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**A STUDY ON SUBOPTIMAL FILTERING FOR
NONLINEAR SYSTEMS: THE OBSERVER
FOLLOWER FILTER**

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Abstract

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Key words: Nonlinear systems; Observers; Kalman filtering; Polynomial filtering.

A study on suboptimal filtering for nonlinear systems: the Observer Follower Filter

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Abstract

This paper investigates the state estimation problem for a class of stochastic nonlinear differential systems. A novel algorithm is proposed, denoted as *Observer Follower Filter (OFF)*, based on a mixed approach that can be summarized in two steps: the first step makes use of a high-gain observer-based estimator for nonlinear systems, applied to the system equations in order to provide the trajectory around which a ν -degree Carleman approximation of the stochastic differential system is achieved. This second step defines an approximation of the stochastic nonlinear system as a bilinear system (i.e. linear drift and multiplicative noise), for which a suboptimal filtering algorithm is available from the literature. In principle, the first step can be performed by using other high-gain estimators, but in this note we prove that the proposed estimator provides an estimate with a bounded mean square error. Numerical simulations show the effectiveness of the proposed methodology, and the improvements of the OFF algorithm with respect to the standard Extended Kalman-Bucy filter obtained by increasing the order of the Carleman approximation.

Key words: Nonlinear systems; Observers; Kalman filtering; Polynomial filtering.

1 Introduction

This work considers the filtering problem for nonlinear stochastic differential systems described by the Itô equations:

$$\begin{aligned} dx_t &= \phi(x_t)dt + g(x_t)(u_t dt + FdW_t^1), & x_0 &= \bar{x}, \\ dy_t &= h(x_t)dt + GdW_t^2, & y_0 &= 0, \text{ a.s.}, \end{aligned} \quad (1)$$

defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $x_t \in \mathbb{R}^n$ is the state vector, $y_t \in \mathbb{R}^q$ is the measured output, $u_t \in \mathbb{R}^s$ is a deterministic input, $W_t^1 \in \mathbb{R}^p$ and $W_t^2 \in \mathbb{R}^q$ are independent standard Wiener processes with respect to a family of increasing σ -algebras $\{\mathcal{F}_t, t \geq 0\}$ (i.e., the components of vectors W_t^1 and W_t^2 are a set of independent standard Wiener processes). $\phi : \mathbb{R}^n \mapsto \mathbb{R}^n$,

$g : \mathbb{R}^n \mapsto \mathbb{R}^{n \times s}$ and $h : \mathbb{R}^n \mapsto \mathbb{R}^q$ are analytical nonlinear maps, smooth enough to ensure the existence and uniqueness of the stochastic solution of (1). It is moreover required the finiteness of the moments of the state variable x_t . The initial state \bar{x} is an \mathcal{F}_0 -measurable random vector, independent of both W_t^1 and W_t^2 . In order to avoid singular filtering problems, see [5], the standard assumption of nonsingular output-noise covariance is made here, i.e. $\text{rank}(GG^T) = q$.

It is well known that the minimum variance state estimate requires the knowledge of the conditional probability density, whose computation, in the general nonlinear case, is a difficult infinite-dimensional problem (see, e.g., [6], [22], [23], [24], [28]). Only in few cases the optimal filter has a finite dimension, such as in [27] or in the recent papers on polynomial systems (see [3] and references therein). For this reason a great deal of work has been devoted to devise suboptimal implementable filtering algorithms (see, e.g., [14], [15], [20]). In a recent paper, [18], a generalization of the classical Extended Kalman-Bucy Filter (EKF) has been proposed, based

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on the ν -degree Carleman approximation (see [21], [26]) of the nonlinear stochastic differential system, instead of the usual first-order approximation exploited in the EKBF. The Carleman approximation provides an approximate representation of the system by means of a bilinear system (i.e. linear drift and multiplicative noise) with respect to a suitably defined extended vector, made of the Kronecker powers, up to a given degree ν , of the original state. The Carleman approximation has been successfully applied also to filter nonlinear discrete-time systems by the same authors [17].

A recent research area is to investigate the application of high-gain observers as filters of stochastic systems [1,2,4]. The aim of these works is to use the global convergence property of high-gain observers to overcome the limitations due to the local nature of the EKBF. Conditions of convergence of the high-gain observer have been studied in [1,2]. The common feature of these works is that the correction gain is computed offline. Since there is a trade-off between a large gain, required to ensure convergence, and the amplification of the measurement noise due to the observer gain, an adaptive gain observer has been proposed in [4].

In the present work we extend this approach by using a two-step method called *Observer Follower Filter (OFF)*. The first step makes use of a high-gain stochastic state estimator, inspired to state observers for nonlinear deterministic systems. The second step uses a suboptimal polynomial filter based on the Carleman approximation around the approximate state trajectory provided by the high-gain estimator. The idea behind of the OFF algorithm is to use the high-gain estimate to ensure global convergence, whereas the filter has the task of providing more accurate estimates. Within this research line we may cite [13], concerning a standard EKBF applied to a first-order approximation around the state estimate provided by a state observer.

An important result of the paper is that the high-gain state estimator has a bounded error in the mean square sense: this is a crucial result for the effective application of the proposed methodology, since the high-gain estimator provides the trajectory around which the filter is used. Another contribution of our work is to show that, using the extended state needed by the Carleman approximation, the dynamics of the estimate error of the high-gain observer can be represented as a bilinear stochastic system, for which it is possible to use existing suboptimal filters.

It is worth highlighting that the proposed methodology can be straightforwardly extended to the more general case of different input maps $g(x)$ for the deterministic and stochastic state disturbances, corresponding to the state equation:

$$dx_t = \phi(x_t)dt + g_1(x_t)u_tdt + g_2(x_t)Fdw_t^1. \quad (2)$$

However, for the sake of simplicity, in this paper we consider the more restrictive system (1), which has the following interpretation: equations (1) model the case of a deterministic plant where both inputs and outputs are affected by a measurement noise: therefore, in this framework u_t is a *measured input* and it is affected by the Wiener process W_t^1 (see Fig. 1). Preliminary versions of the present paper have been presented in [7,8].

The paper is organized as follows. Section 2 provides an overview of the method, endowed with the adopted notations, whereas the high-gain estimator is presented in Section 3. Section 4 describes the filtering part and presents some features of the approach. Numerical simulations are reported in Section 5 to show the effectiveness of the proposed methodology and the improvements of the OFF with respect to the standard Kalman-Bucy Filter applied to the linear approximation of the stochastic nonlinear system.

2 Overview and basic notations

A sketch of the method is illustrated in Fig. 1. A high-gain estimator algorithm is used at first as a filter for the stochastic system, by suitably exploiting the noisy measurements. Besides a necessary observability condition concerning the inversion of the observability map, a set of sufficient hypotheses guarantees the boundedness, in the mean square sense, of the estimate error. Among these, the *strong relative degree property* will deserve some interest, since it may be considered as a special version of the usual *relative degree property*.

From the high-gain estimate, denoted \hat{x}_t^o , we define $\psi_t = x_t - \hat{x}_t^o$, whose estimate is the task of the *displacement filter*. To this end, we derive the approximate stochastic equation of ψ_t by suitably applying the ν -degree Carleman approximation of the original nonlinear system around the high-gain estimate \hat{x}_t^o . The Carleman approximation provides a bilinear system (i.e. linear drift and multiplicative noise), with respect to an *extended state* $\Psi_t = [\psi_t, \psi_t^{[2]}, \dots, \psi_t^{[\nu]}]$, where $\psi_t^{[i]}$ denotes the i -th Kronecker power of ψ_t . Then, the optimal linear filter available from the literature is applied, providing the estimate $\hat{\psi}_t$. The final estimate of x_t is obtained as $\hat{x}_t^o + \hat{\psi}_t$, where $\hat{\psi}_t$ is the first component of $\hat{\Psi}_t$.

