{a,p}. \quad (44)$$

By Eqns. (37) and (43) and by condition (iv') of Definition 2.3 applied to $\mathcal{R}_{c,a}$, there exists $x_c^+ \in X_c$ such that

$$x_c \xrightarrow[u]{c} x_c^+, \quad (45)$$

$$(x_c^+, x_a^+) \in \mathcal{R}_{c,a}. \quad (46)$$

By Eqns. (37), (43), (45) and (46) we get:

$$(x_a, x_c) \xrightarrow[u]{a,c} (x_a^+, x_c^+). \quad (47)$$

By Eqns. (38) and (47) there exists a transition

$$x_{q,a} \xrightarrow[u_q]{q,a} x_{q,a}^+ \quad (48)$$

such that:

$$((x_a^+, x_c^+), x_{q,a}^+) \in \mathcal{R}_{a,c,q}. \quad (49)$$

By Eqns. (40) and (48) there exists a transition $x_q \xrightarrow[q]{u_q} x_q^+$ such that:

$$(x_{q,a}^+, x_q^+) \in \mathcal{R}_q. \quad (50)$$

By combining Eqns. (46), (49), (44) and (50) we get $((x_p^+, x_c^+), x_q^+) \in \mathcal{R}_q$ and condition (iv) of Definition 2.2 holds. Hence, condition (c.1) of Problem 2 is proven.

We now show condition (c.2). Consider any transitions sequence

$$(x_{0,p}, x_{0,c}) \xrightarrow[p,c]{u_0} \dots \xrightarrow[p,c]{u_{l-1}} (x_{l,p}, x_{l,c}) \quad (51)$$

of $S_{\mathcal{X}_0, \mathcal{X}_f}(P) \times_\varepsilon^{\mathcal{R}_{c,p}} S_c$. By definition of $\mathcal{R}_{c,p}$ and Definition 2.4 applied to $S_a \times_0^{\mathcal{R}_{c,a}} S_c$, there exist a transitions sequence

$$(x_{0,a}, x_{0,c}) \xrightarrow[a,c]{u_0} \dots \xrightarrow[a,c]{u_{l-1}} (x_{l,a}, x_{l,c}) \quad (52)$$

of $S_a \times_0^{\mathcal{R}_{c,a}} S_c$ such that

$$(x_{t,a}, x_{t,p}) \in \mathcal{R}_{a,p}, \forall t \in [0; l]. \quad (53)$$

By condition (c.5) of Problem 3, either $(x_{l,a}, x_{l,c}) \in X_{m,a,c}$, or there exists a continuation of the transitions sequence in Eqn. (52) with ending state $(x_{l',a}, x_{l',c}) \in X_{m,a,c}$. Suppose without loss of generality that $l' = l$. By Definition 2.4 applied to $S_a \times_0^{\mathcal{R}_{c,a}} S_c$, we get

$$x_{l,a} \in X_{m,a}, \quad (54)$$

$$x_{l,c} \in X_{m,c}. \quad (55)$$

By Eqn. (54), Eqn. (53) for $t = l$, and condition (ii) of Definition 2.3 applied to $\mathcal{R}_{c,a}$, we get

$$x_{l,p} \in X_{m,p}. \quad (56)$$

By Eqn. (51), we get $(x_{l,p}, x_{l,c}) \in X_{p,c}$ which, combined with Eqns. (55) and (56), implies $(x_{l,p}, x_{l,c}) \in X_{m,p,c}$.

