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VERTEX-COLOURING OF 3-CHROMATIC CIRCULANT GRAPHS

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Abstract

A circulant graph $C_n(a_1, \ldots, a_k)$ is a graph with n vertices $\{v_0, \ldots, v_{n-1}\}$ such that each vertex v_i is adjacent to vertices $v_{(i+a_j) \bmod n}$, for $j = 1, \ldots, k$. In this paper we investigate the vertex colouring problem on circulant graphs. We approach the problem in a purely combinatorial way based on an array representation and propose an exact algorithm for a subclass of 3-chromatic $C_n(a_1, \ldots, a_k)$'s with $k \geq 2$, which are characterized in the paper.

Key words: circulant graphs, vertex-colouring, chromatic number

1. Introduction

Consider $k + 1$ integers $n, a_1, \ldots, a_k$ such that $n > 0$, $k \geq 2$, $a_i \bmod n \neq 0$ for $i = 1, \ldots, k$, and $a_i \not\equiv \pm a_j \pmod{n}$ for all $i = 1, \ldots, k$, $j = 1, \ldots, k$, $i \neq j$. The *circulant graph* $C_n(a_1, \ldots, a_k)$ is the (simple undirected) graph with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ and edge set $\{(v_i, v_{(i+a_j) \bmod n}),$ for $i = 0, \ldots, n-1$, $j = 1, \ldots, k\}$ (see Fig. 1). The integers $a_1, \ldots, a_k$ are called *entries*. For simplicity reasons, in the sequel, a_1 and a_2 will be often denoted by a and b, respectively.

A circulant graph $C_n(a_1, \ldots, a_k)$ is always regular: it is $2k$-regular if $a_i \neq \frac{n}{2}$ for all i, and $(2k-1)$-regular, otherwise. In addition, its adjacency matrix is symmetric and circulant.

Throughout the paper we shall assume w.l.o.g. that $a_i \in \{1, \ldots, n-1\}$ for $i = 1, \ldots, k$: under this hypothesis, the condition $a_i \not\equiv \pm a_j \pmod{n}$ is equivalent to require $a_i \neq a_j$ and $a_i + a_j \neq n$. These conditions imply $n \geq 2k$.

Consider an arbitrary entry $a_t \in \{a_1, \ldots, a_k\}$. If $(x \pm a_t) \bmod n = y$, we say that $v_x, v_y \in V$ are a_t-*adjacent* and that (v_x, v_y) is an a_t-*edge*. By a_t-path and a_t-cycle we shall denote a path and a cycle, respectively, made of a_t-edges only. We observe that a circulant graph $C_n(a_1, \ldots, a_k)$ contains $\gcd(n, a_t)$ distinct a_t-cycles with $\frac{n}{\gcd(n, a_t)}$ vertices each.

In this paper we investigate the vertex colouring problem on circulant graphs. It consists in finding an assignment of colours to the vertices of a given graph in such a way that adjacent vertices receive different colours and the number of colours is minimized. When adjacent vertices receive different colours, the colouring is *feasible*; a *k-colouring* is a vertex colouring which uses k colours; a graph G is *k-colourable* if it admits a feasible k-colouring; the smallest k such that G is k-colourable is the *chromatic number* $\chi(G)$ and G is said to be $\chi(G)$-*chromatic*.

The vertex colouring problem on arbitrary graphs is known to be $\mathcal{NP}$-hard [18].

In [11], the vertex colouring problem on circulant graphs $C_n(a_1, \ldots, a_k)$ is proved to be $\mathcal{NP}$-hard and not approximable better than a certain factor. The same authors, using spectral techniques, also show that $\lceil \log p \rceil$ eigenvectors are necessary and sufficient to feasibly colour p-chromatic circulant graphs of degree less than 5. The computational complexity of the problem suggests to focus on circulant graphs with a fixed number k of entries.
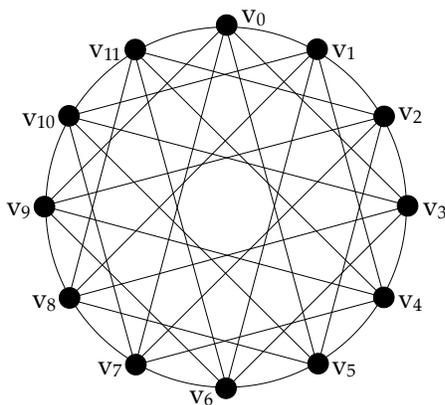

Figure 1: The circulant graph $C_{12}(1, 3, 5)$, with $k = 3$ entries.

For $k = 2$ the problem is studied in [16, 17, 25], where it is shown that it can be solved in polynomial time. In particular, in [16] the authors propose colouring algorithms for some

4.

3-chromatic $C_n(1, b)$'s. In [25] the authors prove a conjecture by [12] which characterizes all the 3-chromatic $C_n(1, b)$'s. In [22] it is proved the existence of a n_0 such that $\chi(C_n(a, b)) \leq 3$ when $n \geq n_0$, $1 \leq a < b$, and $b \neq 2a$. In [17] optimal colouring algorithms are proposed for all $C_n(a, b)$'s: many different subcases are identified, depending on the value of some parameters, a few of which related to the Hermite Normal Form associated to the given $C_n(a, b)$, and an optimal colouring algorithm is proposed for each subcase. The following theorem fully characterizes the chromatic number of a circulant graph with two entries.

Theorem 1.1. [16, 17, 25] *Consider a circulant graph $C_n(a, b)$ with $\gcd(n, a, b) = 1$ and $\gcd(n, a) \leq \gcd(n, b)$. Then*

$$\chi(C_n(a, b)) = \begin{cases} 2, & \text{if } n \text{ even, } a, b \text{ odd} \\ 5, & \text{if } n = 5 \\ 4, & \text{if } n = 13 \text{ AND } \left(b \equiv \pm 5a (\bmod 13) \text{ OR } a \equiv \pm 5b (\bmod 13)\right) \\ 4, & \text{if } n \neq 5, n \bmod 3 \neq 0, \text{ AND } \left(b \equiv \pm 2a \, (\bmod \, n) \text{ OR } a \equiv \pm 2b \, (\bmod \, n)\right) \\ 3, & \text{otherwise.} \end{cases}$$

When $k = 3$ some partial results are known: the authors of [25] characterize 4-chromatic 6-regular $C_n(1, b, b + 1)$'s, proving a conjecture by [12]; the chromatic number of $C_n(a, b, a + b)$'s is studied in [19], where optimal colouring algorithms are also proposed; in [2] the chromatic number of all circulant graphs $C_n(a_1, a_2, a_3)$ with $a_1 < a_2 < a_3$ and $n \geq 4a_2a_3$ is computed: the results are summarized in their Theorem 6, which shows that most of those graphs are 3-chromatic, except some particular cases which are 4- or 5-chromatic. The method by [17] for colouring circulant graphs $C_n(a_1, a_2)$ is extended in [20], resulting in the characterization of the chromatic number of 5-regular $C_n(a_1, a_2, \frac{n}{2})$'s and in optimal colouring algorithms for such graphs. However, as pointed out in [20], the "proof is fairly complicated" and "it does not appear feasible to determine the chromatic number of 6-valent circulants" $C_n(a_1, a_2, a_3)$'s by generalizing the method of [17]. Other results in [1, 5, 7, 23].

No matter about the number k of entries, a circulant graph $C_n(a_1, \ldots, a_k)$ is bipartite (thus 2-chromatic) if and only if n is even and all the entries are odd. Using the spectral properties of a circulant graph, the authors of [13] propose heuristic algorithms to colour the vertices of an arbitrary circulant graph (resulting in an upper bound for the minimum number of colours necessary to obtain a feasible vertex colouring of a circulant graph). The authors of [22] prove the existence of a n_0 such that $\chi(C_n(a_1, \ldots, a_k)) \leq 3$ when $n \geq n_0$ and $a_1 < \cdots < a_k < 2a_1$. In [2] the authors discuss some upper bounds on the chromatic number of circulant graphs whose set of entries has particular properties. Finally, the author of [27] states that upper bounds for the circular chromatic number and the chromatic number of circulant graphs can be obtained by means of the regular colouring method proposed in that paper for distance graphs (a *distance graph* $G_{\mathbb{Z}}(a_1, a_2, \ldots, a_k)$ is a graph with an infinite number of vertices $\{\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots\}$, where two vertices v_x and v_y are adjacent if and only if $|x - y| \in \{a_1, a_2, \ldots, a_k\}$, see [8, 9, 14]).

To our knowledge, no characterization of the chromatic number of circulant graphs $C_n(a_1, \ldots, a_k)$ with $k \geq 4$ has been proposed, yet.

In this paper, we propose a purely combinatorial approach for the vertex-colouring problem on circulant graphs. We shall consider the circulant graphs $C_n(a_1, \ldots, a_k)$'s with two entries a_i, a_j verifying $\gcd(n, a_i, a_j) = 1$. Under this hypothesis, any other entry a_t can be expressed as a weighted combination of a_i and a_j. We shall prove that the non-bipartite circulant graphs where, for each entry, the sum of the weights is odd and the weights themselves undergo some constraints are 3-chromatic. We shall also discuss an exact colouring algorithm for them.

The paper is organized as follows. Preliminary definitions and results can be found in Section 2. In Section 3 we define the subclass $\mathcal{T}$ of circulant graphs admitting a certain structure, the *staircase*, having prescribed properties. We also propose an exact 3-colouring algorithm for the circulant graphs $C_n(a_1, a_2, \ldots, a_k)$ belonging to $\mathcal{T}$: the algorithm starts by finding a (possibly infeasible) 2-colouring of the graph and then modifies it into a feasible 3-colouring for the graph. The correctness and the computational complexity of the algorithm are discussed. We then study the problem of detecting when a graph belongs to $\mathcal{T}$: in Sections 4 and 5 we define some properties for the staircases which will be useful, in Section 6, to characterize the circulant graphs on which our 3-colouring algorithm finds an exact solution. Section 7 concludes.

2. Preliminary definitions and results

In this paper we shall focus on circulant graphs belonging to a certain class $\mathcal{T} \subseteq \mathcal{T}'' \subseteq \mathcal{T}'$. We begin with the definition of the subclass $\mathcal{T}'$.

Definition 2.1. $\mathcal{T}'$ *is the subclass of circulant graphs* $C_n(a_1, \ldots, a_k)$ *admitting two distinct entries* a_i *and* a_j *verifying* $\gcd(n, a_i, a_j) = 1$.

Notice that if there esists an entry a_i such that $\gcd(n, a_i) = 1$ then $\gcd(n, a_i, a_j) = 1$ for any other entry $a_j \in \{a_1, \ldots, a_k\}$.

If $C_n(a_1, \ldots, a_k) \in \mathcal{T}'$ the entries can be rearranged in such a way that a_i and a_j are the first two in the new ordering. In the sequel, we will always deal with circulant graphs $C_n(a, b, a_3, \ldots, a_k) \in \mathcal{T}'$ where we assume w.l.o.g. that $\gcd(n, a, b) = 1$.

As proved in [4], $C_n(a, b)$ is connected if and only if $\gcd(n, a, b) = 1$. Therefore, an arbitrary circulant graph $C_n(a, b, a_3, \ldots, a_k) \in \mathcal{T}'$ is connected.

The circulant graphs $C_n(a, b, a_3, \ldots, a_k) \in \mathcal{T}'$ can be represented by $M_n(a, b)$, a suitable sub-array with n elements of the infinite array $M^*(a, b)$, which we now describe.

$M^*(a, b)$ is an array with an infinite number of rows and columns, which suitably organizes the vertices and the edges of $C_n(a, b, a_3, \ldots, a_k)$. Each element of $M^*(a, b)$ corresponds to a vertex of the graph (the condition $\gcd(n, a, b) = 1$ ensures that every vertex of the graph is represented in $M^*(a, b)$; notice that each vertex is represented infinitely many times in $M_n^*(a, b)$). See Fig. 3, where the value of an element is the index of the corresponding vertex.

Consider an arbitrary element $m_{i,j}^* \in M^*(a, b)$ and let v_x be the corresponding vertex. Then, the elements $m_{i,j-1}^*$ and $m_{i,j+1}^*$ correspond to the vertices $v_{(x-a) \bmod n}$ and $v_{(x+a) \bmod n}$, respectively, which are both a-adjacent to v_x, and the elements $m_{i-1,j}^*$ and $m_{i+1,j}^*$ correspond to the vertices $v_{(x-b) \bmod n}$ and $v_{(x+b) \bmod n}$, respectively, which are both b-adjacent to v_x (similar structures are defined in [3, 6, 10, 15, 24, 26]). Thus, the adjacency of two elements in the same row (column, respectively) represents an a-edge (a b-edge, respectively).

As it will be clear in the sequel, some edges play a peculiar role in the colouring phase. To our extent, we shall draw each of these a-edges as a vertical segment, *a-segment* (and each of these b-edges as a horizontal segment, *b-segment*, respectively) between the matrix elements corresponding to its endpoints.

Consider an arbitrary vertex v_x corresponding to element $m_{i,j}^* \in M^*(a, b)$, and consider its a_t-adjacent vertices $v_{(x+a_t) \bmod n}$ and $v_{(x-a_t) \bmod n}$ for some $t \in \{1, \ldots, k\}$. By construction, the elements of $M^*(a, b)$ corresponding to $v_{(x+a_t) \bmod n}$ and $v_{(x-a_t) \bmod n}$ are infinitely many (and regularly-placed): denote them by m_{i_1,j_1}^*, m_{i_2,j_2}^*, $m_{i_3,j_3}^*, \ldots$. For any of them, say m_{i_p,j_p}^*, the

following relation holds: $a_t = ((j_p - j)a + (i_p - i)b) \bmod n$, and it corresponds to saying that the two a_t-adjacent vertices v_x and $v_{(x+a_t) \bmod n}$ are connected by an *ab-path* consisting of $|j_p - j|$ a-edges and $|i_p - i|$ b-edges. If $i_p - i \geq 0$ ($j_p - j \geq 0$, respectively), $m^*_{i_p,j_p}$ is found $i_p - i$ rows below $m^*_{i,j}$ ($j_p - j$ columns on the right of $m^*_{i,j}$, respectively), that is to say, the b-edges (a-edges, respectively) of such ab-path are taken in the positive direction, (i.e. downwards (rightwards, respectively)). If $i_p - i \leq 0$ ($j_p - j \leq 0$, respectively), $m^*_{i_p,j_p}$ is found $i_p - i$ rows above $m^*_{i,j}$ ($j_p - j$ columns on the left of $m^*_{i,j}$, respectively), that is to say, the a-edges (b-edges, respectively) of such ab-path are taken in the negative direction, (i.e. leftwards (upwards, respectively)).