We introduce the basic notations adopted throughout the paper. Given a set of matrices $\{M_j \in \mathbb{R}^{n_j \times m_j}, j = 1, \dots, r\}$ we define $\text{diag}_{j=1}^r \{M_j\} \in \mathbb{R}^{\bar{n}_r \times \bar{m}_r}$, with $\bar{n}_r = \sum_{j=1}^r n_j$, $\bar{m}_r = \sum_{j=1}^r m_j$ as the block diagonal matrix composed by the blocks M_j .

Let $L_f^k \lambda(x)$ denote the Lie derivative of order $k \geq 0$ of the scalar function $\lambda(x) : \mathbb{R}^n \mapsto \mathbb{R}$ along the vector field

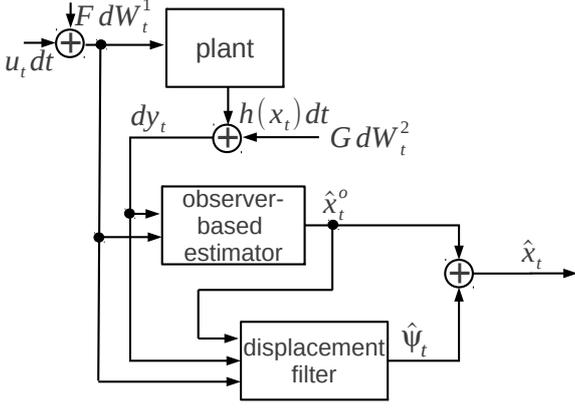


Fig. 1. Overview of the proposed approach.

$f(x) : \mathbb{R}^n \mapsto \mathbb{R}^n$, recursively defined as

$$L_f^0 \lambda(x) = \lambda(x), \quad L_f^k \lambda(x) = \frac{dL_f^{k-1} \lambda}{dx} f(x). \quad (3)$$

Define the multi-index $\bar{s} = \{s_1, \dots, s_q\}$, $s_j = 1, 2, \dots$ such that $\sum_{j=1}^q s_j = n$. Then, for each $j = 1, \dots, q$ and for each integer $s_j \in \bar{s}$, consider the scalar output functions $h_j(x) : \mathbb{R}^n \mapsto \mathbb{R}$ and define the vector function $\Theta_j^{s_j} : \mathbb{R}^n \mapsto \mathbb{R}^{s_j}$

$$\Theta_j^{s_j}(x) = [h_j(x) \quad L_\phi h_j(x) \quad \dots \quad L_\phi^{s_j-1} h_j(x)]^T, \quad (4)$$

according to which the vector map $\Theta_{\bar{s}} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is built:

$$\Theta_{\bar{s}}(x) = [\Theta_1^{s_1 T}(x) \quad \dots \quad \Theta_q^{s_q T}(x)]^T \quad (5)$$

Further notations involving the multi-index \bar{s} are $L_{\phi}^{\bar{s}} h : \mathbb{R}^n \mapsto \mathbb{R}^q$ such that

$$L_{\phi}^{\bar{s}} h(x) = [L_{\phi}^{s_1} h_1(x) \quad \dots \quad L_{\phi}^{s_q} h_q(x)]^T, \quad (6)$$

and $L_g L_{\phi}^{\bar{s}-1} h : \mathbb{R}^n \mapsto \mathbb{R}^{q \times s}$ such that

$$L_g L_{\phi}^{\bar{s}-1} h(x) = \begin{bmatrix} L_g L_{\phi}^{s_1-1} h_1(x) \\ \vdots \\ L_g L_{\phi}^{s_q-1} h_q(x) \end{bmatrix}, \quad (7)$$

where, given a scalar function $\lambda : \mathbb{R}^n \mapsto \mathbb{R}$, it is:

$$L_g \lambda(x) = \frac{d\lambda}{dx} [g_1 \quad \dots \quad g_s] = [L_{g_1} \lambda(x) \quad \dots \quad L_{g_s} \lambda(x)], \quad (8)$$

with g_i the i -th column of g .

3 The high-gain observer-based state estimator

Consider the state-transformation $z_t = \Theta_{\bar{s}}(x_t)$ defined by (4-5). Function $\Theta_{\bar{s}}$ is said to be an *observability map* in a set $\Omega \subseteq \mathbb{R}^n$ if it is a diffeomorphism in an open set that contains or coincides with Ω . In case system (1) admits an observability map $\Theta_{\bar{s}}$ in $\Omega \subseteq \mathbb{R}^n$ for a given \bar{s} it is said to be *drift-observable* in Ω , since the invertibility of $\Theta_{\bar{s}}(x)$ guarantees the state reconstruction from the vector of the output derivatives, in a deterministic framework (see [12] for more details).

Denote \hat{x}_t^o the high-gain state estimate and assume that $\Theta_{\bar{s}}(x)$ is an observability map. Then the following equations for the high-gain estimator are well defined for \hat{x}_t^o :

$$\begin{aligned} d\hat{x}_t^o &= \phi(\hat{x}_t^o) dt + g(\hat{x}_t^o) u_t dt + Q^{-1}(\hat{x}_t^o) \mathcal{K} (dy_t - h(\hat{x}_t^o) dt) \\ &\quad + \frac{1}{2} Q^{-1}(\hat{x}_t^o) \mathcal{B}_{\bar{s}} \mathcal{H}_{\phi h}(\hat{x}_t^o) dt, \end{aligned} \quad (9)$$

where $Q(x)$ is the Jacobian of the observability map (4-5), and $\mathcal{H}_{\phi h}(x)$ is defined as:

$$\mathcal{H}_{\phi h}(x) = \sum_{i=1}^p \text{diag}_{j=1}^q \{F_i^T g^T(x)\} H_{\phi h}(x) g(x) F_i \quad (10)$$

with F_i , $i = 1, \dots, p$ columns of F , and $H_{\phi h}(x)$ a matrix of the Hessians of $L_{\phi}^{s_j-1} h_j$, $j = 1, \dots, q$:

$$H_{\phi h}(x) = [H_{\phi h_1}^T(x) \quad \dots \quad H_{\phi h_q}^T(x)]^T, \quad (11)$$

$$[H_{\phi h_j}(x)]_{(i,k)} = \frac{\partial^2}{\partial x_i \partial x_k} L_{\phi}^{s_j-1} h_j(x), \quad i, k = 1, \dots, n. \quad (12)$$

Matrix $\mathcal{B}_{\bar{s}}$ belongs to a triple of Brunowski-block matrices $\{\mathcal{A}_{\bar{s}}, \mathcal{B}_{\bar{s}}, \mathcal{C}_{\bar{s}}\}$ defined as:

$$\begin{aligned} \mathcal{A}_{\bar{s}} &= \text{diag}_{j=1}^q \{A_b^{s_j}\}, \quad \mathcal{B}_{\bar{s}} = \text{diag}_{j=1}^q \{B_b^{s_j}\}, \\ \mathcal{C}_{\bar{s}} &= \text{diag}_{j=1}^q \{C_b^{s_j}\}, \end{aligned} \quad (13)$$

with $A_b^{s_j}$, $B_b^{s_j}$, $C_b^{s_j}$ Brunowski matrices of size s_j .

Finally, the gain matrix \mathcal{K} is a block-diagonal matrix

$$\mathcal{K} = \text{diag}_{j=1}^q \{K_j\} \quad (14)$$

where the gain matrices $K_j \in \mathbb{R}^{s_j \times 1}$, $j = 1, \dots, q$ assign the eigenvalues of $A_b^{s_j} - K_j C_b^{s_j}$. The choice of K_j is related to the stability in the square mean sense of the estimate error as discussed in Section 4.2.

The high-gain estimator equations (9)-(14) have been derived according to the following conceptual scheme:

- i) rewrite (1) in the transformed state coordinates z_t ;
- ii) define the estimate \hat{z}_t of z_t , and let \hat{x}_t^o be the inverse transformation $\hat{x}_t^o = \Theta_{\bar{s}}^{-1}(\hat{z}_t)$;
- iii) derive an algorithm to compute \hat{x}_t^o without the intermediate step of \hat{z}_t .

In order to ensure that the equations for z_t do not contain the derivatives of the input – both deterministic and stochastic – the usual full relative degree property is not sufficient. Instead, the following property is necessary.