We finally show condition (c.3) of Problem 2. Consider any pair of states $x, x' \in X_p$ with $[x]_\mu = [x']_\mu$. By definition of $\mathcal{R}_{c,p}$ and in view of (36) we get $\mathcal{R}_{c,p}^{-1}(\{x\}) = \mathcal{R}_{c,a}^{-1}(\mathcal{R}_{a,p}^{-1}(\{x\})) = \mathcal{R}_{c,a}^{-1}(\mathcal{R}_{a,p}^{-1}(\{x'\})) = \mathcal{R}_{c,p}^{-1}(\{x'\})$, which concludes the proof. ■

4.3. Symbolic models for the specification and the plant

Theorem 4.4 requires to find symbolic systems $S_{q,a}$ and S_a satisfying conditions (c.6) and (c.8) and relations $\mathcal{R}_{a,p}$ and \mathcal{R}_q satisfying conditions (c.7) and (c.9). This is the object of investigation of this section. We start with the following

Definition 4.5. *Given the specification symbolic system S_q in (15) and $\mu \in \mathbb{R}^+$ define symbolic system $S_{q,a}$ in (33) by $X_{q,a} = [X_q]_\mu$, $X_{0,q,a} = [X_{0,q}]_\mu$, $U_{q,a} = \{u_q\}$, $[x_q]_\mu \xrightarrow[q,a]{u_q} [x_q^+]_\mu$ if $x_q \xrightarrow[q]{u_q} x_q^+$, $X_{m,q,a} = [X_{m,q}]_\mu$, $Y_{q,a} = Y_q$ and $H_{q,a}(x_q) = x_q$.*

System $S_{q,a}$ is metric with metric \mathbf{d} in (13). From the above construction it is readily seen that:

Proposition 4.6. $S_{q,a} \preceq_\mu S_q$ with μ -approximate simulation relation

$$\mathcal{R}_q = \{(x_{q,a}, x_q) \in X_{q,a} \times X_q \mid x_{q,a} = [x_q]_\mu\}.$$

From the definitions of $S_{q,a}$ and \mathcal{R}_q it is readily seen that conditions (c.7) and (c.9) of Theorem 4.4 are fulfilled.

We now proceed a step further by defining a symbolic system approximating the plant system $S_{\mathcal{X}_0, \mathcal{X}_f}(P)$. To this purpose, we first extend to discrete-time control systems the notion of incremental forward complete (δ -FC) Lyapunov function given in [13] for continuous-time control systems.

Definition 4.7. A smooth function $V : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_0^+$ is a δ -FC Lyapunov function for the plant control system P in (5) if there exist \mathcal{K}_∞ functions $\underline{\alpha}$, $\bar{\alpha}$, \mathcal{K} (or zero) functions σ_u and σ_d and a real $\kappa \in \mathbb{R}$ such that the following inequalities are satisfied for any $x_1, x_2 \in \mathcal{X}$, any $u_1, u_2 \in \mathcal{U}$ and any $d_1, d_2 \in \mathcal{D}$:

$$(i) \quad \underline{\alpha}(\|x_1 - x_2\|) \leq V(x_1, x_2) \leq \bar{\alpha}(\|x_1 - x_2\|),$$

$$(ii) \quad V(f(x_1, u_1, d_1), f(x_2, u_2, d_2)) \leq \kappa V(x_1, x_2) + \sigma_u(\|u_1 - u_2\|) + \sigma_d(\|d_1 - d_2\|).$$

In the sequel we assume that there exists a δ -FC Lyapunov function for the plant control system P . Let γ be a \mathcal{K}_∞ function¹ such that

$$V(x, x') - V(x, x'') \leq \gamma(\|x' - x''\|),$$

for every $x, x', x'' \in \mathcal{X}$. We assume without loss of generality that V is symmetric, i.e. $V(x_1, x_2) = V(x_2, x_1)$ for all $x_1, x_2 \in \mathcal{X}$. In fact, if V is not symmetric, function V' defined by $V'(x_1, x_2) = V(x_1, x_2) + V(x_2, x_1)$ for all $x_1, x_2 \in \mathcal{X}$ is a δ -FC Lyapunov function for P and symmetric.