Let $m^*_{i_q,j_q}$ be an element connected to $m_{i,j}$ with a shortest ab-path, and let $(\alpha_t, \beta_t) = (i_q - i, j_q - j)$.

Definition 2.2. *The pair* (α_t, β_t) *will be called the* coordinates *of entry* a_t *w.r.t.* a, b.

As an example, the coordinates of a are always $(0, 1)$ or $(0, -1)$ indifferently, those of b are always $(1, 0)$ or $(-1, 0)$, while the coordinates of entry $a_3 = 23$ w.r.t. $6, 35$ in $M^*(6, 35)$ (see Fig. 3) are $(-1, 2)$ or $(1, -2)$.

Notice that if either one among $m^*_{i+\alpha_t,j+\beta_t}$ and $m^*_{i-\alpha_t,j-\beta_t}$ corresponds to vertex $v_{(x+a_t) \bmod n}$, then the other one corresponds to vertex $v_{(x-a_t) \bmod n}$ (that is to say, $a_t = (\beta_t a + \alpha_t b) \bmod n$): clearly, the ab-paths of these two elements have the same number of a-edges and the same number of b-edges, but in opposite direction.

We are now ready to state the second hypothesis the circulant graphs we focus on must verify.

Definition 2.3. $\mathcal{T}''$ *is the subclass of circulant graphs* $C_n(a, b, a_3, \ldots, a_k) \in \mathcal{T}'$ *verifying* $\alpha_t + \beta_t$ *odd for* $t = 1, \ldots, k$.

By definition, $\mathcal{T}'' \subseteq \mathcal{T}'$. Observe that the circulant graphs $C_n(a, b, a_3)$ with $a_3 = m(a + b)$ for $m \geq 1$ [19, 27] do not belong to $\mathcal{T}''$, as the coordinates of a_3 are $(\alpha_3, \beta_3) = (m, m)$, resulting in $\alpha_t + \beta_t$ even.

The subclass $\mathcal{T}''$ plays an important role. In fact, the main result of the paper is Theorem 6.6, where it is proved that if one of the sets of conditions of the table in Fig.14 is verified by every entry a_t of a circulant graph $C_n(a_1, \ldots, a_k) \in \mathcal{T}''$, then the graph is 3-chromatic, and the algorithm proposed in the paper yields a feasible 3-colouring (C, C', C'', R, R', R'' and λ, λ', π are easily computable parameters of the graph, which will be defined in Section 4).

In $M^*(a, b)$ we identify infinitely many copies of a connected and, generally speaking, irregularly-shaped n-element sub-array $M_n(a, b)$ whose elements $m_{i,j}$ are in one-to-one correspondence with the vertices of $C_n(a, b, a_3, \ldots, a_k)$. Many are the possible shapes of $M_n(a, b)$ (see the "closed areas" in Fig. 2), but any of them tessellates the whole $M^*(a, b)$.

Whatever the shape of $M_n(a, b)$, we can assume w.l.o.g. that element $m_{0,0}$ is the leftmost element of the topmost row of $M_n(a, b)$ (rows and columns of $M_n(a, b)$ can be consequently numbered).

Consider two elements adjacent in $M^*(a, b)$ but not adjacent within the same copy of $M_n(a, b)$. The segment separating them will be called *boundary segment* (BS, for short). The corresponding edge will be called *boundary edge* (BE, for short). Observe that if we draw all the boundary segments in $M^*_n(a, b)$, we get the "geometric" boundary of each copy of $M_n(a, b)$.

Figure 2: Examples of tessellation of $M^*(6,35)$: every closed areas in the figures above represents feasible shapes $M_{150}(6,35)$ for the circulant graph $C_{150}(6,35,a_3,\dots,a_k)$.

3. Colouring algorithm

In this section we describe an algorithm which outputs a feasible (and optimal) 3-colouring for some 3-chromatic $C_n(a,b,a_3,\dots,a_k) \in \mathcal{T}''$. In the sequel $B, W,$ and R will denote the colours black, white, and red, respectively.

The algorithm has two phases. In the first phase (Section 3.1) we start with a $B\&W$-colouring of (every copy of) $M_n(a,b)$. In the second phase (Section 3.2) we suitably modify into red the colour of one endpoint of each *infeasible* edge, that is, of each edge whose endpoints happen to have the same colour.

3.1. Phase 1

In the first phase of our approach we determine a 2-colouring of the subgraph $C_n(a,b)$. The $B\&W$-colouring of $M_n(a,b)$ ($B\&W(M_n(a,b))$, for short) is obtained by applying the following algorithm:

```
B&W(M_n(a,b))
For all elements m_{i,j} ∈ M_n(a,b) do
assign colour black to m_{i,j} iff i+j is even, and
assign colour white to m_{i,j} iff i+j is odd.
```

By definition of $B\&W$-colouring, two elements which are adjacent within the same copy of $M_n(a,b)$ receive different colours. Thus, the infeasible a- and b-edges, if any, are all BE's. In
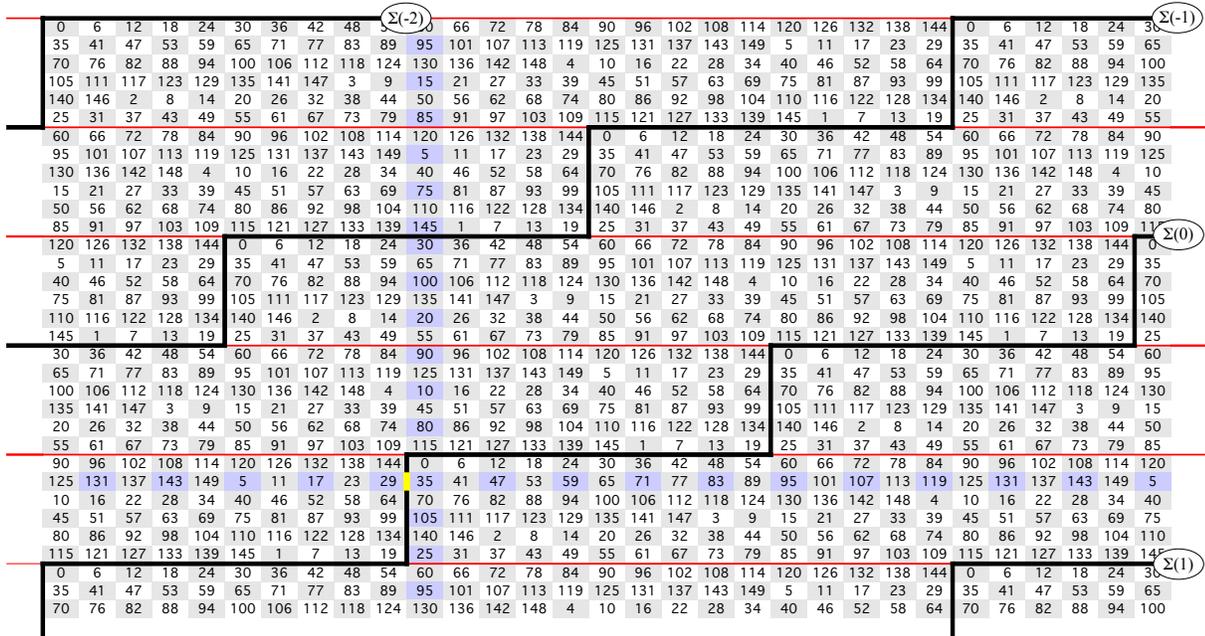
8.

Figure 3: The array $M^*_{150}(6,35)$ and a 6×25 rectangular sub-array $M_n(a,b)$: the feasible boundary segments of the chosen rectangular shape are highlighted in red, and the infeasible ones in black (in black, the staircase, as well). The elements highlighted in blue identify row 0 and column 0, thus $m_{0,0} = 35$, and the yellow segment on its left is x.

Fig. 3 we choose a 6×25 rectangular sub-array $M_{150}(6,35)$, and propose the $B\&W$-colouring of it; the infeasible boundary segments are highlighted in black, and the feasible ones in red. In the same figure observe that the set of all the infeasible boundary segments form infinitely many "parallel" *staircases* of infinitely many *steps* each.

For a given circulant graph $C_n(a,b,a_3,\ldots,a_k)$ and w.r.t. $B\&W(M_n(a,b))$, all the copies of the staircase have the same shape, which will be called Σ. Refer to Fig. 3. Consider an arbitrary element in $M^*_n(a,b)$, and w.l.o.g. let its row and column indices be $(0,0)$. If Σ contains vertical segments, choose a vertical one, say x, and let $\Sigma(0)$ be the copy of Σ where x coincides with the vertical segment separating $m^*_{0,-1}$ from $m^*_{0,0}$; then go downstairs along $\Sigma(0)$ and call $\Sigma(i)$, with $i > 0$ ($i < 0$, respectively) and integer, the i-th copy of Σ on your left (on your right respectively) of $\Sigma(0)$. If Σ contains only horizontal segments, let $\Sigma(0)$ be the copy of Σ containing the horizontal segment separating $m^*_{-1,0}$ from $m^*_{0,0}$; then go from left to right along $\Sigma(0)$ and call $\Sigma(i)$, with $i > 0$ ($i < 0$, respectively) and integer, the i-th copy of Σ above (below, respectively) $\Sigma(0)$.

Observe that many are the shapes of $M_n(a,b)$ giving rise to the same staircase.