Definition 1 *The j th output h_j of system (1) is said to have a **strong k_j relative degree property** if:*

$$\left(\frac{d}{dx}L_\phi^i h_j(x)\right)g(\xi) = 0, \quad \forall x, \xi, \in \mathbb{R}^n, \quad 0 \leq i \leq k_j - 2 \quad (15)$$

and

$$\left(\frac{d}{dx}L_\phi^{k_j-1} h_j(x)\right)g(\xi) \neq 0, \quad \forall x, \xi, \in \mathbb{R}^n \quad (16)$$

Remark 1 *Notice that Definition 1 defines a stronger property than the usual relative degree property, which requires conditions (15-16) hold true for $x = \xi$. Nevertheless, there are many significant cases for which k_j -relative degree is equivalent to strong k_j -relative degree, such as the case of a constant function $g(x) = \bar{g}$, $\forall x$.*

Definition 1 is used in the following Lemma, that solves step i).

Lemma 1 *Consider the vector function $z_t = \Theta_{\bar{s}}(x_t)$ defined in (4-5) and assume the following hypothesis on the relative degree property holds true:*

- H0)** *for each j – th output of system (1) either the strong r_j relative degree property holds with $r_j \geq s_j$, or (15) holds for any i .*

Then, the differential dz_t can be written as:

$$dz_t = (\mathcal{A}_{\bar{s}}z_t + \mathcal{B}_{\bar{s}}L_\phi^{\bar{s}}h(x_t))dt + \frac{1}{2}\mathcal{B}_{\bar{s}}\mathcal{H}_{\phi h}(x_t)dt + \mathcal{B}_{\bar{s}}L_gL_\phi^{\bar{s}-1}h(x_t)(u_t dt + FdW_t^1) \quad (17)$$

Proof. See Appendix.

Step ii) uses the following estimator of z_t :

$$d\hat{z}_t = \left(\mathcal{A}_{\bar{s}}\hat{z}_t + \mathcal{B}_{\bar{s}}L_\phi^{\bar{s}}h(\Theta_{\bar{s}}^{-1}(\hat{z}_t)) + \mathcal{B}_{\bar{s}}L_gL_\phi^{\bar{s}-1}h(\Theta_{\bar{s}}^{-1}(\hat{z}_t))u_t\right)dt + \frac{1}{2}\mathcal{B}_{\bar{s}}\mathcal{H}_{\phi h}(\Theta_{\bar{s}}^{-1}(\hat{z}_t))dt + \mathcal{K}(dy_t - \mathcal{C}_{\bar{s}}\hat{z}_t dt). \quad (18)$$

In step iii) the observer equations (9-14) are derived from (18), as detailed in Appendix (proof of Theorem 2).

Note that checking the observability condition requires to invert the observability map, which is equivalent to solve a set of nonlinear equations. This is not necessary because the observer equations (9) only require the matrix inversion of the Jacobian $Q(\hat{x}_t^o)$.

4 The OFF estimate

4.1 Filter equations

The displacement $\psi_t = x_t - \hat{x}_t^o$ has equation

$$d\psi_t = dx_t - d\hat{x}_t^o = \left(\phi(x_t) - \phi(\hat{x}_t^o) + (g(x_t) - g(\hat{x}_t^o))u_t - Q^{-1}(\hat{x}_t^o)\mathcal{K}(h(x_t) - h(\hat{x}_t^o))\right)dt + g(x_t)F dW_t^1 - Q^{-1}(\hat{x}_t^o)\mathcal{K}G dW_t^2 - \frac{1}{2}Q^{-1}(\hat{x}_t^o)\mathcal{B}_{\bar{s}}\mathcal{H}_{\phi gh}(\hat{x}_t^o)dt. \quad (19)$$

The proposed OFF algorithm makes use of the ν -degree Carleman approximation of the displacement equation (19) and of the output equation (1) around the high-gain estimator trajectory \hat{x}_t^o generated by (9).

To this aim, we extend the state space with the variables $\psi_t^{[k]}$ to exploit the Taylor expansion of the nonlinear maps $\phi(\cdot)$, $g(\cdot)$, $h(\cdot)$ in (1). Under standard analyticity hypotheses, this Taylor expansions around \hat{x}_t^o can be written as $g(\cdot)$ and $h(\cdot)$:

$$\begin{aligned} \phi(x_t) &= \phi(\hat{x}_t^o + \psi_t) \\ &= \sum_{i=0}^{\infty} \frac{(\nabla_x^{[i]} \otimes \phi(x))}{i!} \Big|_{x=\hat{x}_t^o} \psi_t^{[i]} = \sum_{i=0}^{\infty} \Phi_i(\hat{x}_t^o) \psi_t^{[i]}, \quad (20) \end{aligned}$$

$$\begin{aligned} g(x_t)u_t &= \sum_{j=1}^s g_j(\hat{x}_t^o + \psi_t)u_{t,j} \\ &= \sum_{j=1}^s \sum_{i=0}^{\infty} \tilde{\Gamma}_{ij}(\hat{x}_t^o) \psi_t^{[i]} u_{t,j} = \sum_{i=0}^{\infty} \Gamma_i(\hat{x}_t^o, u_t) \psi_t^{[i]}, \quad (21) \end{aligned}$$

$$g^{[l]}(x_t)F_j^{[l]} = g^{[l]}(\hat{x}_t^o + \psi_t)F_j^{[l]} = \sum_{i=0}^{\infty} J_{ij}^{(l)}(\hat{x}_t^o) \psi_t^{[i]}, \quad (22)$$

$$h(x_t) = h(\hat{x}_t^o + \psi_t) = \sum_{i=0}^{\infty} H_i(\hat{x}_t^o) \psi_t^{[i]}, \quad (23)$$

$l = 1, 2$, according to the following matrices definitions:

$$\Phi_i(x) = \frac{\nabla_x^{[i]} \otimes \phi(x)}{i!}, \quad H_i(x) = \frac{\nabla_x^{[i]} \otimes h(x)}{i!}, \quad (24)$$

$$\tilde{\Gamma}_{ij}(x) = \frac{\nabla_x^{[i]} \otimes g_j(x)}{i!}, \quad \Gamma_i(x, u) = \sum_{j=1}^s \tilde{\Gamma}_{ij}(x) u_j, \quad (25)$$

$$J_{ij}^{(l)}(x) = \frac{\nabla_x^{[i]} \otimes (g^{[l]}(x) F_j^{[l]})}{i!}, \quad l = 1, 2. \quad (26)$$

By using eq.s (20)-(23) we can rewrite eq. (19) as

$$\begin{aligned} d\psi_t &= \sum_{i=1}^{\infty} M_i(\hat{x}_t^o, u_t) \psi_t^{[i]} dt + D(\hat{x}_t^o) dt \\ &+ \sum_{j=1}^s \sum_{i=0}^{\infty} J_{ij}^{(1)}(\hat{x}_t^o) \psi_t^{[i]} dW_{t,j}^1 + \sum_{j=1}^q N_j(\hat{x}_t^o) dW_{t,j}^2, \end{aligned} \quad (27)$$

where

$$M_i(\hat{x}_t^o, u_t) = \Phi_i(\hat{x}_t^o) + \Gamma_i(\hat{x}_t^o, u_t) - Q^{-1}(\hat{x}_t^o) \mathcal{K} H_i(\hat{x}_t^o), \quad (28)$$

$$D(\hat{x}_t^o) = -\frac{1}{2} Q^{-1}(\hat{x}_t^o) \mathcal{B}_s \mathcal{H}_{\phi h}(\hat{x}_t^o), \quad (29)$$

$$N_j(\hat{x}_t^o) = -Q^{-1}(\hat{x}_t^o) \mathcal{K} G_j. \quad (30)$$

The differential of the Kronecker power $\psi_t^{[k]}$, $k > 1$ can be written, following [18], as

