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REPETITIVITY OF SEMIGROUPS AND A RESULT OF CASSAIGNE, CURRIES, SCHAEFFER AND SHALLIT

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1. Introduction

The recent remarkable result of Cassaigne et al. [2] about existence of an infinite word without three consecutive factors 'of the same sum and size' let appear the importance of the theory of 'repetitive semigroups' in that problematics.

This seems an occasion to recall the main definitions and results in the theory of repetitivity, which had been shortly presented in the first Lothaire book [8]. We recall the progresses since this epoch and in the last section formulate the results based on [2].

2. Repetitity of semigroups

Here, $\mathbb{Z}$, (resp. $\mathbb{P}$, resp. $\mathbb{N}$) will denote the sets of the integers, (resp. positive integers, resp. non-negative integers), viewed as additive semigroups.

Terminology about words is the common one (see, for example, the book of Lothaire in reference [8]). The set of finite non-empty words $a_1 a_2 \cdots a_n$, $a_i \in A$, $n \geq 1$, on the alphabet A is denoted A^+; the set of infinite words $a_1 a_2 \cdots$, $a_i \in A$, is denoted A^ω. A finite or infinite word is represented by the sequence of its letters, e.g., $w = w(1)w(2)\cdots$, where $w(i)$ is the letter occurring in position i.

Let us recall the following definitions.

Definition 2.1. *Let $\phi : A^+ \to S$ be a morphism , A a finite alphabet, S a semigroup. A k-power* $\bmod \phi$ *is a word $f = f_1 f_2 \cdots f_k \in A^+$, $f_i \in A^+$, such that $\phi(f_1) = \phi(f_2) = \cdots = \phi(f_k)$. We say that*

i) $s \in A^\omega$ is repetitive (resp. k-repetitive $\bmod \phi$*) if for any $k \in \mathbb{P}$ (resp. for the considered k), s contains a factor which is a k-power* $\bmod \phi$*;*

ii) ϕ is repetitive (resp. k-repetitive) if any $s \in A^\omega$ is repetitive (resp. k-repetitive $\bmod \phi$*).*

Definition 2.2. *Semigroup S is repetitive (resp. k-repetitive) if for any finite alphabet A and morphism $\phi : A^+ \to S$, ϕ is repetitive (resp. k-repetitive).*

Note 1. *If in addition we impose that the f_i's in Definition 2.1 have the same length, i.e., $|f_1| = |f_2| \cdots = |f_k|$ we obtain the useful notions of uniform k-power* $\bmod \phi$ *and uniform repetitivity. Indeed, this is a shortcut to saying that $f_1 \cdots f_k$ is a k-power modulo the morphism $\psi : A^+ \to S \times \mathbb{P}$ given by $\psi(u) = (\phi(u), |u|)$.*

Remark 2.3. Definitions above were given using infinite words to simplify formulation, but clearly all could be done with finite words only.

2.1. Reminder of main results in the 1980's

We recall the short survey given in [8] and its references.

- Finite semigroups are repetitive.

- $\mathbb{Z}$ is repetitive.

-A finitely generated (in short *f.g.*) commutative semigroup is repetitive if and only if it does not contain a subsemigroup isomorphic to $\mathbb{P} \times \mathbb{P}$.

-Let S, T be two semigroups and $\phi : S \to T$ be a morphism with bounded classes, i.e., there exists m such that $|\phi^{-1}(t| \leq m$ for all $t \in \phi(S)$, then if T is repetitive S is also.

This powerful theorem allows to prove repetitivity in several cases (for instance the bicyclic semigroup and even the plactic monoid on two letters).

Also, using deep results of Milnor, Tits, Gromov, we have

-A f.g. group with polynomial growth is repetitive if and only if it is cyclic-by-finite.

Remark 2.4. All 'negative' results above, i.e., of the form 'something is not repetitive' come from the existence of infinite words without some kind of factors. These infinite words have been constructed using iterations of suitable morphisms. More precisely,

-A^+ is not repetitive [11]

-$\mathbb{P}^4$ is not 2-repetitive [7]

-$\mathbb{P}^2$ is not 4-repetitive [3] and last but not least

-$\mathbb{P}$ is not uniformly 3-repetitive [2] (which implies that $\mathbb{P}^2$ is not 3-repetitive).

On the opposite all 'positive' results come from the van der Waerden theorem on arithmetic progressions, which amounts to '$\mathbb{P}$ is repetitive'.

Remark 2.5. Among several variations on repetitivity, let us mention [4] ramseyanity, ω-repetitivity and, probably the most interesting, 'strong repetitivity' which is in relation with the so-called lemma of Brown [1].

Definition 2.6. *A semigroup S is strongly repetitive if for any finite alphabet A, for any morphism $\phi : A^+ \to S$ and for any infinite word $s \in A^\omega$ there exists a positive integer p such that s contains for any k a k-power $\mod \phi$, $f_1 f_2 \cdots f_k$, with $|f_i| \leq p$.*

This property is a finiteness condition for any f.g. semigroup [4]. We shall see now another remarkable other finiteness condition for f.g. semigroups.

3. Progresses since the 1980's

Indeed the Dekking result that $\mathbb{P}^2$ is not 4-repetitive allowed to prove more.