If $B\&W(M_n(a,b))$ does not produce infeasible edges, then $C_n(a,b)$ is bipartite and, by Definition 2.3, the whole $C_n(a,b,a_3,\ldots a_k)$ is bipartite and belongs to $\mathcal{T}''$ (in fact, by Theorem 1.1, a,b are odd, and since $\alpha_t + \beta_t$ is assumed to be odd, then every $a_t = \beta_t a + \alpha_t b$ is odd). If there are infeasible edges, we proceed with the second phase, where we suitably modify into red the colour of one endpoint of each infeasible edge, without inducing new infeasibilities. In order to describe the *Zone Modification* (ZM, for short) which characterizes the second phase of the approach, we need to introduce the following definition.
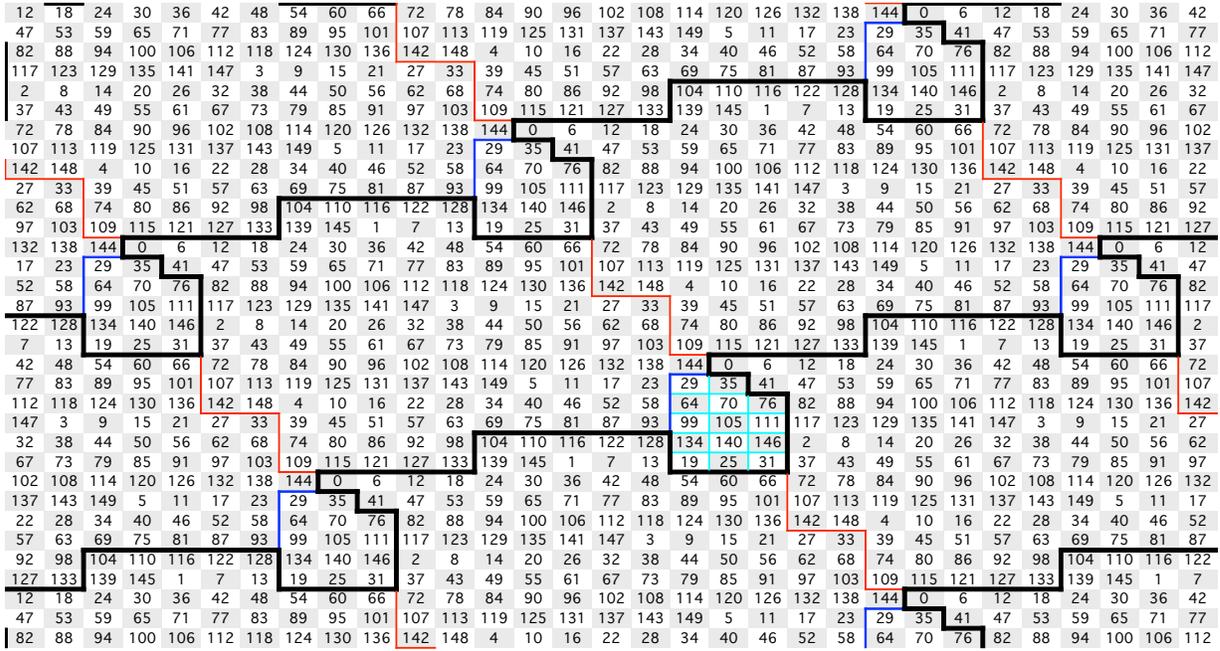
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 47  53  59  65  71  77  83  89  95 101 107 113 119 125 131 137 143 149   5  11  17  23  29  35  41  47  53  59  65  71  77
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 22  28  34  40  46  52  58  64  70  76  82  88  94 100 106 112 118 124 130 136 142 148   4  10  16  22  28  34  40  46  52
 57  63  69  75  81  87  93  99 105 111 117 123 129 135 141 147   3   9  15  21  27  33  39  45  51  57  63  69  75  81  87
 92  98 104 110 116 122 128 134 140 146   2   8  14  20  26  32  38  44  50  56  62  68  74  80  86  92  98 104 110 116 122
127 133 139 145   1   7  13  19  25  31  37  43  49  55  61  67  73  79  85  91  97 103 109 115 121 127 133 139 145   1   7
 12  18  24  30  36  42  48  54  60  66  72  78  84  90  96 102 108 114 120 126 132 138 144   0   6  12  18  24  30  36  42
 47  53  59  65  71  77  83  89  95 101 107 113 119 125 131 137 143 149   5  11  17  23  29  35  41  47  53  59  65  71  77
 82  88  94 100 106 112 118 124 130 136 142 148   4  10  16  22  28  34  40  46  52  58  64  70  76  82  88  94 100 106 112
```

Figure 4: In black, the staircase, i.e. the infeasible BS's of $M_{150}(6,35)$, in red the feasible BS's of $M_{150}(6,35)$; in blue, examples of shortcuts w.r.t. the black staircase; with a pale blue border, in the lower right corner of one copy of $M_{150}(6,35)$, the elements of Q.

Definition 3.1. *A staircase is* monotone *if any two (infeasible boundary) segments in it are connected by a minimum length (finite) sequence of consecutive (infeasible boundary) segments.*

Given x and y, two arbitrary BS's of a staircase, by *shortcut* we denote a minimum length (finite) sequence of consecutive segments connecting x and y: a staircase is monotone if it does not admit shortcuts (see Fig. 4, where, in black, many copies of a staircase are drawn and, in blue, many copies of a shortcut are indicated; remark that a staircase might admit different shortcuts). In order to transform a non-monotone staircase into a monotone one, determine a shortcut between two segments, insert the shortcut, remove the corresponding portion of the staircase, and repeat until the staircase is monotone (in Fig. 5, in black, the staircase resulting from the application of the blue shortcut to the staircase in Fig. 4). Clearly, these modifications apply to all the infinitely many copies of the staircase.

Recalling that a staircase is the collection of infeasible BS's, the application of a shortcut to a staircase corresponds to forcing the segments in the shortcut to become infeasible BS's, and the BS's belonging to the removed portion of the staircase to become feasible segments. This is easily achieved by complementing the colour of (the infinitely many copies of) a suitable (connected) subset $Q \subset M_n(a,b)$ of elements, that is to say, turning into black the colour of the white elements in Q, and viceversa. Precisely, Q is the subset of the elements between the staircase and the corresponding shortcut (the non-monotone black staircase of Fig. 4 is transformed into the monotone staircase of Fig. 5 by complementing the colour of the set Q of elements with a pale blue border).

Thus, applying a shortcut can also be seen as a modification of the shape of $M_n(a,b)$. Compare Fig. 4 and Fig. 5: the subset Q, with a pale blue border, is found in the lower right part of $M_{150}(6,35)$ (Fig. 4) and in the upper right part of $\overline{M}_{150}(6,35)$ (Fig. 5).
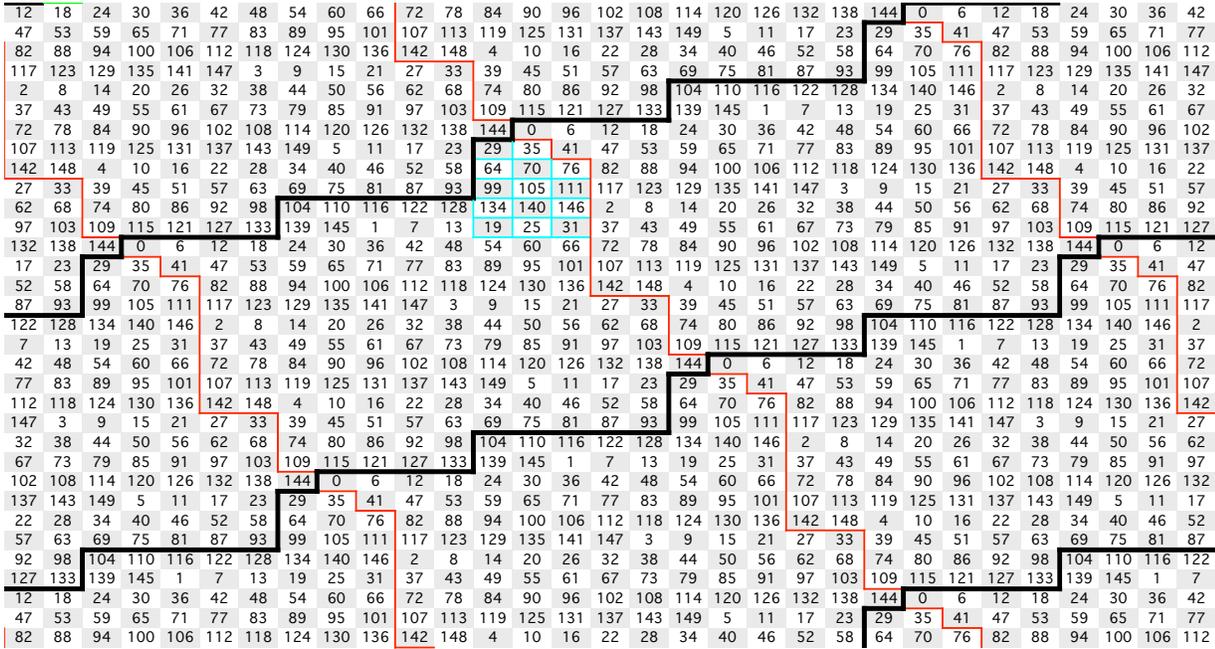
10.

Figure 5: In black, the staircase obtained by applying the shortcut of Fig.4; that is to say, in black the infeasible BS's of $\overline{M}_{150}(6,35)$, in red the feasible BS's of $\overline{M}_{150}(6,35)$; with a pale blue border, in the upper right corner of one copy of $\overline{M}_{150}(6,35)$, the elements of S.

Consider a circulant graph $C_n(a,b,a_3,\ldots,a_k) \in \mathcal{T}''$, and let $\Sigma(p)$ be an arbitrary copy of a monotone staircase for it. Consider an arbitrary entry a_t (with coordinates (α_t, β_t)), an arbitrary element $m^*_{i,j}$ of $M^*(a,b)$, and the a_t-edge connecting the elements $m^*_{i,j}$ and $m^*_{i+\alpha_t,j+\beta_t}$. By construction, the considered a_t-edge is infeasible if and only if $m^*_{i,j}$ and $m^*_{i+\alpha_t,j+\beta_t}$ are separated by $\Sigma(p)$. Going downstairs along $\Sigma(p)$, exactly one among $m^*_{i,j}$ and $m^*_{i+\alpha_t,j+\beta_t}$ lays on the left of $\Sigma(p)$, and the other one lays on the right of it: vary a_t and $m^*_{i,j}$ in all possible ways, and collect into $\mathcal{L}(p)$ ($\mathcal{R}(p)$, respectively) all the endpoints of infeasible edges laying on the *left* (on the *right*, respectively) of $\Sigma(p)$, going downstairs (if the staircase is horizontal, arbitrarily choose the leftwards or rightwards direction). In Fig. 6 the elements of $C_{150}(6,35,117)$ belonging to $\mathcal{L}(p)$ are highlighted with a pale blue border and those belonging to $\mathcal{R}(p)$ with a green border, for $p = \ldots, 1, 2, \ldots$

From now on, w.l.o.g. we shall also focus on the set $\mathcal{I}$ of elements laying within two arbitrary consecutive copies, say $\Sigma(1)$ and $\Sigma(2)$, of a monotone staircase for the given circulant graph. $\mathcal{L}(1)$ and $\mathcal{R}(2)$ are contained in $\mathcal{I}$ (see Fig. 6).

Let $\mathcal{L}'(1) \subseteq \mathcal{L}(1)$ denote the subset of all vertices in $\mathcal{L}(1)$ which are endpoint of a feasible edge $(v_x, v_{(x+a_t) \bmod n})$ with one endpoint in $\mathcal{L}(1)$ and one in $\mathcal{R}(2)$, if any. The following definition characterizes the class $\mathcal{T}$ of the circulant graphs we are interested in.

Definition 3.2. $\mathcal{T}$ is the subclass of circulant graphs $C_n(a,b,a_3,\ldots,a_k) \in \mathcal{T}''$ which admit a monotone staircase Σ such that the vertices in $\mathcal{L}'(1)$, if any, are all assigned the same colour. Such a monotone staircase will be called τ-staircase.

Observe that, by definition, $\mathcal{T} \subseteq \mathcal{T}'' \subseteq \mathcal{T}'$ and that no bipartite graph belongs to $\mathcal{T}$, as no staircase is defined for them.

Finally notice that being a τ-staircase depends both on the considered staircase and on the value of the given entries: the staircase in Fig. 6 is monotone but not a τ-staircase for the circulant graph $C_{150}(6, 35, 117)$ as the vertices in $\mathcal{L}'(1) = \{v_{66}, v_{101}, v_{136}\}$ are assigned different colours. The staircase in Fig. 8 is a τ-staircase for the circulant graph $C_{150}(6, 35, 117)$ because $\mathcal{L}'(1) = \emptyset$.
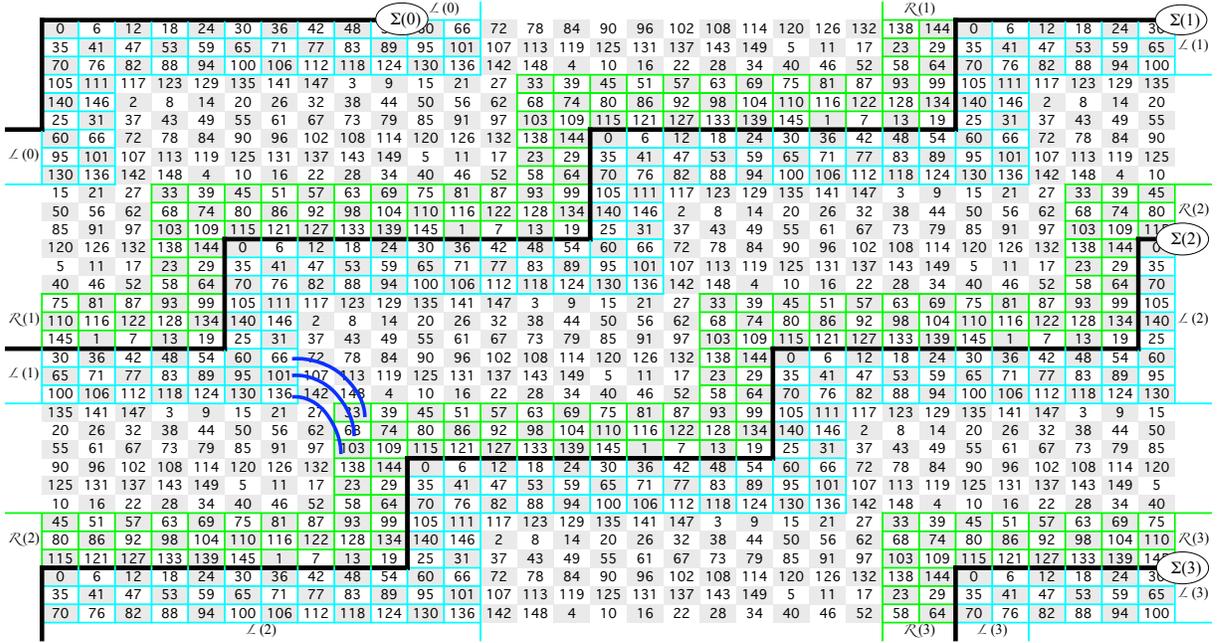

Figure 6: With a pale blue (green, respectively) border the elements in $\mathcal{L}(p)$ ($\mathcal{R}(p)$, respectively) for $C_{150}(6, 35, 117)$, for $p = 0, 1, 2, 3$; in blue the edges with one endpoint in $\mathcal{L}(1)$ and one in $\mathcal{R}(2)$. The considered staircase does not allow $C_{150}(6, 35, 117)$ to belong to $\mathcal{T}$.

We can prove the following result:

Lemma 3.3. If $C_n(a, b, a_3, \ldots, a_k) \in \mathcal{T}$, then $\mathcal{L}(1) \cap \mathcal{R}(2) = \emptyset$ and $\mathcal{L}(1) \cup \mathcal{R}(2) \subset V$.

Proof. Assume by contradiction that $\mathcal{L}(1) \cap \mathcal{R}(2) \neq \emptyset$. By construction, the elements in $\mathcal{L}(1)$ ($\mathcal{R}(2)$, respectively) are a connected set, thus every element in $\mathcal{L}(1)$ is adjacent to at least a different element in $\mathcal{L}(1)$ ($\mathcal{R}(2)$, respectively). Consider an arbitrary element $m^*_{i_p, j_p} \in \mathcal{L}(1) \cap \mathcal{R}(2)$. Since $m^*_{i_p, j_p}$ belongs to $\mathcal{L}(1)$, it is a- or b-adjacent to at least one other element in $\mathcal{L}(1)$, say $m^*_{i_q, j_q}$. But $m^*_{i_p, j_p}$ belongs to $\mathcal{R}(2)$, too, hence it is a- or b-adjacent to at least one other element in $\mathcal{R}(2)$, say $m^*_{i_s, j_s}$. By construction, the two edges $(m^*_{i_q, j_q}, m^*_{i_p, p})$ and $(m^*_{i_p, j_p}, m^*_{i_s, j_s})$ are two feasible edges with the first endpoint in $\mathcal{L}(1)$ and the second endpoint in $\mathcal{R}(2)$. In addition we observe that both vertices $m^*_{i_q, j_q}$ and $m^*_{i_p, p}$ belong to $\mathcal{L}'(1)$. Since edge $(m^*_{i_q, j_q}, m^*_{i_p, p})$ is feasible, its endpoints are assigned different colour. As a consequence $m^*_{i_q, j_q}$ and $m^*_{i_p, p}$ are two vertices with different colours belonging to $\mathcal{L}'(1)$, contradicting the hypothesis that $C_n(a, b, a_3, \ldots, a_k) \in \mathcal{T}$. For similar reasons, if we assume that $\mathcal{L}(1) \cup \mathcal{R}(2) = V$, the hypothesis that the considered circulant graph belongs to $\mathcal{T}$ is contradicted again. ∎