$$\begin{aligned} d(\psi_t^{[k]}) &= \left(\nabla_{\psi} \otimes \psi_t^{[k]} \right) \left(\sum_{i=1}^{\infty} M_i(\hat{x}_t^o, u_t) \psi_t^{[i]} \right) dt \\ &+ \left(\nabla_{\psi} \otimes \psi_t^{[k]} \right) D(\hat{x}_t^o) dt \\ &+ \frac{1}{2} \left(\nabla_{\psi}^{[2]} \otimes \psi_t^{[k]} \right) \left(\sum_{j=1}^p g^{[2]}(\hat{x}_t^o + \psi_t) F_j^{[2]} + N_0(\hat{x}_t^o) \right) dt \\ &+ \left(\nabla_{\psi} \otimes \psi_t^{[k]} \right) \sum_{j=1}^p \sum_{i=0}^{\infty} J_{ij}^{(1)}(\hat{x}_t^o) \psi_t^{[i]} dW_{t,j}^1 \\ &+ \left(\nabla_{\psi} \otimes \psi_t^{[k]} \right) \sum_{j=1}^q N_j(\hat{x}_t^o) dW_{t,j}^2, \end{aligned} \quad (31)$$

with $N_0(\hat{x}_t^o) = \sum_{j=1}^q N_j(\hat{x}_t^o)$. Through suitably defined matrices O_n^k, U_n^k , whose explicit computation is reported in the Appendix for the ease of the reader (see also [18]), together with the properties of Kronecker algebra (69) and (70) reported in the Appendix as well, the differen-

tial $d(\psi_t^{[k]})$ in (67) can finally be written as:

$$\begin{aligned} d(\psi_t^{[k]}) &= U_n^k \sum_{i=1}^{\infty} \left(M_i(\hat{x}_t^o, u_t) \otimes I_{n^{k-1}} \right) \psi_t^{[k+i-1]} dt \\ &+ U_n^k \left(D(\hat{x}_t^o) \otimes I_{n^{k-1}} \right) \psi_t^{[k-1]} dt \\ &+ \frac{1}{2} O_n^k \sum_{i=0}^{\infty} \sum_{j=1}^p \left(J_{ij}^{(2)}(\hat{x}_t^o) \otimes I_{n^{k-2}} \right) \psi_t^{[k+i-2]} dt \\ &+ \frac{1}{2} O_n^k \left(N_0(\hat{x}_t^o) \otimes I_{n^{k-2}} \right) \psi_t^{[k-2]} dt \\ &+ U_n^k \sum_{j=1}^p \sum_{i=0}^{\infty} \left(J_{ij}^{(1)}(\hat{x}_t^o) \otimes I_{n^{k-1}} \right) \psi_t^{[k+i-1]} dW_{t,j}^1 \\ &+ U_n^k \sum_{j=1}^q \left(N_j(\hat{x}_t^o) \otimes I_{n^{k-1}} \right) \psi_t^{[k-1]} dW_{t,j}^2, \end{aligned} \quad (32)$$

The ν -degree Carleman approximation of (19) around the trajectory \hat{x}_t^o given by the high-gain estimator equations (9)-(14) is obtained by using the new state vector $\Psi_t = [\Psi_t^1{}^T \dots \Psi_t^{\nu}{}^T]^T$, whose components $\Psi_t^k \in \mathbb{R}^{n^k}$ are such that

$$\begin{aligned} d\Psi_t^k &= \sum_{i=1}^{\nu-k+1} \mathcal{M}_{ki}(\hat{x}_t^o, u_t) \Psi_t^{k+i-1} dt + \Upsilon_k(\hat{x}_t^o) \Psi_t^{k-1} dt \\ &+ \sum_{i=0}^{\nu-k+2} \mathcal{G}_{ki}(\hat{x}_t^o) \Psi_t^{k+i-2} dt + \mathcal{N}_k(\hat{x}_t^o) \Psi_t^{k-2} dt \\ &+ \sum_{j=1}^p \sum_{i=0}^{\nu-k+1} \mathcal{J}_{ki}^j(\hat{x}_t^o) \Psi_t^{k+i-1} dW_{t,j}^1 \\ &+ \sum_{j=1}^q \mathcal{U}_{kj}(\hat{x}_t^o) \Psi_t^{k-1} dW_{t,j}^2, \end{aligned} \quad (33)$$

with:

$$\mathcal{M}_{ki}(\hat{x}_t^o, u_t) = U_n^k \left(M_i(\hat{x}_t^o, u_t) \otimes I_{n^{k-1}} \right) \quad (34)$$

$$\Upsilon_k(\hat{x}_t^o) = U_n^k \left(D(\hat{x}_t^o) \otimes I_{n^{k-1}} \right),$$

$$\mathcal{G}_{ki}(\hat{x}_t^o) = \begin{cases} \frac{1}{2} \sum_{j=1}^p O_n^k \left(J_{i-1,j}^{(2)}(\hat{x}_t^o) \otimes I_{n^{k-2}} \right), & 1 < k \\ 0_{n^k \times n^{k+i-2}}, & k = 1 \end{cases}$$

$$\mathcal{N}_k(\hat{x}_t^o) = \begin{cases} \frac{1}{2} O_n^k \left(N_0(\hat{x}_t^o) \otimes I_{n^{k-2}} \right), & 1 < k \\ 0_{n^k \times n^{k-2}}, & k = 1. \end{cases} \quad (35)$$

$$\mathcal{J}_{ki}^j(\hat{x}_t^o) = U_n^k \left(J_{ij}^{(1)}(\hat{x}_t^o) \otimes I_{n^{k-1}} \right) \quad (36)$$

$$\mathcal{U}_{kj}(\hat{x}_t^o) = U_n^k \left(N_j(\hat{x}_t^o) \otimes I_{n^{k-1}} \right)$$

Comparing (33) with (32), it is clear that $\Psi_t^k \in \mathbb{R}^{n^k}$ is aimed to approximate $\psi_t^{[k]}$, $k = 1, \dots, \nu$ by neglecting in (27) and (32) the terms of order higher than ν . Therefore, the extended state Ψ_t evolves according to the following differential system:

$$\begin{aligned} d\Psi_t &= \mathcal{A}(\hat{x}_t^o, u_t)\Psi_t dt + \mathcal{D}(\hat{x}_t^o)dt \\ &+ \sum_{j=1}^p \left(\mathcal{B}_1^j(\hat{x}_t^o)\Psi_t + \mathcal{F}_1^j(\hat{x}_t^o) \right) dW_{t,j}^1 \\ &+ \sum_{j=1}^q \left(\mathcal{B}_2^j(\hat{x}_t^o)\Psi_t + \mathcal{F}_2^j(\hat{x}_t^o) \right) dW_{t,j}^2, \end{aligned} \quad (37)$$

where:

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \cdots & \mathcal{A}_{1\nu} \\ \vdots & \ddots & \vdots \\ \mathcal{A}_{\nu 1} & \cdots & \mathcal{A}_{\nu\nu} \end{bmatrix} \quad \mathcal{D} = \begin{bmatrix} \mathcal{D}_1 \\ \vdots \\ \mathcal{D}_\nu \end{bmatrix} \quad (38)$$

with $\mathcal{A}_{ki} \in \mathbb{R}^{n^k \times n^i}$, $\mathcal{D}_k \in \mathbb{R}^{n^k \times 1}$ given by

$$\mathcal{A}_{ki}(\hat{x}_t^o, u_t) = \begin{cases} \mathcal{M}_{ki}(\hat{x}_t^o, u_t) + \mathcal{G}_{ki}(\hat{x}_t^o), & i \geq k \\ \mathcal{G}_{ki}(\hat{x}_t^o) + \Upsilon_k(\hat{x}_t^o), & i = k - 1 \\ \mathcal{G}_{ki}(\hat{x}_t^o) + \mathcal{N}_k(\hat{x}_t^o), & i = k - 2 \\ 0, & \text{otherwise.} \end{cases} \quad (39)$$

$$\mathcal{D}_k(\hat{x}_t^o) = \begin{cases} \Upsilon_1(\hat{x}_t^o), & k = 1, \\ \mathcal{G}_{20}(\hat{x}_t^o) + \mathcal{N}_2(\hat{x}_t^o), & k = 2, \\ 0_{n^k \times 1}, & k > 2 \end{cases} \quad (40)$$