We can now define symbolic system S_a :

Definition 4.8. Consider quantization parameters $\mu \leq \mu_{\mathcal{X}}$ and $\eta \leq \eta_{\mathcal{D}}$ and define the symbolic system S_a in (32) by $X_a = [\mathcal{X}]_\mu$, $X_{0,a} = X_a \cap \mathcal{X}_0$, $U_a = \mathcal{U}$, $x_a \xrightarrow[u]{a} x_a^+$ if there exists $d_a \in [\mathcal{D}]_\eta$ such that

$$V([f(x_a, u, d_a)]_\mu, x_a^+) \leq (\kappa + 2)\gamma(\mu) + \sigma_d(\eta), \quad (57)$$

$X_{m,a} = X_a \cap \mathcal{X}_f$, $Y_a = \mathcal{X}$ and $H_a(x) = x$.

System S_a is metric with metric \mathbf{d} in (13). We can now give the following result:

Theorem 4.9. Suppose that P admits a δ -FC Lyapunov function V . Then, for any desired accuracy $\varepsilon \in \mathbb{R}^+$ and for any quantization parameters $\mu \leq \min\{\mu_{\mathcal{X}}, \varepsilon\}$ we get $S_a \preceq_\varepsilon^{s, \text{alt}} S_{\mathcal{X}_0, \mathcal{X}_f}(P)$ with strong $A\varepsilon A$ simulation relation

$$\mathcal{R}_{a,p} = \{(x_a, x_p) \in X_a \times X_p \mid x_a = [x_p]_\mu\}.$$

Proof. For any $x_a \in X_{0,a}$ by picking $x = x_a \in X_{0,p}$ we get $(x_a, x) \in \mathcal{R}_{a,p}$ and condition (i) of Definition 2.2 holds. For any $x_a \in X_{m,a}$ by picking $x = x_a \in X_{m,p}$ we get $(x_a, x) \in \mathcal{R}_{a,p}$ and condition (ii) of Definition 2.2 holds. Next, for any $(x_a, x) \in \mathcal{R}_{a,p}$ we get $\mathbf{d}(x_a, x) \leq \mu \leq \varepsilon$ from

¹Since V is smooth and \mathcal{X} is bounded, one can always choose $\gamma(\|w - z\|) = \left(\sup_{x, y \in \mathcal{X}} \left\| \frac{\partial V}{\partial y}(x, y) \right\| \right) \|w - z\|$.

which, condition (iii) holds. We now show condition (iv'). Consider any $(x_a, x) \in \mathcal{R}_{a,p}$, pick any $u \in U_a = \mathcal{U}$ and consider any transition

$$x \xrightarrow[p]{u} x^+. \quad (58)$$

By picking $x_a^+ = [x^+]_\mu$ we get $(x_a^+, x^+) \in \mathcal{R}_{a,p}$. We finally need to show that $x_a \xrightarrow[a]{u} x_a^+$. By (58), there exists $d \in \mathcal{D}$ such that $x^+ = f(x, u, d)$. Pick $d_a = [d]_\eta$. By using conditions in Definition 4.7 and by definition of function γ we get:

$$\begin{aligned} & V([f(x_a, u, d_a)]_\mu, x_a^+) \leq \\ & V(f(x_a, u, d_a), x_a^+) + \gamma(\mu) \leq \\ & V(f(x_a, u, d_a), x^+) + 2\gamma(\mu) = \\ & V(f(x_a, u, d_a), f(x, u, d)) + 2\gamma(\mu) \leq \\ & \kappa V(x_a, x) + \sigma_d(\|d_a - d\|) + 2\gamma(\mu) \leq \\ & \kappa(V(x_a, [x]_\mu) + \gamma(\mu)) + \sigma_d(\eta) + 2\gamma(\mu) = \\ & (\kappa + 2)\gamma(\mu) + \sigma_d(\eta) \end{aligned}$$

from which, we get Eqn. (57) and hence $x_a \xrightarrow[a]{u} x_a^+$. ■

From the definitions of S_a and $\mathcal{R}_{a,p}$ it is readily seen that conditions (c.6) and (c.8) of Theorem 4.4 are fulfilled.