Notice that, by the lemma above, $\mathcal{L}(1)$, $\mathcal{R}(2)$, and the subset of all the remaining vertices are a 3-partition of the vertex set of the considered circulant.

12.

We observe that if the considered circulant graph in $\mathcal{T}$ has $k = 2$ entries, the sets $\mathcal{L}(1)$ and $\mathcal{R}(2)$ contain the endpoints of a boundary a- or b-edge, only, that is to say only elements of $M^*(a, b)$ adjacent to the staircase. When $k \geq 3$, the infeasible zone may also contain elements of $M^*(a, b)$ that are not directly adjacent to any staircase.

3.2. Phase 2

We are now ready to describe the second phase of our approach. It consists in applying the *Zone Modification* algorithm to $M_n(a, b)$ (for short, $ZM(M_n(a, b))$), which modifies the colour of a suitable set of elements.

> $ZM(M_n(a, b))$
>
> ```
> Modify into red the colour
> of the black elements in L(1),
> and of the white elements in R(2).
> ```

Notice that, w.l.o.g., we can exchange the role of black and white in the algorithm above without affecting all the results we are going to prove.
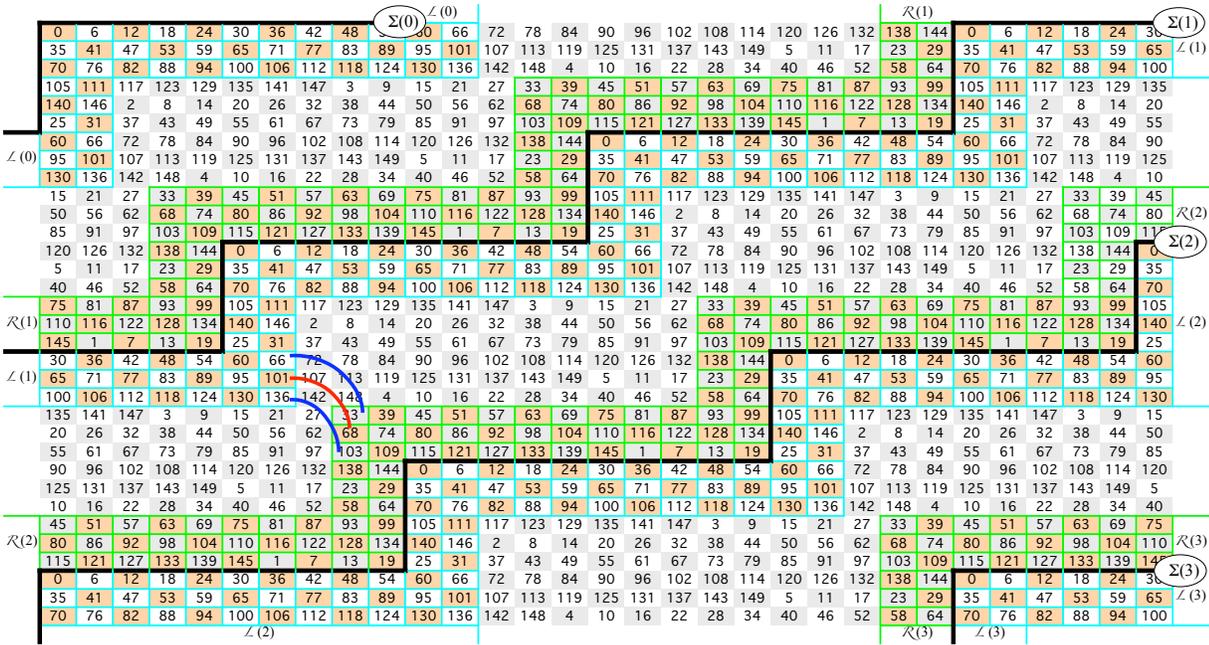

Figure 7: An infeasible 3-colouring for $C_{150}(6, 35, 117)$: the blue edges and the red one have one endpoint in $\mathcal{L}_1$ and one in $\mathcal{R}_2$; $ZM(M_{150}(6, 35))$ assigns colour red to both endpoints of the red edge.

3.3. Correctness

Let $B\&W + ZM(M_n(a, b))$ denote the algorithm obtained applying $B\&W$ and then ZM to the sub-array $M_n(a, b)$.

Theorem 3.4. *Let* $C_n(a, b, a_3, \ldots, a_k) \in \mathcal{T}$, *and let* $M_n(a, b)$ *be a sub-array originating a* τ*-staircase. Then* $B\&W + ZM(M_n(a, b))$ *outputs an optimal 3-colouring.*

Proof. Consider the colouring output by $B\&W(M_n(a,b))$. Since no graph in $\mathcal{T}$ is bipartite, $B\&W(M_n(a,b))$ originates a staircase, say Σ. We claim that the application of ZM removes all the infeasibilities generated by the $B\&W$-colouring without introducing any new one. W.l.o.g. we focus our attention on two consecutive copies of Σ, say $\Sigma(1)$ and $\Sigma(2)$. By construction, all the infeasible edges have one endpoint in $\mathcal{L}(1)$ and one in $\mathcal{R}(2)$. ZM modifies the colour of exactly one endpoint of each infeasible edge, precisely: if both endpoints are white, it modifies into red the only endpoint in $\mathcal{L}(1)$; if both are black, the only one in $\mathcal{R}(2)$. Thus, ZM removes all the infeasibilities generated by $B\&W$ on $M_n(a,b)$, as claimed. In addition, ZM does not introduce any new infeasibility: a new infeasibility would arise only if both endpoints of a feasible edge got the colour red. We distinguish two cases: $\mathcal{L}'(1) = \emptyset$ and $\mathcal{L}'(1) \supset \emptyset$. If $\mathcal{L}'(1) = \emptyset$, at most one endpoint of every feasible edge may get colour red, and we are done. In the other case, by Def. 3.2, all the elements in $\mathcal{L}'(1)$ have the same colour. If they are all white, then, by definition of ZM, the endpoints of all the feasible edges with one endpoint in $\mathcal{L}(1)$ and one in $\mathcal{R}(2)$ do not modify their colour into red. If the elements in $\mathcal{L}'(1)$ are all black, then, exchange the role of black and white in ZM: by doing so, the endpoints of all the feasible edge with one endpoint in $\mathcal{L}(1)$ and one in $\mathcal{R}(2)$ do not modify their colour into red, and the theorem is proved. ∎

Clearly, if the elements in $\mathcal{L}'(1)$ do not have all the same colour, ZM introduces new infeasibilities, which can not be avoided by exchanging the role of black and white in ZM. This fact shows that the condition in Def. 3.2 is necessary.

Two different colourings output by $B\&W + ZM$ are found in Figures 7 and 8 for the non-bipartite $C_{150}(6, 35, 117)$: the colouring of Figure 7 is infeasible because the considered staircase is not a τ-staircase; the one in in Figure 8 is feasible because the considered staircase is a τ-staircase. As a consequence, $C_{150}(6, 35, 117) \in \mathcal{T}$.

The following corollary to Theorem 3.4 holds.

Corollary 3.5. *If $C_n(a, b, a_3, \ldots, a_k) \in \mathcal{T}$, then $\chi(C_n(a, b, a_3, \ldots, a_k)) = 3$.*

In order to apply Theorem 3.4, the most important problem we have to face is that of detecting whether a circulant graph belonging to $\mathcal{T}''$ also belongs to $\mathcal{T}$ or not. This problem is twofold: we have to find a monotone staircase, and we have to check whether it is a τ-staircase or not. Sections 4 is devoted to these two aspects.

3.4. Computational complexity

A feasible 3-colouring of the vertices of a circulant graph $C_n(a_1, \ldots, a_k)$ belonging to $\mathcal{T}$ can be obtained in $O(n)$ time. In fact, algorithm $B\&W$ takes $O(n)$ to colour the vertices in black and white. Also, the subsets $\mathcal{L}(1), \mathcal{L}'(1)$, and $\mathcal{R}(2)$ can be efficiently determined in $O(n)$. Then, ZM has to suitably colour in red at most half the number of vertices of the graph. All together, the computational complexity of algorithm $B\&W + ZM$ for a graph belonging to $\mathcal{T}$ is $O(n)$.

More difficult is to check whether the given circulant graph belongs to $\mathcal{T}$. This is done in three steps: we first have to check if it belongs to $\mathcal{T}'$, then we have to check if it belongs to $\mathcal{T}''$, and finally we have to check if it belongs to $\mathcal{T}$.

The first task consists in finding two distinct entries a_i and a_j, if any, such that $\gcd(n, a_i, a_j) = 1$, and it can be accomplished in $O(k^2 \log n)$ time.

Once we have found two such entries, call them a and b, recalling that $a_t = \alpha_t b + \beta_t a$, we have to verify if $\alpha_t + \beta_t$ is odd, for $t = 1, \ldots, k$. This operation takes $O(\log n)$ for each entry.
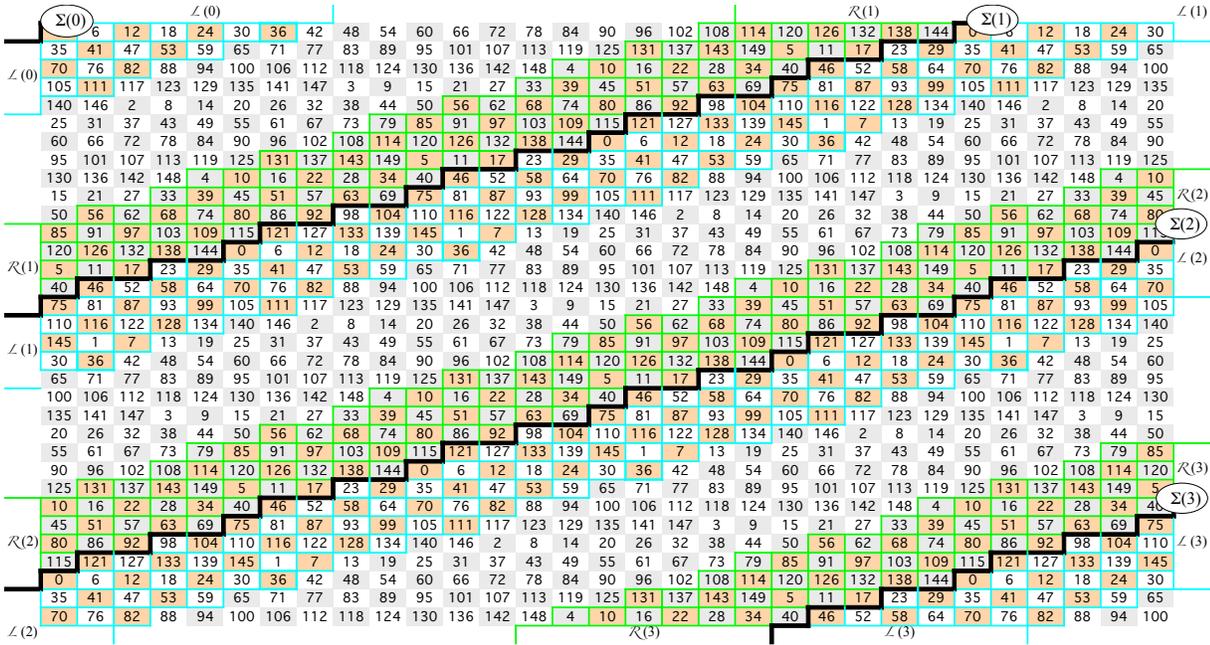
14.

Figure 8: A smooth staircase for $C_{150}(6, 35, 117)$ and a feasible 3-colouring for it.

Thus, in $O(k^3 \log^2 n)$ computational complexity we may know whether a given circulant graph belongs to $\mathcal{T}''$ or not.

In order to know whether the given circulant graph also belongs to $\mathcal{T}$ or not, we have to find a monotone staircase Σ having the properties required in Definition 3.2, if any. This is clearly not an easy task, generally speaking. In Theorem 6.6 we will characterize a subclass of graphs admitting such a staircase, hence a subclass of graphs belonging to $\mathcal{T}$. The characterization consists in verifying a set of conditions: this task can be accomplished in $O(\log a_2 + k)$ time.

Since $a_2 < n$, we get an overall computational complexity of $O(k^3 \log^2 n + n)$ to know whether the given circulant graph is 3-chromatic and can be coloured with the proposed algorithm, or not.

4. Monotone staircases

In this section we shall see how to get two (generally different) monotone staircases for a graph $C_n(a, b, a_3, \ldots, a_k) \in \mathcal{T}''$, by looking at its representation $M^*(a, b)$ (two staircases are different if no rigid shift exists which brings one staircase to coincide with the other one). If one of these two monotone staircases fits into the definition of τ-staircase (Def. 3.2), we are done (in fact we get an optimal 3-colouring of $C_n(a, b, a_3, \ldots, a_k)$ applying $B\&W + ZM$, according to Theorem 3.4). If none of them is a τ-staircase, it is sometimes possible to derive from them a different monotone staircase, the *smooth* staircase (see Section 5). Smooth staircases are important because necessary an sufficient conditions can be defined to check whether they are τ-staircases for the given graph, or not (see Theorem 6.3). When it is the case, we are done, and we get an optimal 3-colouring of $C_n(a, b, a_3, \ldots, a_k)$ applying $B\&W + ZM$, otherwise our algorithm outputs an infeasible 3-colouring for it.

Recalling that two circulant graphs $C_n(a, b, a_3, \ldots, a_k)$ and $C_n(b, a, a_3, \ldots, a_k)$ are isomorphic,

in the sequel w.l.o.g. we assume $\gcd(n,a) \leq \gcd(n,b)$.

We shall describe three different sub-arrays, called $W_n(a,b), Y_n(a,b)$, and $X_n(a,b)$, and the properties of their BS's. $W_n(a,b)$ is a sub-array which can be used to get a monotone staircase when a circulant graph $C_n(a,b,a_3,\ldots,a_k) \in \mathcal{T}'$ verifies $\gcd(n,a) \geq 2$; $Y_n(a,b)$ when $\gcd(n,b) \geq 2$; and $X_n(a,b)$ when $\gcd(n,a) = 1$.

4.1. Array $W_n(a,b)$ originating staircases $\Sigma_1, \Sigma_2, \Sigma_3$

In the present section we describe $W_n(a,b) = [w_{i,j}]$, a rectangular sub-array of $M_n^*(a,b)$ which can be defined for all the circulant graphs $C_n(a,b,a_3,\ldots,a_k) \in \mathcal{T}'$ with $\gcd(n,a) \geq 2$. $W_n(a,b)$ has n elements, $R = \gcd(n,a)$ rows, numbered from 0 to $R-1$, and $C = n/\gcd(n,a)$ columns, numbered from 0 to $C-1$. Notice that each row of $W_n(a,b)$ contains all the vertices of an a-cycle.

If we use $W_n(a,b)$ to tessellate $M_n^*(a,b)$ and colour it with the $B\&W$ algorithm, and if the considered graph is non-bipartite, we get a monotone staircase.

The following preliminary result holds (see Fig. 9).