and

$$\mathcal{B}_1^j = \begin{bmatrix} \mathcal{J}_{11}^j & \mathcal{J}_{12}^j & \cdots & \cdots & \mathcal{J}_{1\nu_j}^j \\ \mathcal{J}_{20}^j & \mathcal{J}_{21}^j & \cdots & \cdots & \mathcal{J}_{2\nu_{j-1}}^j \\ 0_{n^3 \times n} & \mathcal{J}_{30}^j & \ddots & & \mathcal{J}_{3\nu_{j-2}}^j \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{n^\nu \times 1} & \cdots & 0_{n^\nu \times n^{\nu-2}} & \mathcal{J}_{\nu 0}^j & \mathcal{J}_{\nu 1}^j \end{bmatrix} \quad \mathcal{F}_1^j = \begin{bmatrix} \mathcal{J}_{10}^j \\ 0_{n^2 \times 1} \\ \vdots \\ 0_{n^\nu \times 1} \end{bmatrix} \quad (41)$$

$$\mathcal{B}_2^j = \begin{bmatrix} 0_{n \times n} & 0_{n \times n^2} & \cdots & \cdots & 0_{n \times n^\nu} \\ \mathcal{U}_2^j & 0_{n^2 \times n^2} & \cdots & \cdots & 0_{n^2 \times n^\nu} \\ 0_{n^3 \times n} & \mathcal{U}_3^j & \ddots & & 0_{n^3 \times n^\nu} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{n^\nu \times n} & \cdots & 0_{n^\nu \times n^{\nu-2}} & \mathcal{U}_\nu^j & 0_{n^\nu \times n^\nu} \end{bmatrix} \quad \mathcal{F}_2^j = \begin{bmatrix} \mathcal{U}_1^j \\ 0_{n^2 \times 1} \\ \cdots \\ 0_{n^\nu \times 1} \end{bmatrix}. \quad (42)$$

The measurement equation in (1) can be written using (23) as

$$dy_t = h(\hat{x}_t^o + \psi_t)dt + GdW_t^2 = \sum_{i=0}^{\infty} H_i(\hat{x}_t^o)\psi_t^{[i]}dt + GdW_t^2 \quad (43)$$

The ν -degree Carleman approximation of the output equation (43) around the trajectory \hat{x}_t^o given by the high-gain estimator (9)-(14) is obtained using the state vector Ψ_t together with a new output vector $Y_t \in \mathbb{R}^q$

$$\begin{aligned} dY_t &= \sum_{i=1}^{\nu} H_i(\hat{x}_t^o)\Psi_t^i dt + h(\hat{x}_t^o)dt + GdW_t^2 \\ &= \mathcal{C}(\hat{x}_t^o)\Psi_t dt + h(\hat{x}_t^o)dt + \sum_{j=1}^q G_j dW_{t,j}^2, \end{aligned} \quad (44)$$

with $\mathcal{C}(\hat{x}_t^o) = [H_1(\hat{x}_t^o) \ H_2(\hat{x}_t^o) \ \cdots \ H_\nu(\hat{x}_t^o)]$. It is straightforward that vector Y_t is aimed to approximate the original measurement vector y_t .

Using the ν -degree Carleman approximation (37) of the original nonlinear system around the high-gain estimator state-trajectory \hat{x}_t^o , we formally define the OFF estimate as follows.

Definition 2 *The OFF estimate of the state of the nonlinear system (1) is given by*

$$\hat{x}_t = \Pi \hat{\Psi}_t + \hat{x}_t^o, \quad \Pi = [I_n \ 0_{n \times n^2 + \cdots + n^\nu}] \quad (45)$$

where $\hat{\Psi}_t$ is the optimal linear estimate of the extended state Ψ_t in (37), with respect to the output vector Y_t given by (44), that is the projection of Ψ_t onto the space $L(Y_t)$ of all the affine transformations of the random variables $\{Y_\tau, 0 \leq \tau \leq t\}$.

Motivations for Definition 2 stems from the fact that the Carleman approximation (37), endowed with the approximate measurement equation (44), is a bilinear system (i.e. linear drift and multiplicative noise), whose optimal linear filter is available in the literature, [10]. Let $m_t = \mathbb{E}[\Psi_t]$ be the mean value of Ψ_t , that obeys the equation

$$\dot{m}_t = \mathcal{A}(\hat{x}_t^o, u_t)m_t + \mathcal{D}(\hat{x}_t^o), \quad (46)$$

and let $\Xi_t = \text{Cov}(\Psi_t)$ be the covariance matrix $\mathbb{E}[(\Psi_t - m_t)(\Psi_t - m_t)^T]$, for which:

$$\dot{\Xi}_t = \mathcal{A}(\hat{x}_t^o, u_t)\Xi_t + \Xi_t \mathcal{A}^T(\hat{x}_t^o, u_t) + \mathcal{Q}(m_t, \Xi_t) \quad (47)$$

where

$$\begin{aligned} \mathcal{Q}(m_t, \Xi_t) &= \sum_{i=1}^s \mathcal{B}_1^i(\hat{x}_t^o) \Xi_t \mathcal{B}_1^{iT}(\hat{x}_t^o) + \sum_{i=1}^q \mathcal{B}_2^i(\hat{x}_t^o) \Xi_t \mathcal{B}_2^{iT}(\hat{x}_t^o) \\ &\quad + \sum_{i=1}^s \mu_{t,j}^1(\hat{x}_t^o) \mu_{t,j}^{1T}(\hat{x}_t^o) + \sum_{i=1}^q \mu_{t,j}^2(\hat{x}_t^o) \mu_{t,j}^{2T}(\hat{x}_t^o), \end{aligned} \quad (48)$$

with $\mu_{t,j}^1(\hat{x}_t^o) = \mathcal{B}_1^j(\hat{x}_t^o) m_t + \mathcal{F}_1^j(\hat{x}_t^o)$ and $\mu_{t,j}^2(\hat{x}_t^o) = \mathcal{B}_2^j(\hat{x}_t^o) m_t + \mathcal{F}_2^j(\hat{x}_t^o)$.

The optimal linear estimate $\widehat{\Psi}_t$ of the process Ψ_t is provided by

$$\begin{aligned} d\widehat{\Psi}_t &= \mathcal{A}(\hat{x}_t^o, u_t) \widehat{\Psi}_t dt + \mathcal{D}(\hat{x}_t^o) dt + \left(P_t \mathcal{C}^T(\hat{x}_t^o) + \Phi_t \right) \\ &\quad \cdot R^{-1} \left(dY_t - \mathcal{C}(\hat{x}_t^o) \widehat{\Psi}_t dt - h(\hat{x}_t^o) dt \right), \end{aligned} \quad (49)$$

with

$$R = GG^T, \quad \Phi(\hat{x}_t^o) = \sum_{j=1}^q \mu_{t,j}^2(\hat{x}_t^o) G_j^T \quad (50)$$

and $P_t = \mathbb{E}[(\Psi_t - \widehat{\Psi}_t)(\Psi_t - \widehat{\Psi}_t)^T]$ the error covariance matrix evolving according to the following equation

$$\begin{aligned} \dot{P}_t &= \mathcal{A}(\hat{x}_t^o, u_t) P_t + P_t \mathcal{A}(\hat{x}_t^o, u_t) + \mathcal{Q}(m_t, \Xi_t) \\ &\quad - \left(\Phi(\hat{x}_t^o) + P_t \mathcal{C}^T(\hat{x}_t^o) \right) R^{-1} \left(\Phi(\hat{x}_t^o) + P_t \mathcal{C}^T(\hat{x}_t^o) \right)^T, \\ P_0 &= \Xi_0 \end{aligned} \quad (51)$$

Remark 2 Note that when implementing the $\widehat{\Psi}_t$ -filter equations, one has to substitute the available measurement process y_t in (49) instead of the fictitious output Y_t .