Theorem 3.1 ([6]) *Let S be a f.g. semigroup. Then S is uniformly repetitive if and only if it is finite.*

Many years have now passed since we began our work on this subject. To honor the memory of our departed colleague Stefano, from this point on we would like to refer to the technique we pioneered with him as "Varricchio's argument".

This argument have then been exploited for obtaining some other results ([10] in particular) and this culminated in the theorem

Theorem 3.2 ([9]) *If $S \times T$ is repetitive with S, T f.g. semigroups then at least one of S, T is finite. Even, condition '$S \times T$ is 4-repetitive' is sufficient.*

Some interesting applications to repetitivity of groups appeared in [5]. One concerns f.g. groups of 'intermediate growth' (i.e., neither polynomial nor exponential).

Theorem 3.3. *The 'Grigorchuk groups' G_ω with intermediate growth are not repetitive.*

As is seems plausible that groups with exponential growth are not repetitive and in view of the result given in Section 2.1 we are led to the conjecture

Conjecture 3.4. *A f.g. infinite group is repetitive if and only it is cyclic-by-finite.*

The other result answers a natural question inspired by a similar famous one about infinite groups with all their proper subgroups finite.

Theorem 3.5. *There exist infinite f.g. non-repetitive groups whose all proper subgroups are repetitive. In particular if G is an infinite f.g. group whose all proper subgroups are finite then G or $G \times G$ is such a group.*

4. An application of the Cassaigne et al. result

In [2] the authors construct an infinite word $t = t(1)t(2)\cdots$ on the alphabet $B = \{a, b, c, d, \}$ which contains no uniform 3-power modulo the morphism $\psi : B^+ \to \mathbb{P}$ given by $a \mapsto 1$, $b \mapsto 2$, $c \mapsto 4$, $d \mapsto 5$ (indeed they used $a, b, c, d \mapsto 0, 1, 3, 4$, which trivially amounts to the same).

This allows to get the following improvement of Theorem 3.1

Theorem 4.1. *If a f.g. semigroup S is uniformly 3-repetitive, then it is finite.*

Proof. As noted in [2] this follows directly by applying Proposition 1 of [10], but we choose to repeat the argument here. Let A be a finite set of generators of S and $\phi : A^+ \to S$ be the canonical morphism defined by $\phi(x) = x$ for all $x \in A$. By hypothesis, S is uniformly 3-repetitive.

Suppose by contradiction S is infinite. Then the set L of irreducible words on A is infinite (a word $w \in A^+$ is irreducible if $\phi(w) \neq \phi(w')$ for any $w' \in A^+$ such that $|w'| < |w|$). Thus by the König lemma, as L is closed by factors, there exists an infinite word, $m = m(1)m(2)\cdots \in A^\omega$, whose all factors are irreducible.

Consider the infinite word t introduced above and factorize m in the form $m = x_1x_2\cdots$, with $x_i \in A^+$ and $|x_i| = \psi(t(i))$ for all i (this is the Varricchio argument). Let C be an alphabet in bijection with the subset $X = A \cup A^2 \cup A^3 \cup A^4 \cup A^5$ of A^+ and $\beta : C \to X$ be this bijection. Let $\xi : C^+ \to A^+$ be the morphism generated by the bijection β. Consider the morphism $\theta : C^+ \to S$ defined by $\theta(w) = \phi(\xi(w))$ for all $w \in C^+$.

Consider the infinite word $r = r(1)r(2)\cdots$ where $r(i) = \beta^{-1}(x_i)$ for all $i \in \mathbb{P}$. Since θ is uniformly 3-repetitive, there exists integers, $n \in \mathbb{N}$, $p \in \mathbb{P}$ such that $\theta(r(n+1)r(n+2)\cdots r(n+p)) = \theta(r(n+p+1)r(n+p+2)\cdots r(n+2p)) = \theta(r(n+2p+1)r(n+2p+2)\cdots r(n+3p))$. By the definition of θ we have that $\theta(r(j)) = \phi(x_j)$ for all $j \in \mathbb{P}$. This implies that $\phi(x_{n+1}x_{n+2}\cdots x_{n+p}) = \phi(x_{n+p+1}x_{n+p+2}\cdots x_{n+2p}) = \phi(x_{n+2p+1}x_{n+2p+2}\cdots x_{n+3p})$. As m and its factors are irreducible, and as $|x_j| = \psi(t_j)$ for all $j \in \mathbb{P}$ it follows that $\psi(t(n+1)\cdots t(n+p)) = \psi(t(n+p+1)\cdots t(n+2p)) = \psi(t(n+2p+1)\cdots t(n+3p))$, thus $t(n+1)\cdots t(n+3p)$ is a uniform 3-power modulo ψ of t, a contradiction.
∎

In the same way (see [9]) we have the improvement of Theorem 3.2

Theorem 4.2. *If $S \times T$ is 3-repetitive with S, T f.g. semigroups then at least one of S, T is finite.*

In conclusion, one (difficult) problem that remains, already mentioned in [10] is 'can we replace 3-repetitive in Theorem 4.2 (resp. uniformly 3-repetitive in Theorem 4.1) by 2-repetitive (resp. uniformly 2-repetitive)?

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