Lemma 4.1. [21] *Consider a $C_n(a,b,a_3,\ldots,a_k) \in \mathcal{T}'$ and an arbitrary element $w_{R-1,h}$ in the last row of $W_n(a,b)$. Then, there exists a unique element $w_{0,k}$ in the first row which is b-adjacent to it, where $k = (h+\lambda) \bmod C$ and $\lambda = \min\{z \in \mathbb{Z}^+ : \gcd(n,a)b \equiv za \pmod{n}\}$.*

We remark that $0 \leq \lambda \leq C-1$, by definition, and that $\lambda = 0$ if and only if $n = \gcd(n,a)\gcd(n,b)$.
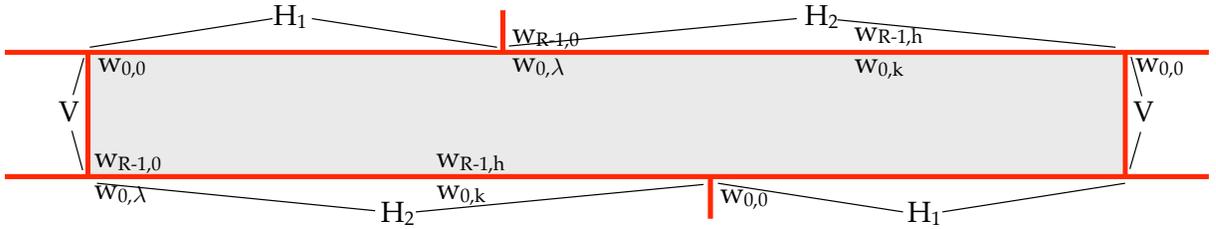

Figure 9: H_1, H_2, V and λ w.r.t. the gray copy of $W_n(a,b)$.

The set of boundary segments of $W_n(a,b)$ can be partitioned into "homogeneous" subsets: in Lemma 4.2 we shall prove, in fact, that a segment in a subset is infeasible w.r.t. $B\&W(M_n(a,b))$ iff the whole subset is infeasible w.r.t. $B\&W(M_n(a,b))$. These homogeneous subsets are denoted H_1, H_2, and V (they appear twice in Fig. 9, as each boundary edge of the graph is represented by two boundary segments of $W_n(a,b)$).

We shall represent each boundary segment by the two elements of $W_n(a,b)$ which are endpoints of the corresponding boundary edge. Therefore, the subsets H_1, H_2, and V are defined as follows:

$$H_1 = \begin{cases} \{(w_{0,0}, w_{R-1,C-\lambda}), (w_{0,1}, w_{R-1,C-\lambda+1}), \ldots, (w_{0,\lambda-1}, w_{R-1,C-1})\} & \text{when } \lambda \neq 0 \\ \emptyset & \text{when } \lambda = 0 \end{cases}$$

$$H_2 = \{(w_{0,\lambda}, w_{R-1,0}), (w_{0,\lambda+1}, w_{R-1,1}), \ldots, (w_{0,C-1}, w_{R-1,C-\lambda-1})\}$$

$$V = \{(w_{0,0}, w_{0,C-1}), (w_{1,0}, w_{1,C-1}), \ldots, (w_{R-1,0}, w_{R-1,C-1})\}$$

16.

Clearly, $|H_1| = \lambda$, $|H_2| = C - \lambda$, and $|V| = R$, as shown in Fig. 9.

Lemma 4.2. *Consider a boundary segment $s \in B$, for $B \in \{H_1, H_2, V\}$. Then s is infeasible w.r.t. the B&W-colouring if and only if each boundary segment in B is.*

Proof. Consider $B = V$ and let $(w_{i,0}, w_{i,C-1}) \in L$ for an arbitrary $i \in \{0, \ldots, R-1\}$. Since this boundary segment is infeasible, $w_{i,0}$ and $w_{i,C-1}$ have the same colour. Hence i and $i + C - 1$ are either both even or both odd. This implies that also h and $h + C - 1$, for $h = 0, \ldots, R-1$, do always have the same parity. In a similar way one can prove the result for the other three sets. ∎

Each subset among H_1, H_2, V is feasible or infeasible w.r.t. $B\&W(W_n(a,b))$, depending on some conditions on C and $R + \lambda$, as stated in the following lemma.

Lemma 4.3. *The following results hold:*
- *H_1 is infeasible w.r.t. the B&W-colouring of $W_n(a,b)$ if and only if $R + C + \lambda$ is odd.*
- *H_2 is infeasible w.r.t. the B&W-colouring of $W_n(a,b)$ if and only if $R + \lambda$ is odd.*
- *V is infeasible w.r.t. the B&W-colouring of $W_n(a,b)$ if and only if C is odd.*
- *No subset among H_1, H_2, and V is infeasible w.r.t. the B&W-colouring of $W_n(a,b)$ if and only if $R + \lambda$ and b are both even.*

Proof. Let us prove the first result. Consider an arbitrary boundary edge in H_1, say $(w_{0,0}, w_{R-1,C-\lambda})$. It is infeasible if and only if its endpoints have the same colour, that is to say, if and only if $0 + 0$ and $R - 1 + C - \lambda$ have the same parity, as claimed. By the same technique, one can prove the other results. ∎

From the lemma above it follows that four cases arise, depending on the parity of $R + \lambda$, b, and C. The last result, in particular, ensures that no staircase is originated by the $B\&W$ colouring; in fact, it identifies the bipartite $C_n(a,b)$, thus also the bipartite $C_n(a,b,a_3,\ldots,a_k) \in \mathcal{T}''$, by Def. 2.3. The other three results, summarized in the table below, identify three types of steps resulting from $B\&W(W_n(a,b))$. In the last two columns of the table, in particular, we find the number of infeasible horizontal segments (a-edges) and the number of infeasible vertical segments (b-edges) of the corresponding step (recall that $C - \lambda \geq 1$, by definition). We adopt the following notation. Go downwards along a staircase: the vertical segments are always found in rows with increasing indices, by definition; if the horizontal segments are found in columns of increasing indices, their number in the table is positive, otherwise it is negative.

$R + \lambda$	C	STEP	H_1	H_2	V	number of infeasible a-edges	number of infeasible b-edges
odd	even	S_1	infeas.	infeas.	feas.	0	C
even	odd	S_2	infeas.	feas.	infeas.	R	$-\lambda$
odd	odd	S_3	feas.	infeas.	infeas.	R	$C - \lambda$

The staircases originated by steps S_1, S_2, and S_3 are monotone and will be called Σ_1, Σ_2, and Σ_3 (Σ_1 and Σ_3 are shown in Fig. 10; Σ_2 is depicted in Fig. 7, where $\lambda = 10$).

Notice that the $B\&W$-colouring of $W_n(a,b)$ corresponds to alternating the two colours along each a-cycle. Hence each a-cycle has exactly one infeasible edge if and only if C is odd, if any,

and $W_n(a,b)$ turns out to be a sub-array which minimizes the number of infeasible a-edges. As for the b-edges, the only infeasible ones are some of those connecting the a-cycle in the first row of $W_n(a,b)$ and the one in the last row. Different steps are obtained by focussing on b-cycles, as explained in the next section.
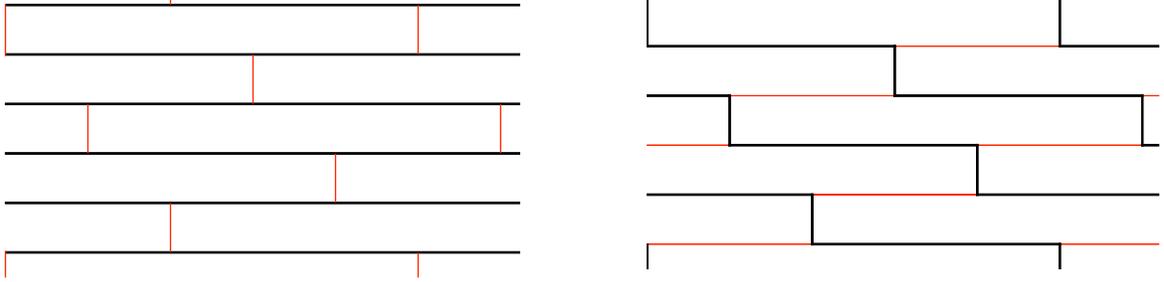

Figure 10: In black, the staircases Σ_1 (left) and Σ_3 (right) originated by steps S_1 and S_3, respectively.

4.2. Array $Y_n(a,b)$ originating staircases $\Sigma_4, \Sigma_5, \Sigma_6$

If we colour b-cycles suitably alternating colours black and white along them, we are actually defining $Y_n(a,b) = [y_{i,j}]$, a rectangular sub-array of $M_n^*(a,b)$. $Y_n(a,b)$ can be defined for all the circulant graphs $C_n(a,b,a_3,\ldots,a_k) \in \mathcal{T}'$ with $\gcd(n,b) \geq 2$, and has n elements, $R' = n/\gcd(n,b)$ rows, numbered from 0 to $R'-1$, and $C' = \gcd(n,b)$ columns, numbered from 0 to $C'-1$. It is important to note that each column of $Y_n(a,b)$ contains all the vertices of a b-cycle of $C_n(a,b)$ thus also of $C_n(a,b,a_3,\ldots,a_k)$. If we use $Y_n(a,b)$ to tessellate $M_n^*(a,b)$, and colour it in $B\&W$, and if the considered graph is non-bipartite, we get a monotone staircase for the considered graph.

A result similar to Lemma 4.1 holds: consider a $C_n(a,b,a_3,\ldots,a_k) \in \mathcal{T}'$ and an arbitrary element $y_{h,C'-1}$ in the last column of $Y_n(a,b)$; then, there exists a unique element $w'_{k,0}$ in the first column which is a-adjacent to it, where $k = (h+\lambda') \bmod R'$ and $\lambda' = \min\{z \in \mathbb{Z}^+ : \gcd(n,b)a \equiv zb \pmod{n}\}$.

Define

$$V'_1 = \begin{cases} \{(y_{0,0}, y_{R'-\lambda',C'-1}), (y_{1,0}, y_{R'-\lambda'+1,C'-1}), \ldots, (y_{\lambda'-1,0}, y_{R'-1,C'-1})\} & \text{when } \lambda' \neq 0 \\ \emptyset & \text{when } \lambda' = 0 \end{cases}$$

$$V'_2 = \{(y_{\lambda',0}, y_{0,C'-1}), (y_{\lambda'+1,0}, y_{1,C'-1}), \ldots, (y_{R'-1,0}, y_{R'-\lambda'-1,C'-1})\}$$

$$H' = \{(y_{0,0}, y_{R'-1,0}), (y_{0,1}, y_{R'-1,1}), \ldots, (y_{0,C'-1}, y_{R'-1,C'-1})\}.$$

Then, the following table can be written (recall that $R'-\lambda \geq 1$), where the number of horizontal segments is negative iff they are found in columns of increasing indices, when going downwards along the corresponding staircase.

$C'+\lambda'$	R'	STEP	V'_1	V'_2	H'	number of infeasible a-edges	number of infeasible b-edges
odd	even	S_4	infeas.	infeas.	feas.	R'	0
even	odd	S_5	infeas.	feas.	infeas.	λ'	$-C'$
odd	odd	S_6	feas.	infeas.	infeas.	$(R'-\lambda')$	C'

18.

The staircases Σ_4, Σ_5, and Σ_6 originated by the steps S_4, S_5, and S_6, respectively, are shown in Fig. 11.

For example, $Y_{150}(6, 35)$ has $R' = 30$ rows, $C' = 5$ columns, $\lambda' = 18$, $V_1' = \{(y_{0,0}, y_{12,4}), \ldots, (y_{17,0}, y_{29,4})\}$, $V_2' = \{(y_{18,0}, y_{0,4}), \ldots, (y_{29,0}, y_{11,4})\}$, $H' = \{(y_{0,0}, y_{29,0}), \ldots, (y_{0,4}, y_{29,4})\}$, and the resulting step is S_4.

The $B\&W$-colouring of $Y_n(a, b)$ is a colouring which minimizes the number of infeasible b-edges, and concentrates the infeasible a-edges among those connecting the b-cycles corresponding to the first and the last columns of $Y_n(a, b)$.
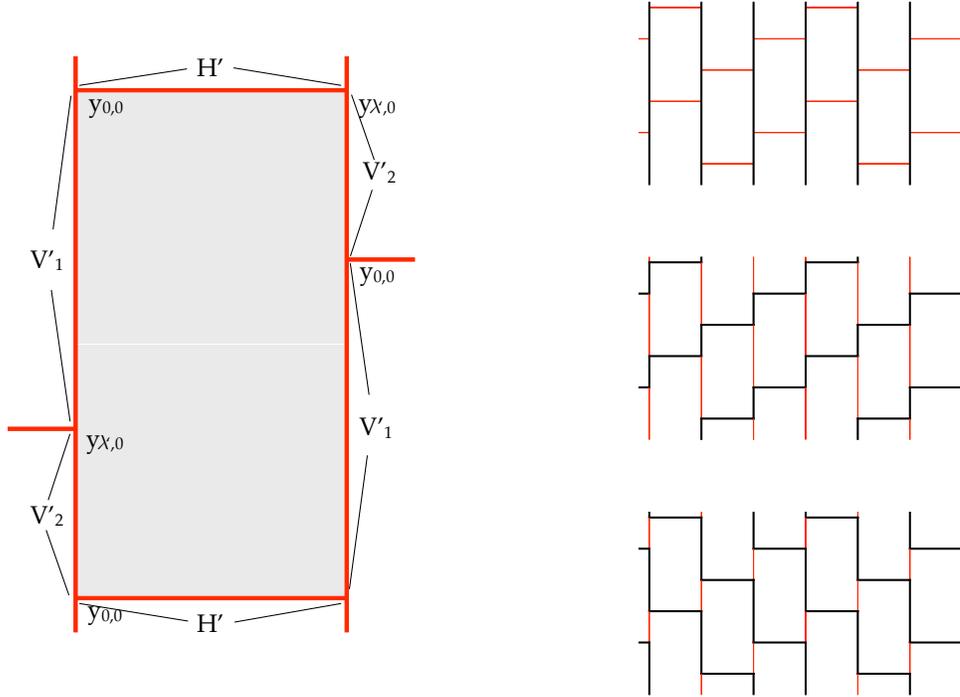
Figure 11: on the left, V_1', V_2', H' and λ' w.r.t. the gray copy of $Y_n(a, b)$; on the right, in black, the staircases Σ_4 (top), Σ_5 (center) and Σ_6 (bottom).

4.3. Array $X_n(a, b)$ originating staircases $\Sigma_7, \Sigma_8, \Sigma_9$

In the present section we describe $X_n(a, b) = [x_{i,j}]$, an irregularly-shaped sub-array of $M_n^*(a, b)$ which can be defined for all the circulant graphs $C_n(a_1, \ldots, a_k) \in \mathcal{T}'$ admitting an entry a_i such that $\gcd(n, a_i) = 1$. If we use $X_n(a, b)$ to tessellate $M_n^*(a, b)$, and colour it in $B\&W$, and if the considered graph is non-bipartite, we get a monotone staircase for the considered graph.

To start with, we recall that the entries of a circulant graph $C_n(a_1, \ldots, a_k)$ admitting an entry a_i verifying $\gcd(n, a_i) = 1$ can be rearranged in such a way that a_i is the first one in the new ordering, and it will be denoted a. The second entry in this new ordering can be chosen arbitrarily as $\gcd(n, a, a_j) = 1$ for all $a_j \in \{a_2, \ldots, a_k\}$. Whatever the second entry we choose, we shall denote it b.

Consider a graph $C_n(a, b, a_3, \ldots, a_k) \in \mathcal{T}'$ such that $\gcd(n, a) = 1$, and let t be an integer such