4.2 Structural properties

According to the OFF estimate definition, the error estimate in the mean square sense obeys the following inequality:

$$\begin{aligned} \mathbb{E} [\|x_t - \hat{x}_t\|^2] &= \mathbb{E} [\|(x_t - \hat{x}_t^o) + (\hat{x}_t^o - \hat{x}_t)\|^2] \\ &\leq 2\mathbb{E} [\|x_t - \hat{x}_t^o\|^2] + 2\mathbb{E} [\|\Pi \widehat{\Psi}_t\|^2] \quad (52) \end{aligned}$$

Note that the error is split in two contributes: the former is directly related to the error of the high-gain estimator and the latter is related to the estimate of the displacement filter with the ν -degree Carleman approximation around the high-gain estimator trajectory. In this section we provide sufficient conditions that ensure that the first term is bounded in the mean square sense (*i.e.* boundedness of the high-gain estimate error in the

mean square sense). This result motivates the use of the OFF estimate algorithm: indeed, besides using a higher order degree polynomial approximation (with respect to the standard linear one of the EKBF), it suitably exploits the approximation around a trajectory whose displacement from the real state is stable in the mean square sense. This property cannot be guaranteed, and, on the contrary, it does not hold in general, when the trajectory is given by a linear approximation as in the EKBF. Of course, the closer is the high-gain estimator trajectory to the real one, the smaller are both the contributes in inequality (52). However, a convergence result for OFF cannot be provided in the general case, as it also happens for the EKBF.

The following Theorem provides an upper bound for the error of the high-gain state estimate in the mean square sense.

Theorem 2 Assume that **H0**) and the following hypotheses hold true:

- H1)** the map $z_t = \Theta_{\bar{s}}(x_t)$ defined in (4-5) is a global diffeomorphism, with the inverse map $\Theta_{\bar{s}}^{-1}(\cdot)$ globally Lipschitz, with Lipschitz coefficient γ_θ ;
- H2)** function $L_g L_\phi^{\bar{s}-1} h(x)$ is bounded in the mean square sense, that is: there exists a positive constant γ_1 such that
$$\sup_x \|L_g L_\phi^{\bar{s}-1} h(x)\|^2 \leq \gamma_1 \quad (53)$$
- H3)** functions $L_\phi^{\bar{s}} h(x)$, $L_g L_\phi^{\bar{s}-1} h(x)$ and $\mathcal{H}_{\phi h}(x)$ are globally Lipschitz, with Lipschitz coefficients γ_2 , γ_3 and γ_4 , respectively;
- H4)** u_t is uniformly bounded, that is: there exists a positive constant U_M such that

$$\sup_{t \geq 0} \|u_t\|^2 \leq U_M. \quad (54)$$

Then, there exists a gain vector $\mathcal{K} \in \mathbb{R}^{n \times q}$ such that the high-gain estimator defined by (9-14) has a bounded error (in the mean square sense), that is: there exists a positive constant L such that

$$\mathbb{E} [\|x_t - \hat{x}_t^o\|^2] \leq L. \quad (55)$$

Proof. See Appendix.

Remark 1 It has to be stressed that the global Lipschitz hypotheses in Theorem 2 guarantee also existence and uniqueness of the solutions of (9).

Remark 2 The global Lipschitz property is restrictive. However, when the state evolution is bounded to a compact set in \mathbb{R}^n , as frequently happens in practical applications, the hypotheses of Theorem 2 can be substituted by a local Lipschitz requirement.

As a first approximation, the algorithmic complexity of the OFF filter can be characterized through the size of the resulting filter as a function of n and ν (spatial complexity). Let $S_{\hat{\Psi}}$ be the size of $\hat{\Psi}$,

$$S_{\hat{\Psi}} = \sum_{i=1}^{\nu} n^i \quad (56)$$

Since the high-gain estimator has size n , it is easy to show that the overall dimension of OFF is $S(n, \nu) = n + S_{\hat{\Psi}}(S_{\hat{\Psi}} + 2)$, where the evolution of the state and error covariance matrices, as well as of the system estimate, has been accounted for. The function $S(n, \nu)$ is polynomial with respect to n and exponential with respect to ν , yielding a much larger filter (for example $S(5, 3) = 24340$) than the EKBF, whose size $n + n(n + 1)/2$ increases quadratically with n . However, since the filter matrices (38),(41),(42) have a vast majority of null elements and the estimate $\hat{\Psi}_t$, due to the properties of the Kronecker power, contains many repeated terms, the computational burden can be drastically reduced by means of an efficient implementation based on sparse matrices, as well as on the properties of the *reduced* Kronecker algebra, that aims at eliminating the redundant terms [10].

The time complexity of the algorithm depends on the complexity of each step, since the number of iterations will approximately be the same for any algorithm for time continuous system. Each iteration step in OFF only involves multiplications between matrices and matrices and vectors, thus it is proportional to the size of the data objects and then to $S_{\hat{\Psi}}$. Thus, also in this case we have that it is polynomial with respect to the system size n and exponential with respect to ν . Notice that the independent variable is n , whereas ν is a design parameter,

5 Simulations

5.1 HIV model

Numerical simulations are reported here in order to show the effectiveness of the proposed OFF algorithm. We consider a basic mathematical model, [25], that has been widely employed to describe the virus dynamics of primary HIV infection through a system of nonlinear differential equations

$$\begin{aligned} dx_{1_t} &= (s - d_v x_{1_t} - \beta x_{1_t} x_{3_t}) dt \\ dx_{2_t} &= (\beta x_{1_t} x_{3_t} - \delta x_{2_t}) dt \\ dx_{3_t} &= (p x_{2_t} - c x_{3_t}) dt \end{aligned} \quad (57)$$

where x_{1_t} is concentration of target cells, s is the constant influx rate of target cells, d_v the target cell loss

rate, β the target cells infection rate, x_{2_t} the concentration of infected cells, δ their loss rate, x_{3_t} the serum viral concentration, p the viral production rate and c the viral clearance rate. With a realistic choice of parameters and for δ large enough the model predicts that the viral load reaches steady state through damped oscillations. With respect to the original deterministic model we have added a noise term on x_{1_t} and on the measurement equation so that, the first equation becomes:

$$dx_{1_t} = (s - d_v x_{1_t} - \beta x_{1_t} x_{3_t}) dt + F dW_t^1 \quad (58)$$

and, moreover, considered a measurement equation:

$$dy_t = x_{3_t} dt + G dW_t^2, \quad (59)$$

where $W_t^1, W_t^2 \in \mathbb{R}$ are independent standard Wiener processes. The resulting output y_t clearly satisfies the strong full (i.e. equal to 3) relative degree hypothesis. Notice that in the physical system all state variables must be positive, thus is it necessary to choose the amplitude F of the state noise small enough to satisfy this constraint, as it actually happens in the simulations.

The results have been obtained simulating the system (57-58) in the time interval $[0, 100]$ according to the Euler-Maruyama method, [19], with integration step $\Delta = 10^{-2}$. The parameters were set at $s = 10^2$, $d_v = 10^{-3}$, $\delta = 1$, $\beta = 1.3 \cdot 10^{-6}$, $p = 10^3$, and $c = 3$. The initial state was set at $x_0^T = [5 \cdot 10^3, 5 \cdot 10^{-1}, 1.4 \cdot 10^2]$. We have compared the performance of the observer-based estimator and the second order ($\nu = 2$) OFF state estimator, with respect to the standard Extended Kalman-Bucy Filter.

In Fig. 2 the estimate error and its variance, for a time sample of 1 sec and averaged over 100 simulations, are shown for variables x_1 and x_2 of the system. Notice that the scale of vertical axes is logarithmic, due to large interval spanned by the values of the system variables, whose time course is shown in the top left boxes. We have used $t > 10$ to exclude the transient phase. In these simulations we have used $F = 5$ and $G = 0.5$, and a gain corresponding to an largest eigenvalue of $\lambda = -0.2$ for the observer. In these conditions, the observer and the EKBF have comparable performance, whereas the OFF method has the smallest error and variance in the whole time interval. Notice that the error estimate is reduced by a factor of 50-100 (the scale is logarithmic).