In order to solve Problem 3, some preliminary definitions are needed. Given two systems $S_i = (X_i, X_{0,i}, U_i, \xrightarrow{i}, X_{m,i}, Y_i, H_i)$ ($i = 1, 2$), S_1 is a *sub-system* of S_2 if $X_1 \subseteq X_2$, $X_{0,1} \subseteq X_{0,2}$, $U_1 \subseteq U_2$, $\xrightarrow{1} \subseteq \xrightarrow{2}$, $X_{m,1} \subseteq X_{m,2}$, $Y_1 \subseteq Y_2$ and $H_1(x) = H_2(x)$ for any $x \in X_1$. Moreover, given two sub-systems $S_i = (X_i, X_{0,i}, U_i, \xrightarrow{i}, X_{m,i}, Y_i, H_i)$ ($i = 1, 2$) of a system S , define the union system $S_1 \sqcup S_2$ as $(X_1 \cup X_2, X_{0,1} \cup X_{0,2}, U_1 \cup U_2, \xrightarrow{1} \cup \xrightarrow{2}, X_{m,1} \cup X_{m,2}, Y_1 \cup Y_2, H)$, where $H(x) = H_1(x)$ if $x \in X_1$ and $H(x) = H_2(x)$ otherwise. Note that $S_1 \sqcup S_2$ is a sub-system of S . It is easy to see that the union operator enjoys the associative property.

We now give the solution to Problem 3.

Theorem 4.10. *Assume the conditions of Theorem 4.9 hold. Then define $S_{c,a}$ is the union of all non-blocking sub-systems S_c of S_a satisfying*

- 1) S_c is exactly simulated by $S_{q,a}$, i.e. $S_c \preceq_0 S_{q,a}$;
- 2) S_c is alternatingly 0-simulated by S_a , i.e. $S_c \preceq_0^{s,alt} S_a$.

Let $\mathcal{R}_{c,a}$ be the strong A0A simulation relation such that the previous conditions hold. Then $S_{c,a}$ and $\mathcal{R}_{c,a}$ solve Problem 3.

Proof. The proof is sketched for lack of space. Following the arguments included in the proof of Proposition 1 in [2], it is seen that $S_{c,a}$ satisfies conditions 1) and 2) of Theorem 4.10. Furthermore, by using Lemma 1 (iii) in [2], $S_{c,a} \preceq_0^{s,alt} S_a$ implies that $S_a \times_0^{\mathcal{R}_{c,a}} S_{c,a} \preceq_0 S_{c,a} \preceq_0 S_{q,a}$, so that condition (c.4) in Problem 3 is fulfilled as a consequence of Lemma 1 (ii) in [2]. Finally, condition (c.5) in Problem 3 is proved along the lines of the analogous proof of condition 2 of Problem 2 in Theorem 5 of [2]. This concludes the proof. ■

We point out that condition (i) of Definition 4.7 is not used in Definition 4.8 and Theorem 4.9. Hence, any smooth function $V : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_0^+$ satisfying condition (ii) of Definition 4.7 can be used in practice. However, we maintain condition (i) in Definition 4.7 to refer to δ -FC Lyapunov functions whose existence was shown in [13] to ensure the incremental forward completeness property.

We conclude this section by briefly discussing the role of quantization parameter η . While η plays a role in defining S_a , it does not in the above theorem. Hence, it can be chosen arbitrarily but affects computational complexity in deriving S_a : choosing smaller (resp. larger) values of η results in

- decreasing (resp. increasing) cardinality of \xrightarrow{a} and hence, space computational complexity in deriving S_a ;
- increasing (resp. decreasing) cardinality of $[\mathcal{D}]_\eta$ and hence, time computational complexity in deriving S_a .

5. Conclusions

In this paper we proposed a modular approach to solve symbolic control problems for discrete-time nonlinear control systems affected by disturbances, with specifications expressed in terms of regular languages and with partial information on the state given by its quantization. In future work we plan to extend these results to symbolic control design of hybrid systems with state quantized measurements.

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