that $(ta) \bmod n \; = \; 1.$ The circulant graph $C_n(a', b', a'_3, \ldots, a'_k)$ where $a' = (ta) \bmod n = 1$, $b' = (tb) \bmod n$, and $a'_i = (ta_i) \bmod n$, for $i = 3, \ldots, k$, is isomorphic to $C_n(a, b, a_3, \ldots, a_k)$ [17], and verifies $\gcd(n, a', b') = 1$ and $1 = \gcd(n, a') \leq \gcd(n, b')$: thus, $C_n(a', b', a'_3, \ldots, a'_k) \in \mathcal{T}'$.

We also observe that a circulant graph $C_n(a = 1, b', a'_3, \ldots, a'_k) \in \mathcal{T}'$ with $b' \in \{b, n - b\}$, and $a'_t \in \{a_t, n - a_t\}$ for $t = 3, \ldots, k$, is isomorphic to $C_n(1, b, a_3, \ldots, a_k)$. To our extent, it is convenient to consider $b' = \min\{b, n - b\}$, as $\min\{b, n - b\} \leq n/2$.

In the present section, w.l.o.g., we shall deal with circulant graphs $C_n(a, b, a_3, \ldots, a_k)$ with $a = 1$ and $b \leq n/2$.

For a non-bipartite circulant graph $C_n(a, b, a_3, \ldots, a_k)$ with $a = 1$ and $b \leq n/2$, a monotone staircase is originated by $X_n(1, b)$, an irregularly shaped sub-array of $M^*(1, b)$ with n elements. It is a sub-array of $M^*(1, b)$, and its elements are in one-to-one correspondence with the vertices of $C_n(1, b)$. $X_n(1, b)$ is defined on $C'' = b$ consecutive columns and $R'' = \left\lceil \frac{n}{C''} \right\rceil \geq 2$ consecutive rows, the last of which is not complete and contains $\pi = n - (R'' - 1)C'' < C''$ elements. Observe that the assumption $b \leq n/2$ implies $R'' \geq 2$.

We begin by introducing the following preliminary result, similar to Lemma 4.1: consider a $C_n(1, b, a_3, \ldots, a_k)$ and let $x_{q,h}$ denote the bottom-most element of column h of $X_n(1, b)$ (that is to say, either $q = R'' - 1$ and $h \leq \pi - 1$ or $q = R'' - 2$ and $h \geq \pi$); then, there exists a unique element $x_{0,k}$ in the first row of $X_n(1, b)$ which is b-adjacent to $x_{q,h}$, where $k = (h - \pi) \bmod C''$.

Similarly to what described in Section 4.1, the set of edges corresponding to the boundary segments of $X_n(1, b)$ can be partitioned into homogeneous subsets. Precisely,

$$H''_1 = \begin{cases} \{(x_{0,0}, x_{R''-2,\pi}), (x_{0,1}, x_{R''-2,\pi+1}), \ldots, (x_{0,C''-\pi-1}, x_{R''-2,C''-1})\} & \text{when } \pi \neq 0 \\ \emptyset & \text{when } \pi = 0 \end{cases}$$

$$H''_2 = \{(x_{0,C''-\pi}, x_{R''-1,0}), (x_{0,C''-\pi+1}, x_{R''-1,1}), \ldots, (x_{0,C''-1}, x_{R''-1,\pi-1})\}$$

$$V''_1 = \{(x_{0,0}, x_{R''-1,\pi-1})\}$$

$$V''_2 = \{(x_{1,0}, x_{0,C''-1}), (x_{2,0}, x_{1,C''-1}), \ldots, (x_{R''-1,0}, x_{R''-2,C''-1})\}$$

as illustrated in Fig. 12. Notice that V_1 consists of a unique boundary edge (all the other boundary a-edges belong to V_2).

The following table can be written (see Fig. 12), where the number of horizontal segments is negative iff they are found in columns of increasing indices, when going downwards along the corresponding staircase):

$R'' + \pi$	C''	STEP	H''_1 and V''_1	H''_2	V''_2	number of infeasible a-edges	number of infeasible b-edges
even	odd	S_7	infeas.	infeas.	feas.	1	$-C''$
odd	odd	S_8	feas.	infeas.	infeas.	$(R'' - 1)$	π
even	even	S_9	infeas.	feas.	infeas.	R''	$-(C'' - \pi)$

5. Smooth and monotone staircases

If a monotone staircase happens to be a τ-staircase for the considered circulant graph, then we are done and we get a feasible 3-colouring for the graph applying $B\&W + ZM$ to the corresponding shape.
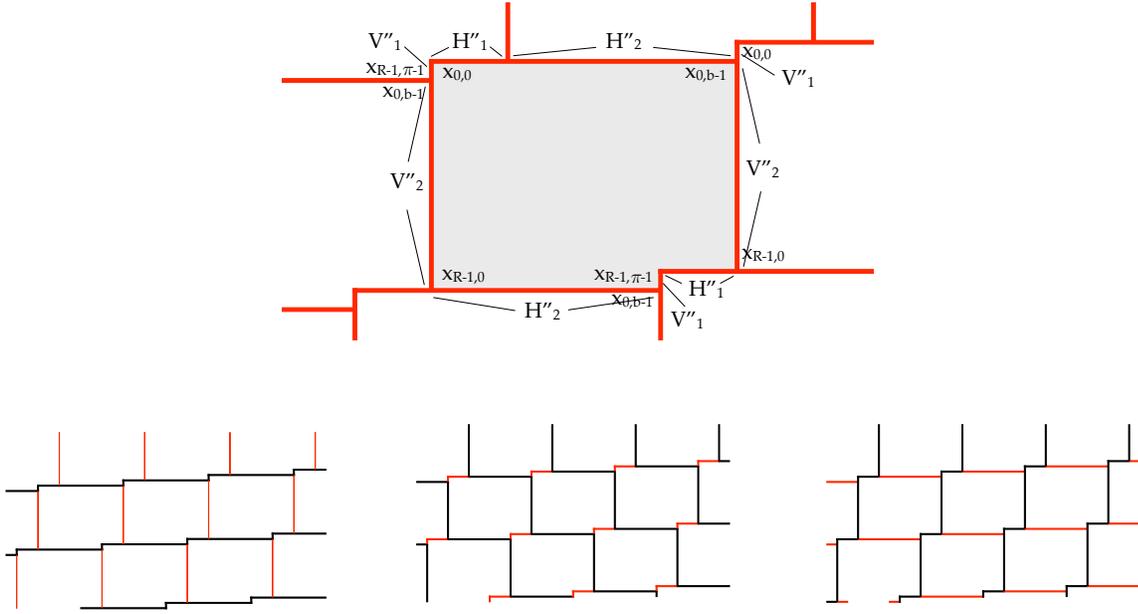
20.

Figure 12: above, $V_1'', V_2'', H_1'', H_2''$ and π w.r.t. the gray copy of $X_n(1,b)$; below, in black, the staircases Σ_7 (left), Σ_8 (center) and Σ_9 (right).

Unfortunately, being a monotone staircase is not sufficient to be a τ-staircase. Even requiring that the facing boundaries of $\mathcal{L}(i)$ and $\mathcal{R}(i+1)$ are sufficiently far away from one another does not imply that a monotone staircase is a τ-staircase.

In fact, consider a circulant graph $C_n(a_1, \ldots, a_h) \in \mathcal{T}''$, a shape $M_n(a_1, a_2)$ originating a monotone staircase Σ, two arbitrary consecutive copies of Σ, say $\Sigma(1)$ and $\Sigma(2)$, and let $\mathcal{I}$ be the set of elements between them. The staircase $\Sigma(1)$ separates $\mathcal{I}$ from the elements in the external region $\mathcal{E}(1)$, while the staircase $\Sigma(2)$ separates $\mathcal{I}$ from $\mathcal{E}(2)$. For each entry a_t, $t = 1, \ldots, k$, and w.r.t. a given monotone staircase Σ, define

$$\delta_\Sigma^{a_t} = \min_{m_{i,j}^* \in \mathcal{E}(1)} \{p \in \mathbb{Z}^+ : m_{i+p\alpha_t, j+p\beta_t}^* \in \mathcal{E}(2)\},$$

which represents the number of edges of a shortest a_t-path connecting an element in $\mathcal{E}(1)$ with an element in $\mathcal{E}(2)$. Notice that $\delta_\Sigma^{a_1}, \delta_\Sigma^{a_2} \geq \delta_\Sigma^{a_t}$ for $t = 3, \ldots, k$. In Fig. 13 two such paths are drawn: let $a_s = 42$ and $a_t = 146$, in blue a shortest a_s-path with $\delta_\Sigma^{a_s} = 4$, in pink a shortest a_t-path with $\delta_\Sigma^{a_t} = 4$.

By definition, if there exists an entry a_t, $t \in \{1, \ldots, k\}$ with $d_\Sigma^{a_t} = 2$ then Σ is not a τ-staircase for $C_n(a_1, \ldots, a_k)$.

If all the entries a_t, for $t = 1, \ldots, k$ verify $\delta_\Sigma^{a_t} \geq 3$, Σ might be a τ-staircase but often this is not the case as we now show with two examples.

Consider a circulant graph with a single entry a_s veryfying $\delta_\Sigma^{a_s} = 3$, and all the other entries a_t with $t \neq s, t = 1, \ldots, k$ verifying $\delta_\Sigma^{a_t} \geq 4$. Let $m_{i,j}^*$ and $m_{p,q}^*$ in $\mathcal{E}(1)$ such that $m_{i+3\alpha_s, j+3\beta_s}^*$ and $m_{p+3\alpha_s, q+3\beta_s}^*$ belong to $\mathcal{E}(2)$. If $m_{i,j}^*$ and $m_{p,q}^*$ are a- or b-adjacent (that is to say, when $i = p$ and $j = q \pm 1$ or $i = p \pm 1$ and $j = q$) they have different colours. Hence algorithm

$BW + ZM(M_n(a_1, a_2))$ does not output a feasible 3-colouring. For example, consider $C_{150}(6, 35, 117)$ and refer to Fig. 7. The chosen shape originating the depicted monotone staircase is $W_{150}(6, 35)$. The coordinates of $a_3 = 117$ are $(\alpha_3, \beta_3) = (3, 2)$, and $\delta_\Sigma^{117} = 3$. Choose $m_{i,j}^*$ as the element corresponding to vertex v_{134} and choose $p = i - 1$ and $q = j$ yielding $m_{p,q}^*$ as the element just above $m_{i,j}$ and corresponding to vertex v_{99}. Both v_{134} and v_{99} belong to $\mathcal{E}(1)$, and their a_3-adjacent vertices v_{101} and v_{66} belong to $\mathcal{L}'(1)$. By construction, also v_{101} and v_{66} are b-adjacent, thus $B\&W$ assigns them different colours. As a consequence, $B\&W + ZM(W_{150}(6, 35))$ outputs an infeasible 3-colouring for it, as shown by the red edge ($W\&B + ZM(W_{150}(6, 35))$ outputs a infeasible 3-colouring for it, too, as shown by the blue edges).

The second example is the following. Assume there exist two distinct entries a_s, a_t such that $\delta_\Sigma^{a_s} = \delta_\Sigma^{a_t} = 3$, and two elements with different colours $m_{i,j}^*$ and $m_{p,q}^*$ in $\mathcal{E}(1)$ such that $m_{i+3\alpha_s, j+3\beta_s}^*$ and $m_{p+3\alpha_t, q+3\beta_t}^*$ belong to $\mathcal{E}(2)$, then $m_{i+\alpha_s, j+\beta_s}^*$ and $m_{p+\alpha_t, q+\beta_t}^*$ belong to $\mathcal{L}'(1)$, and have different colours. By Definition 3.2, Σ is not a τ-staircase for $C_n(a_1, \ldots, a_k)$.

When the graph has only $k = 3$ entries, and $\delta_\Sigma^{a_3} = 4$, then $\mathcal{L}'(1)$ is empty, and the following result can be stated:

Theorem 5.1. *Let $C_n(a, b, a_3) \in \mathcal{T}''$, and let $M_n(a, b)$ be a sub-array originating a monotone staircase Σ. If $\delta_\Sigma^{a_3} \geq 4$, then $B\&W + ZM(M_n(a, b))$ outputs an optimal 3-colouring.*

This result applies to $C_{150}(6, 35, 100)$ when we choose the monotone staircase in Fig. 8.

Unfortunately, when $k \geq 4$, even requiring $\delta_\Sigma^{a_t} \geq 4$ for all $a_t \in \{a_1, \ldots, a_k\}$ is not sufficient to state that the given monotone staircase is a τ-staircase. The reason is that $\mathcal{L}(1)$ and $\mathcal{R}(2)$ depend on all the entries $a_1, \ldots, a_k$, while requiring $\delta_\Sigma^{a_t} \geq 4$ for each entry $a_t \in \{a_1, \ldots, a_k\}$ considers only one entry at a time, and not the interactions between them. In Fig. 13 the example $C_{150}(6, 35, 42, 146)$ is studied: one has $\delta_\Sigma^6 = 25$, $\delta_\Sigma^{35} = 12$, $\delta_\Sigma^{42} = 4$, and $\delta_\Sigma^{146} = 4$, but the edges in red have one endpoint in $\mathcal{L}(1)$ and one in $\mathcal{R}(2)$.

Necessary and sufficient conditions can be defined for a monotone staircase to be a τ-staircase if we ask the monotone staircase to be *smooth*. A smooth (and monotone) staircase can be derived from every monotone staircase. W.l.o.g. we will derive the smooth staircases from the monotone staircases $\Sigma_1, \ldots, \Sigma_9$.

For $i = 1, \ldots, 9$, define the infeasibility ratio IR_i of step S_i as the average number of columns we move rightwards per each row we move downwards: thus $IR_i < 0$ if we move leftwards per each row we move downwards. The infeasibility ratios $IR_1, \ldots, IR_9$ of the steps $S_1, \ldots, S_9$, respectively, are reported in the following table.

IR_1	IR_2	IR_3	IR_4	IR_5	IR_6	IR_7	IR_8	IR_9
∞	$-\frac{\lambda}{R}$	$\frac{C-\lambda}{R}$	0	$-\frac{C'}{\lambda'}$	$\frac{C'}{R'-\lambda'}$	C''	$\frac{\pi}{R''-1}$	$-\frac{C''-\pi}{R''}$

Consider an arbitrary element in $M_n^*(a, b)$, and w.l.o.g. let its row and column indices be $(0, 0)$.