We have compared the performance of the three methods in the same conditions for a set of choices for F and G by using the mean of the squared relative estimation error (MSRE) defined as

$$\mu = \frac{1}{N+1} \sum_{k=1}^N \sum_{i=1}^3 \left(\frac{\hat{x}_{t_k}^i - x_{t_k}^i}{x_{t_k}^i} \right)^2, \quad t_k = k\Delta. \quad (60)$$

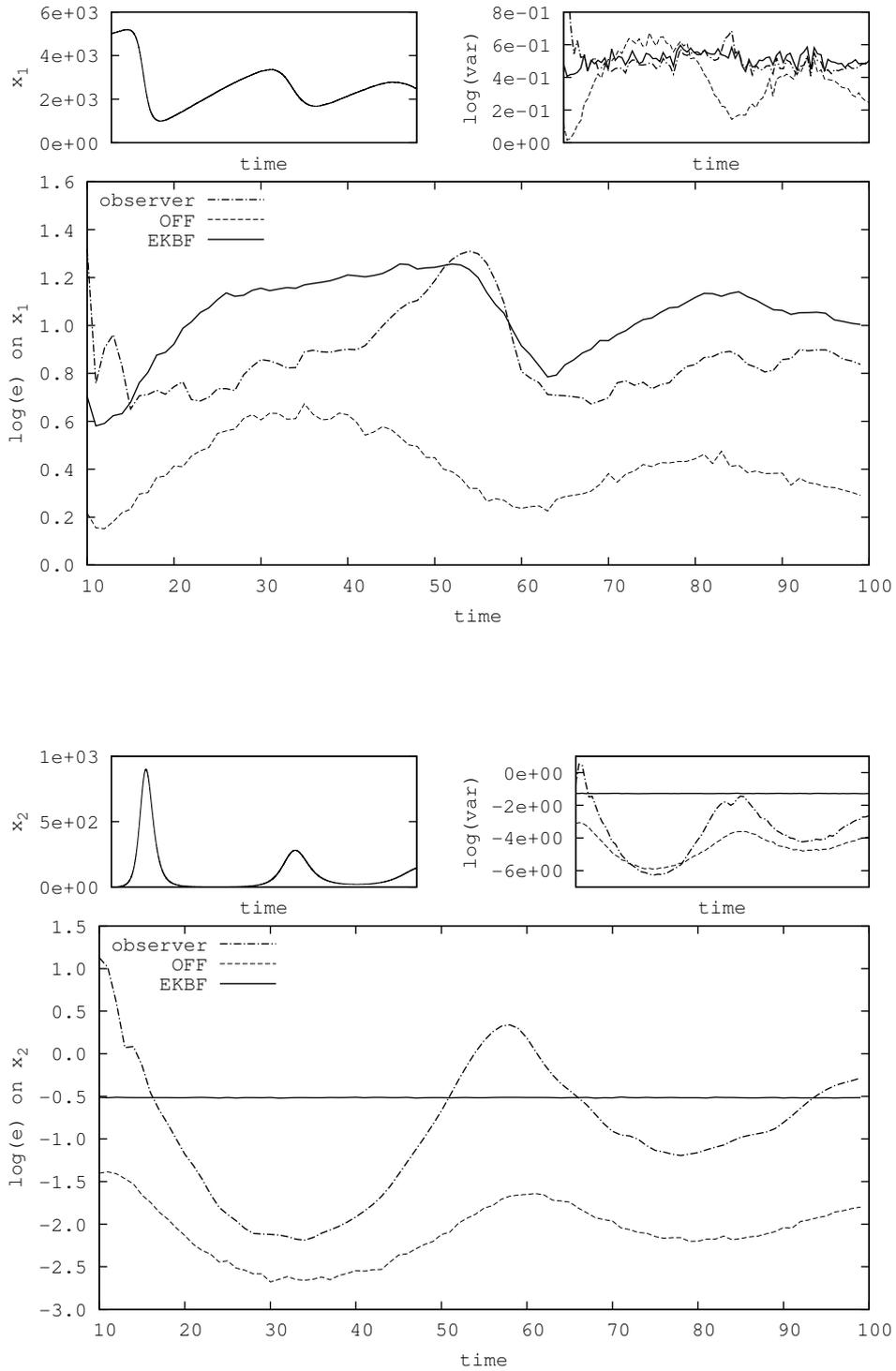


Fig. 2. Estimate error for x_1 (top) and x_2 (bottom) at $F=5, G=0.5$. The evolution of the corresponding system variable and variance of the estimation error is shown in top boxes.

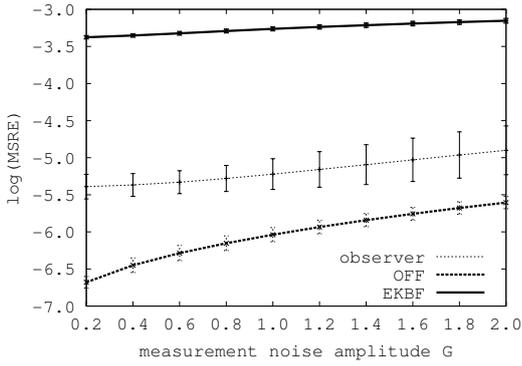


Fig. 3. Comparison of MSRE (logarithmic scale) at $F=1$.

The MSRE for the observer alone (μ^o), the OFF (μ^{OFF}) and the reference EKBF filter (μ^{EKBF}) averaged over 100 simulation runs are reported in Table 1 for a few choices of state and measurement noise amplitudes.

It may be noticed that it is more efficient for the OFF algorithm to use a low gain in the observer, since this reduces the measurement noise amplification introduced by the observer.

A comparison of MSRE at $F = 1$ and $0.2 \leq G \leq 2$ is shown in Fig. 3. It may be noticed the improvement of the filter with respect to the observer alone and that the MSRE is consistently reduced by a factor 10^2 with respect to the EKBF.

5.2 Scalar example

In this section we use a scalar example to investigate the improvements in the performances of the OFF algorithm by increasing the order ν of the Carleman approximation.

Consider the scalar stochastic system

$$dx_t = \frac{kx_t^3}{x_t^2 + \epsilon} dt + \frac{F}{x_t^2 + \epsilon} dW_t^1, \quad (61)$$

$$dy_t = (a_4x_t^4 + a_3x_t^3 + a_2x_t^2 + a_1x_t) dt + GdW_t^2, \quad (62)$$

with a non-gaussian state noise and a nonlinear output function.

With the following choice of parameters: $k = -2$, $\epsilon = 3$, $a_2 = a_3 = a_4 = 1$, $a_1 = 3$, $F = 1$, $G \in [0.08, 0.12]$, $x_0 = 0$, numerical simulations with integration step $\Delta = 5 \cdot 10^{-2}$ show that the estimation error of the EKBF is not bounded, with a failure ratio between 0.80 (for $G = 0.08$) and 0.15 (for $G = 0.12$). On the other hand, by choosing $\lambda = -0.8$, the OFF algorithm is always stable for $\nu \geq 2$, whereas the linear version ($\nu = 1$) displays the

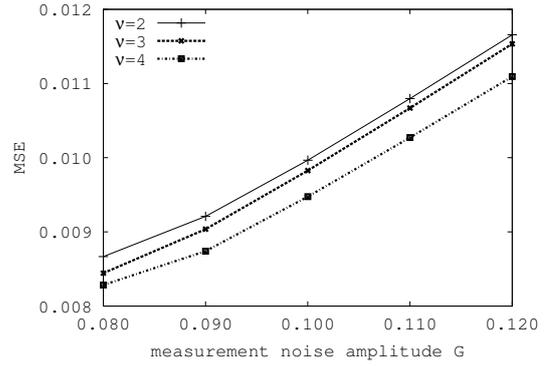


Fig. 4. Comparison of MSE at $F=1$.

same unstable behavior of the EKBF. To investigate the performances of the OFF algorithm at different choices of ν we use the mean squared estimation error (MSE) defined as

$$\mu = \frac{1}{N+1} \sum_{k=1}^N (\hat{x}_k - x_k)^2, \quad (63)$$

which, for a scalar system, is reliable and simpler than the MSRE. The MSE for the observer, μ^o , and the OFF μ^ν , with $\nu = 2, 3, 4$ averaged over 100 simulation runs in the time interval $[0, 100]$ are reported in Table 2. The corresponding MSE for the OFF state estimates are plotted in Figure 4, where the improvement in the error estimate obtained with a higher value of ν can be appreciated.