Definition 5.2. *The staircase made of the vertical segments separating elements $m_{i,-1+\lceil iIR_h \rceil}^*$ and $m_{i,\lceil iIR_h \rceil}^*$, with $i \in \mathbb{Z}$, for $h \in \{2, 3, 5, 6, 8, 9\}$ and of the horizontal segments connecting them is a* smooth *staircase, and will be denoted by Σ_h^s. For $h \in \{1, 4, 7\}$, the staircases Σ_h are already smooth, thus $\Sigma_h^s = \Sigma_h$.*

An example is found in Fig. 8, where $m_{0,0}^* = 46$.

22.
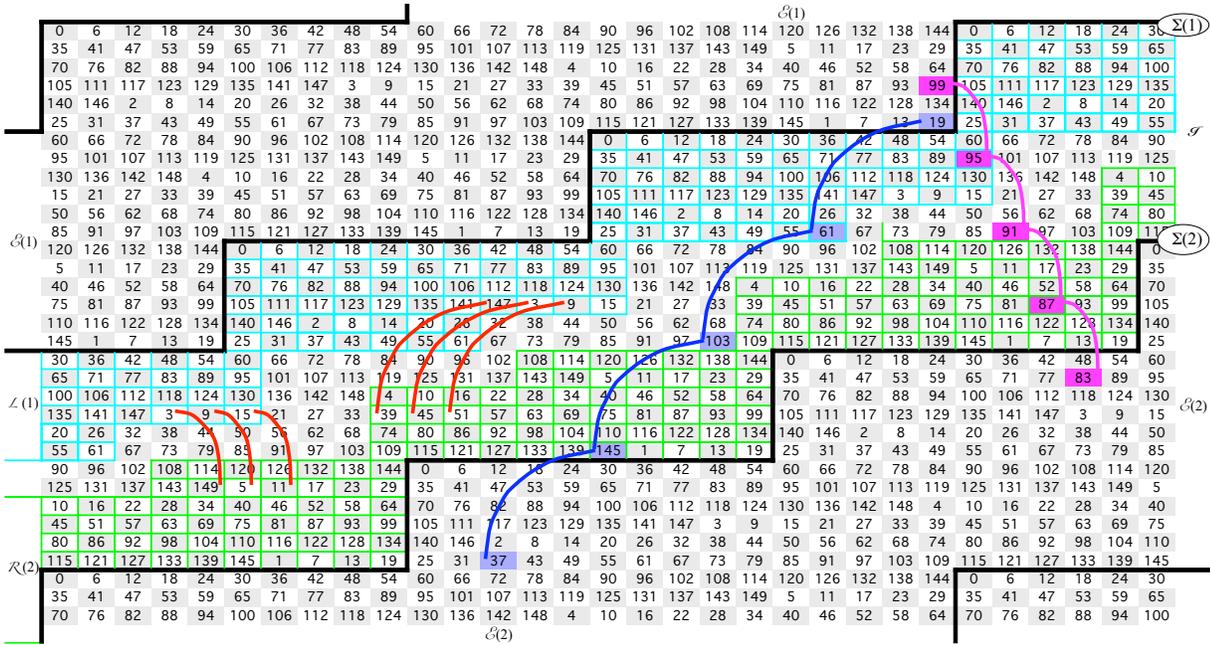
Figure 13: $M^*_{150}(6, 35)$ for the circulant graph $C_{150}(6, 35, 42, 146)$ (the chosen shape is $W_{150}(6, 35)$ which originates the monotone staircase Σ), two consecutive copies of Σ, $\Sigma(1)$ and $\Sigma(2)$, and the regions $\mathcal{I}$, $\mathcal{E}(1)$, and $\mathcal{E}(2)$: in blue, the edges of a shortest a_3-path with $\delta^{a_3}_\Sigma = 4$, in pink, the edges of an a_4-path with $\delta^{a_4}_\Sigma = 4$; in red some edges with one endpoint in $\mathcal{L}(1)$ and the other one in $\mathcal{R}(2)$.

A smooth staircase will have the same infesibility ratio of the monotone staircase it is derived from, and will keep being monotone. In addition, the smooth staircase will have the same number of infeasible a- and b-edges of the corresponding non-smooth staircase: what will differ in the two cases is the number of vertices which are endpoints of the infeasible a- and b-edges, in fact in the smooth staircase this number is not larger than the number of vertices which are endpoints of the infeasible a- and b-edges in the non-smooth staircase.

Like any other staircase, a smooth staircase is generated by some shape, which we will denote $M^s_n(a, b)$.

Smooth staircases are important because if the coordinates of the entries undergo certain conditions, the smooth staircase is a τ-staircase for the graph, as discussed in the next section and proved in Theorem 6.3, and the graph can be optimally coloured by our algorithm, as stated by Corollary 6.5. Similar results, unfortunately, can not be proved for arbitrary monotone staircases.

6. 3-chromatic circulant graphs

In the present section we characterize those 3-chromatic circulant graphs which can be optimally coloured by means of algorithm $B\&W + ZM$. We will state a necessary and sufficient condition for a smooth (and monotone) staircase to be a τ-staircase for a circulant graph in $\mathcal{T}''$: Theorem 3.4 then ensures the our algorithm finds an exact 3-coloring of the given circulant.

For practical reasons, we shall first prove our results on a circulant graph $C_n(a_1, \ldots, a_k) \in \mathcal{T}''$ originating step S_2. In Theorem 6.6 such results are extended to a circulant graph $C_n(a_1, \ldots, a_k) \in \mathcal{T}''$ originating an arbitrary S_h, with $h \in \{1, \ldots, 9\}$.

Consider a circulant graph $C_n(a_1, \ldots, a_k) \in \mathcal{T}''$ originating step S_2, consider the corresponding smooth staircase Σ^2, and two arbitrary consecutive copies of Σ_2^s, say $\Sigma_2^s(1)$ and $\Sigma_2^s(2)$. Once again, consider the internal region $\mathcal{I}^s$, i.e. the set of elements between $\Sigma_2^s(1)$ and $\Sigma_2^s(2)$, and recall that $\Sigma_2^s(1)$ separates $\mathcal{I}^s$ from the elements in the external region $\mathcal{E}^s(1)$, that $\Sigma_2^s(2)$ separates $\mathcal{I}^s$ from $\mathcal{E}^s(2)$, and that (α_t, β_t) and $(-\alpha_t, -\beta_t)$ are both valid coordinates for entry a_t, for $t = 1, \ldots, k$. We can prove the following results.

Lemma 6.1. *Consider a circulant graph $C_n(a_1, \ldots, a_k) \in \mathcal{T}''$ originating step S_2, and consider the corresponding smooth staircase Σ_2^s. One has $\delta_{\Sigma_2^s}^{a_t} \geq 4$ if and only if $0 \leq \beta_t \leq \left\lfloor \frac{C + \lfloor 3\alpha_t IR_2 \rfloor}{3} \right\rfloor$, for $t = 1, \ldots, k$.*

Proof. Consider an arbitrary element $m_{i,j}^* \in \mathcal{E}^s(1)$. Thus $(i, j) \leq (i, \lceil iIR_2 \rceil - 1)$. By definition, $\delta_{\Sigma_2^s}^{a_t} \geq 4$ when $m_{i+3\alpha_t, j+3\beta_t}^* \in \mathcal{I}^s$. That is to say, we have to ensure that $(i + 3\alpha_t, j + 3\beta_t) \leq (i + 3\alpha_t, C - 1 + \lceil (i + 3\alpha_t)IR_2 \rceil)$. The worst case happens when $m_{i,j}^*$ is the rightmost element in row i belonging to $\mathcal{E}^s(1)$, precisely, when $j = \lceil iIR_2 \rceil - 1$. Under this choice, the considered inequality becomes $j + 3\beta_t = (\lceil iIR_2 \rceil - 1) + 3\beta_t \leq C - 1 + \lceil (i + 3\alpha_t)IR_2 \rceil$, which asks $\beta_t \leq \lfloor \frac{C + \lceil (i+3\alpha_t)IR_2 \rceil - \lceil iIR_2 \rceil}{3} \rfloor$ for all $t \in \{1, \ldots, k\}$ and for an arbitrary i. Since $\lceil 3\alpha_t IR_2 \rceil - 1 \leq \lceil (i + 3\alpha_t)IR_2 \rceil - \lceil iIR_2 \rceil \leq \lceil 3\alpha_t IR_2 \rceil$ and considering that, in the worst case, $\lceil (i + 3\alpha_t)IR_2 \rceil - \lceil iIR_2 \rceil = \lceil 3\alpha_t IR_2 \rceil - 1 = \lfloor 3\alpha_t IR_2 \rfloor$, the thesis follows. ∎

Lemma 6.2. *Consider a circulant graph $C_n(a_1, \ldots, a_k) \in \mathcal{T}''$ originating step S_2, and consider the corresponding smooth staircase Σ_2^s. Let $q_{\max}(i) = \max_{h=1,\ldots,k}\{-1 + \beta_h + \lceil (i - \alpha_h)IR_2 \rceil\}$ and $q_{\min}(i) = \min_{h=1,\ldots,k}\{C - \beta_h + \lceil (i + \alpha_h)IR_2 \rceil\}$. Then,*

$$\mathcal{L}(1) \subseteq \{m_{i,\lceil iIR_2 \rceil}^*, \ldots, m_{i,q_{\max}(i)}^*, \text{ for } i \in \mathbb{Z}\}$$
$$\mathcal{R}(2) \subseteq \{m_{i,q_{\min}(i)}^*, \ldots, m_{i,C-1+\lceil iIR_2 \rceil}^*, \text{ for } i \in \mathbb{Z}\}$$

if and only if $0 \leq \beta_t \leq \left\lfloor \frac{C + \lfloor 3\alpha_t IR_2 \rfloor}{3} \right\rfloor$ for $t = 1, \ldots, k$.

Proof. In order to prove the first claim, we will show that $m_{i,q_{max}(i)+1}^* \notin \mathcal{L}(1)$. Assume by contradiction that element $m_{i,q_{max}(i)+1}^*$ belongs to $\mathcal{L}(1)$. By construction, $m_{i,q_{max}(i)+1}^* \in \mathcal{L}(1) \implies \exists a_t \in \{a_1, \ldots, a_k\} : m_{i-\alpha_t,q_{max}(i)+1-\beta_t}^* \in \mathcal{E}(1)$, hence we have to find a $t \in \{1, \ldots, k\}$ such that $(i - \alpha_t, q_{max}(i) + 1 - \beta_t) \leq (i - \alpha_t, \lceil (i - \alpha_t)IR_2 \rceil - 1)$, i.e. that $q_{max}(i) + 1 - \beta_t \leq \lceil (i - \alpha_t)IR_2 \rceil - 1$, which yields $q_{max}(i) \leq -2 + \beta_t \lceil (i - \alpha_t)IR_2 \rceil$. This contradicts our hypothesis and proves the claim. By similar arguments the second claim can be proved. ∎

Theorem 6.3. *Consider a circulant graph $C_n(a_1, \ldots, a_k) \in \mathcal{T}''$ originating step S_2, and consider the corresponding smooth staircase Σ_2^s. Then, Σ_2^s is a τ-staircase for it if and only if $0 \leq \beta_t \leq \left\lfloor \frac{C + \lfloor 3\alpha_t IR_2 \rfloor}{3} \right\rfloor$ for $t = 1, \ldots, k$.*

Proof. We prove that no edge exists connecting a vertex in $\mathcal{L}(1)$ to a vertex in $\mathcal{R}(2)$. Consider two consecutive copies of Σ_2^s, for example $\Sigma_2^s(1)$ and $\Sigma_2^s(2)$ described above Lemma 6.1. Assume by contradiction that there exists an a_j-edge connecting element $m_{p,q}^* \in \mathcal{L}(1)$ to element $m_{p+\alpha_j,q+\beta_j}^* \in \mathcal{R}(2)$. By Lemma 6.2,

$$m_{p,q}^* \in \mathcal{L}(1) \implies q \leq q_{max}(p) = \max_{h=1,\ldots,k}\{-1 + \beta_h + \lceil (p - \alpha_h)IR_2 \rceil\}$$

24.

$$m^*_{p+\alpha_j,q+\beta_j} \in \mathcal{R}(2) \quad \Longrightarrow \quad q + \beta_j \geq q_{min}(p+\alpha_j) = \min_{h=1,\ldots,k}\{C - \beta_h + \lceil(p+\alpha_j+\alpha_h)IR_2\rceil\}.$$

The worst case happens when $q = q_{max}(p)$. We will prove the claim finding a contradiction when proving that

$$q_{\max}(p) + \beta_j \geq q_{\min}(p+\alpha_j) \qquad \text{for all } j = 1,\ldots,k.$$

For all $j = 1,\ldots,k$ and for an arbitrary p, one has

$$\max_{h=1,\ldots,k}\{-1 + \beta_h + \lceil(p-\alpha_h)IR_2\rceil\} + \beta_j \geq \min_{h=1,\ldots,k}\{C - \beta_h + \lceil(p+\alpha_j+\alpha_h)IR_2\rceil\}$$

$$\max_{h=1,\ldots,k}\{\beta_h + \lceil(p-\alpha_h)IR_2\rceil + \beta_j\} - \min_{h=1,\ldots,k}\{-\beta_h + \lceil(p+\alpha_j+\alpha_h)IR_2\rceil\} \geq C + 1$$

$$\max_{h=1,\ldots,k}\{\beta_h + \lceil(p-\alpha_h)IR_2\rceil + \beta_j\} + \max_{h=1,\ldots,k}\{\beta_h - \lceil(p+\alpha_j+\alpha_h)IR_2\rceil\} \geq C + 1$$

$$\max_{h=1,\ldots,k}\{2\beta_h + \beta_j + \lceil(p-\alpha_h)IR_2\rceil - \lceil(p+\alpha_j+\alpha_h)IR_2\rceil\} \geq C + 1.$$

The following relation holds

$$-\lceil(\alpha_j + 2\alpha_h)IR_2\rceil \leq \lceil(p-\alpha_h)IR_2\rceil - \lceil(p+\alpha_j+\alpha_h)IR_2\rceil \leq -\lceil(\alpha_j + 2\alpha_h)IR_2\rceil + 1.$$

Depending on p, the difference $\lceil(p-\alpha_h)IR_2\rceil - \lceil(p+\alpha_j+\alpha_h)IR_2\rceil$ assumes either the smallest or the largest of the two values. The worst case happens when

$$\lceil(p-\alpha_h)IR_2\rceil - \lceil(p+\alpha_j+\alpha_h)IR_2\rceil = -\lceil(\alpha_j + 2\alpha_h)IR_2\rceil,$$

and allows us for writing

$$\max_{h=1,\ldots,k}\{2\beta_h + \beta_j - \lceil(2\alpha_h + \alpha_j)IR_2\rceil\} \geq C + 1 \qquad \text{for } j = 1,\ldots,k.$$

On the other hand, we were assuming that $\beta_t \leq \left\lfloor \frac{C + \lfloor 3\alpha_t IR_2 \rfloor}{3} \right\rfloor$, for $t = 1,\ldots,k$, that is to say $3\beta_t - \lfloor 3\alpha_t IR_2 \rfloor \leq C$, for $t = 1,\ldots,k$, which can also be written as

$$\max_{t=1,\ldots,k}\{3\beta_t - \lfloor 3\alpha_t IR_2 \rfloor\} \leq C.$$

Since for all $j = 1,\ldots,k$ one has

$$\max_{h=1,\ldots,k}\{2\beta_h + \beta_j - \lceil(2\alpha_h + \alpha_j)IR_2\rceil\} \leq \max_{h=1,\ldots,k}\{3\beta_h - \lceil 3\alpha_h IR_2\rceil\} \leq \max_{t=1,\ldots,k}\{3\beta_t - \lfloor 3\alpha_t IR_2 \rfloor\} \leq C$$

we get a contradiction. Having proved that no edge exists connecting a vertex in $\mathcal{L}(1)$ to a vertex in $R(2)$, $\mathcal{L}'(1) = \emptyset$, and, according to definition 3.2, Σ_2^s is a τ-staircase, as claimed. ∎

Corollary 6.4. *Consider a circulant graph $C_n(a_1,\ldots,a_k) \in \mathcal{T}''$ originating step S_2. If $0 \leq \beta_t \leq \left\lfloor \frac{C + \lfloor 3\alpha_t IR_2 \rfloor}{3} \right\rfloor$ for $t = 1,\ldots,k$ then $C_n(a_1,\ldots,a_k) \in \mathcal{T}$.*

Putting together Corollaries 3.5 and 6.4, we get the following result, where $M_n^s(a_1, a_2)$ denotes the shape originating a smooth staircase.

Corollary 6.5. *Consider a circulant graph $C_n(a_1, \ldots, a_k) \in \mathcal{T}''$ originating step S_2. If $0 \leq \beta_t \leq \left\lfloor \frac{C + \lfloor 3\alpha_t IR_2 \rfloor}{3} \right\rfloor$ for $t = 1, \ldots, k$ then $C_n(a_1, \ldots, a_k)$ is 3-chromatic and the algorithm $B\&W + ZM(M_n^s(a_1, a_2))$ outputs a feasible 3-colouring.*

Similar results hold for the smooth staircases Σ_h^s with $h \in \{1, \ldots 9\}$, possibly exchanging the role of rows and columns.

The following is the main result of the paper (recall that the entries of a circulant graph $C_n(a_1, \ldots, a_k)$ admitting two distinct entries a_i and a_j verifying $\gcd(n, a_i, a_j) = 1$ can be rearranged in such a way that a_i and a_j are the first two in the new ordering).

Theorem 6.6. *Consider a circulant graph $C_n(a_1, \ldots, a_k)$ such that $\gcd(n, a_1, a_2) = 1$, and let (α_t, β_t) denote the coordinates of a_t w.r.t. a_1, a_2. If $\alpha_t + \beta_t$ is odd for $t = 1, \ldots, k$, and one of the 9 different sets of conditions in Fig.14 is verified for every $t = 1, \ldots, k$, then $C_n(a_1, \ldots, a_k)$ belongs to $\mathcal{T}$, it is 3-chromatic, and the algorithm $B\&W + ZM(M_n^s(a_1, a_2))$ outputs a feasible 3-colouring.*

In Fig. 15 a feasible 3 colouring for a graph with $k = 28$ distinct entries is drawn.

(STEP 1)	$\gcd(n, a_1) \geq 2$	$(R + \lambda)$ odd	C even	$0 \leq \alpha_t \leq \left\lfloor \frac{R}{3} \right\rfloor$
(STEP 2)	$\gcd(n, a_1) \geq 2$	$(R + \lambda)$ even	C odd	$0 \leq \beta_t \leq \left\lfloor \frac{1}{3}(C + \lfloor 3\alpha_t \frac{-\lambda}{R} \rfloor) \right\rfloor$
(STEP 3)	$\gcd(n, a_1) \geq 2$	$(R + \lambda)$ odd	C odd	$0 \leq \beta_t \leq \left\lfloor \frac{1}{3}(C + \lfloor 3\alpha_t \frac{C-\lambda}{R} \rfloor) \right\rfloor$
(STEP 4)	$\gcd(n, a_2) \geq 2$	$(C' + \lambda')$ odd	R' even	$0 \leq \beta_t \leq \left\lfloor \frac{C'}{3} \right\rfloor$
(STEP 5)	$\gcd(n, a_2) \geq 2$	$(C' + \lambda')$ even	R' odd	$0 \leq \beta_t \leq \left\lfloor \frac{1}{3}(C' + \lfloor 3\alpha_t \frac{-C'}{\lambda'} \rfloor) \right\rfloor$
(STEP 6)	$\gcd(n, a_2) \geq 2$	$(C' + \lambda')$ odd	R' odd	$0 \leq \beta_t \leq \left\lfloor \frac{1}{3}(C' + \lfloor 3\alpha_t \frac{C'}{R'-\lambda'} \rfloor) \right\rfloor$
(STEP 7)	$\gcd(n, a_1) = 1$	$(R'' + \pi)$ even	C'' odd	$0 \leq \alpha_t \leq \left\lfloor \frac{R''-1}{3} \right\rfloor$
(STEP 8)	$\gcd(n, a_1) = 1$	$(R'' + \pi)$ odd	C'' odd	$0 \leq \beta_t \leq \left\lfloor \frac{1}{3}(C'' + \lfloor 3\alpha_t \frac{\pi}{R''-1} \rfloor) \right\rfloor$
(STEP 9)	$\gcd(n, a_1) = 1$	$(R'' + \pi)$ even	C'' even	$0 \leq \beta_t \leq \left\lfloor \frac{1}{3}(C'' + \lfloor 3\alpha_t \frac{C''-\pi}{R''} \rfloor) \right\rfloor$

Figure 14: Conditions of Theorem 6.6

7. Conclusions

In this paper we characterize a subclass of 3-chromatic circulant graphs $C_n(a_1, \ldots, a_k)$ which can be coloured with the proposed exact algorithm $B\&W + ZM$.

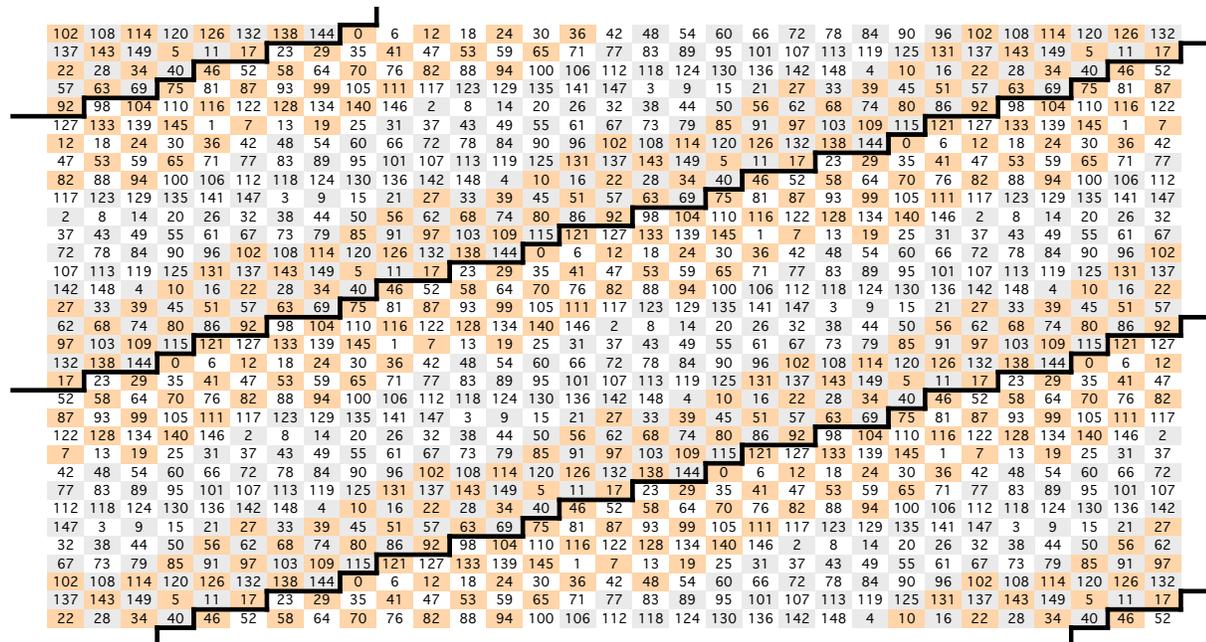| 102	108	114	120	126	132	138	144	0	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120	126	132
137	143	149	5	11	17	23	29	35	41	47	53	59	65	71	77	83	89	95	101	107	113	119	125	131	137	143	149	5	11	17
22	28	34	40	46	52	58	64	70	76	82	88	94	100	106	112	118	124	130	136	142	148	4	10	16	22	28	34	40	46	52
57	63	69	75	81	87	93	99	105	111	117	123	129	135	141	147	3	9	15	21	27	33	39	45	51	57	63	69	75	81	87
92	98	104	110	116	122	128	134	140	146	2	8	14	20	26	32	38	44	50	56	62	68	74	80	86	92	98	104	110	116	122
127	133	139	145	1	7	13	19	25	31	37	43	49	55	61	67	73	79	85	91	97	103	109	115	121	127	133	139	145	1	7
12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120	126	132	138	144	0	6	12	18	24	30	36	42
47	53	59	65	71	77	83	89	95	101	107	113	119	125	131	137	143	149	5	11	17	23	29	35	41	47	53	59	65	71	77
82	88	94	100	106	112	118	124	130	136	142	148	4	10	16	22	28	34	40	46	52	58	64	70	76	82	88	94	100	106	112
117	123	129	135	141	147	3	9	15	21	27	33	39	45	51	57	63	69	75	81	87	93	99	105	111	117	123	129	135	141	147
2	8	14	20	26	32	38	44	50	56	62	68	74	80	86	92	98	104	110	116	122	128	134	140	146	2	8	14	20	26	32
37	43	49	55	61	67	73	79	85	91	97	103	109	115	121	127	133	139	145	1	7	13	19	25	31	37	43	49	55	61	67
72	78	84	90	96	102	108	114	120	126	132	138	144	0	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102
107	113	119	125	131	137	143	149	5	11	17	23	29	35	41	47	53	59	65	71	77	83	89	95	101	107	113	119	125	131	137
142	148	4	10	16	22	28	34	40	46	52	58	64	70	76	82	88	94	100	106	112	118	124	130	136	142	148	4	10	16	22
27	33	39	45	51	57	63	69	75	81	87	93	99	105	111	117	123	129	135	141	147	3	9	15	21	27	33	39	45	51	57
62	68	74	80	86	92	98	104	110	116	122	128	134	140	146	2	8	14	20	26	32	38	44	50	56	62	68	74	80	86	92
97	103	109	115	121	127	133	139	145	1	7	13	19	25	31	37	43	49	55	61	67	73	79	85	91	97	103	109	115	121	127
132	138	144	0	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120	126	132	138	144	0	6	12
17	23	29	35	41	47	53	59	65	71	77	83	89	95	101	107	113	119	125	131	137	143	149	5	11	17	23	29	35	41	47
52	58	64	70	76	82	88	94	100	106	112	118	124	130	136	142	148	4	10	16	22	28	34	40	46	52	58	64	70	76	82
87	93	99	105	111	117	123	129	135	141	147	3	9	15	21	27	33	39	45	51	57	63	69	75	81	87	93	99	105	111	117
122	128	134	140	146	2	8	14	20	26	32	38	44	50	56	62	68	74	80	86	92	98	104	110	116	122	128	134	140	146	2
7	13	19	25	31	37	43	49	55	61	67	73	79	85	91	97	103	109	115	121	127	133	139	145	1	7	13	19	25	31	37
42	48	54	60	66	72	78	84	90	96	102	108	114	120	126	132	138	144	0	6	12	18	24	30	36	42	48	54	60	66	72
77	83	89	95	101	107	113	119	125	131	137	143	149	5	11	17	23	29	35	41	47	53	59	65	71	77	83	89	95	101	107
112	118	124	130	136	142	148	4	10	16	22	28	34	40	46	52	58	64	70	76	82	88	94	100	106	112	118	124	130	136	142
147	3	9	15	21	27	33	39	45	51	57	63	69	75	81	87	93	99	105	111	117	123	129	135	141	147	3	9	15	21	27
32	38	44	50	56	62	68	74	80	86	92	98	104	110	116	122	128	134	140	146	2	8	14	20	26	32	38	44	50	56	62
67	73	79	85	91	97	103	109	115	121	127	133	139	145	1	7	13	19	25	31	37	43	49	55	61	67	73	79	85	91	97
102	108	114	120	126	132	138	144	0	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120	126	132
137	143	149	5	11	17	23	29	35	41	47	53	59	65	71	77	83	89	95	101	107	113	119	125	131	137	143	149	5	11	17
22	28	34	40	46	52	58	64	70	76	82	88	94	100	106	112	118	124	130	136	142	148	4	10	16	22	28	34	40	46	52

Figure 15: A feasible 3-colouring for the circulant graph $C_{150}(6, 35, 18, 30, 42, 23, 47, 59, 71, 52, 64, 76, 88, 100, 81, 93, 105, 117, 98, 110, 122, 134, 146, 127, 139, 1, 13, 25)$ with $k = 28$ entries.

Several open questions remain. Among them:

- Is it true that if none of the monotone staircases discussed in the paper is a τ-staircase for a circulant graph $C_n(a, b, a_3, \ldots, a_k)$, then the graph does not admit a τ-staircase?
- Consider a circulant graph $C_n(a_1, \ldots, a_k)$ with two distinct pairs a_i, a_j and a_h, a_k verifying $\gcd(n, a_i, a_j) = 1$ and $\gcd(n, a_h, a_k) = 1$, and call $\mathcal{T}''(a_i, a_j)$ the subclass $\mathcal{T}''$ obtained when $(a, b) = (a_i, a_j)$, and $\mathcal{T}''(a_h, a_k)$ the subclass $\mathcal{T}''$ obtained when $(a, b) = (a_h, a_k)$. Unfortunately, it is not true that $C_n(a_1, \ldots, a_k) \in \mathcal{T}_n(a_i, a_j)$ if and only if $C_n(a_1, \ldots, a_k) \in \mathcal{T}_n(a_h, a_k)$. For example, consider $C_{30}(1, 6, 10)$. If we choose $(a, b) = (1, 6)$ then $a_3 = 10$ and $\alpha_3 + \beta_3$ is even; if we choose $(a, b) = (1, 10)$ then $a_3 = 6$ and $\alpha_3 + \beta_3$ is odd. So the question is: if there is more than one pair a_i, a_j verifying $\gcd(a_i, a_j) = 1$, how to efficiently choose one giving an odd sum of the coordinates for each entry and originating a τ-staircase so that the algorithm outputs a feasible 3-colouring for it?

Finally, we observe that the results proved in the paper apply also to some non-connected circulant graph $C_{n'}(a'_1, \ldots, a'_k)$: let $\gamma = \gcd(n', a'_1, \ldots, a'_k)$, it is easy to see that the graph has γ components which are all isomorphic to $C_n(a_1, \ldots, a_k)$, where $n = \frac{n'}{\gamma}$, and $a_i = \frac{a'_i}{\gamma}$ for $i = 1, \ldots, k$. If $C_n(a_1, \ldots, a_k)$ verifies the conditions discussed in the paper, then we have obtained an exact colouring for $C_{n'}(a'_1, \ldots, a'_k)$.

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