6 Conclusions

In this paper a novel algorithm has been proposed, denoted as *Observer Follower Filter (OFF)*, for the state estimate of nonlinear stochastic differential systems. It is based on a mixed approach, that combines a high-gain estimator with a filter for bilinear systems (i.e. linear drift and multiplicative noise) based on the Carleman approximation. The main strength of the approach is that the filtering part of the algorithm is applied to the trajectory provided by the high-gain estimator, and this trajectory has a displacement which is stable (in the mean square sense) with respect to the given system. The complexity of the algorithm can be tuned by choosing the degree ν of the Carleman approximation. In the simplest cases of $\nu = 1, 2$ the computational overhead of the OFF approach with respect to the EKBF is negligible, the performance is similar, but the OFF approach yields better stability. This feature makes it a good choice for systems with high nonlinear behavior or poor initial estimates.

F	G	μ^o	μ^{OFF}	μ^{EKBF}
0.2	0.2	$2.3 \pm 1.4 \cdot 10^{-7}$	$3.8 \pm 0.9 \cdot 10^{-8}$	$2.2 \pm 0.1 \cdot 10^{-5}$
0.2	1.2	$3.3 \pm 4.8 \cdot 10^{-6}$	$7.2 \pm 1.4 \cdot 10^{-7}$	$4.7 \pm 0.4 \cdot 10^{-5}$
0.5	0.2	$1.1 \pm 0.5 \cdot 10^{-6}$	$8.9 \pm 2.2 \cdot 10^{-8}$	$1.1 \pm 0.1 \cdot 10^{-4}$
0.5	2.0	$1.0 \pm 1.4 \cdot 10^{-5}$	$2.1 \pm 0.4 \cdot 10^{-6}$	$2.4 \pm 0.2 \cdot 10^{-4}$
1.0	0.2	$4.2 \pm 1.9 \cdot 10^{-6}$	$2.1 \pm 0.4 \cdot 10^{-7}$	$2.6 \pm 0.1 \cdot 10^{-4}$
1.0	2.0	$1.2 \pm 1.2 \cdot 10^{-5}$	$2.5 \pm 0.5 \cdot 10^{-6}$	$7.0 \pm 0.5 \cdot 10^{-4}$

Table 1
Comparison of MSRE for the HIV model.

G	μ^o	μ^2	μ^3	μ^4
0.08	0.0493 ± 0.0111	0.0087 ± 0.0017	0.0084 ± 0.0011	0.0083 ± 0.0013
0.09	0.0493 ± 0.0112	0.0092 ± 0.0019	0.0090 ± 0.0013	0.0087 ± 0.0015
0.10	0.0494 ± 0.0112	0.0100 ± 0.0020	0.0098 ± 0.0015	0.0095 ± 0.0017
0.11	0.0494 ± 0.0112	0.0108 ± 0.0022	0.0107 ± 0.0016	0.0103 ± 0.0019
0.12	0.0495 ± 0.0112	0.0117 ± 0.0024	0.0115 ± 0.0018	0.0111 ± 0.0021

Table 2
Comparison of MRE for the scalar example.

Several points need further investigation, such as the relationship between the amplitude of the state and measurement noises and the optimal gain for the high-gain estimator, the possibility of introducing variable gain as well as extended stability results for the combination of the high-gain estimator with the filter.

Appendix

Given a pair of integer (a, b) , the symbol $C_{a,b}$ denotes a orthonormal *commutation matrix* in $\{0, 1\}^{ab \times ab}$ such that, given any two matrices $A \in \mathbb{R}^{r_a \times c_a}$ and $B \in \mathbb{R}^{r_b \times c_b}$

$$B \otimes A = C_{r_a, r_b}^T (A \otimes B) C_{c_a, c_b} \quad (64)$$

The matrices U_n^h and O_n^h used in Section 4 are recursively computed as

$$\begin{aligned} U_n^h &= I_n^h + C_{n^{h-1}, n}^T (U_n^{h-1} \otimes I_n) \\ O_n^h &= U_n^h C_{n^{h-1}} ((U_n^{h-1} C_{n^{h-2}, n}) \otimes I_n) C_{n^2, n^{h-2}}^T, \end{aligned} \quad (65)$$

with the initial value $U_n^1 = I_n$. From the definition it follows that $U_n^2(I_n \otimes \Psi) = I_n \otimes \Psi + \Psi \otimes I_n$.

Lemma 3 For any $x \in \mathbb{R}^n$ the following identities hold:

$$\begin{aligned} \nabla_x \otimes x^{[h]} &= U_n^h (I_n \otimes x^{[h-1]}), \quad h \geq 1, \\ \nabla_x^{[2]} \otimes x^{[h]} &= O_n^h (I_{n^2} \otimes x^{[h-2]}), \quad h > 1. \quad \square \end{aligned} \quad (66)$$

The use of U_n^h and O_n^h allows to rewrite (31) as

$$\begin{aligned} d(\psi_t^{[k]}) &= U_n^k \left(I_n \otimes \psi_t^{[k-1]} \right) \left(\sum_{i=1}^{\infty} M_i(\hat{x}_t^o, u_t) \psi_t^{[i]} \right) dt \\ &\quad + U_n^k \left(I_n \otimes \psi_t^{[k-1]} \right) D(\hat{x}_t^o) dt \\ &+ \frac{1}{2} O_n^k \left(I_{n^2} \otimes \psi_t^{[k-2]} \right) \cdot \left(\sum_{j=1}^p \sum_{i=0}^{\infty} J_{ij}^{(2)}(\hat{x}_t^o) \psi_t^{[i]} + N_0(\hat{x}_t^o) \right) dt \\ &\quad + U_n^k \left(I_n \otimes \psi_t^{[k-1]} \right) \sum_{j=1}^p \sum_{i=0}^{\infty} J_{ij}^{(1)}(\hat{x}_t^o) \psi_t^{[i]} dW_{t,j}^1 \\ &\quad + U_n^k \left(I_n \otimes \psi_t^{[k-1]} \right) \sum_{j=1}^q N_j dW_{t,j}^2. \end{aligned} \quad (67)$$

The following property of the Kronecker algebra has been used in the derivation of the filter equations. For any matrices A, B, C, D of suitable dimensions it is

$$(A \cdot B) \otimes (C \cdot D) = (A \otimes C) \cdot (B \otimes D), \quad (68)$$

where $A \cdot B$ denotes the standard matrix product.

The following equalities, also used in the filter derivation, hold true for any pair of integers $h, k = 0, 1, \dots$, any

vectors $\chi \in \mathbb{R}^m$, $\xi \in \mathbb{R}^n$ and any matrix $A \in \mathbb{R}^{n \times m^h}$:

$$\begin{aligned} (I_n \otimes \chi^{[k]}) \cdot A \cdot \chi^{[h]} &= (I_n \otimes \chi^{[k]}) \cdot \left((A\chi^{[h]}) \otimes 1 \right) \\ &= (A\chi^{[h]}) \otimes \chi^{[k]} = (A \cdot \chi^{[h]}) \otimes (I_{m^k} \cdot \chi^{[k]}) \\ &= (A \otimes I_{m^k}) \cdot \chi^{[k+h]} \end{aligned} \quad (69)$$

$$\begin{aligned} (I_n \otimes \chi^{[k]}) \cdot \xi &= (I_n \otimes \chi^{[k]}) \cdot (\xi \otimes 1) = \xi \otimes \chi^{[k]} \\ &= (\xi \cdot 1) \otimes (I_{m^k} \cdot \chi^{[k]}) = (\xi \otimes I_{m^k}) \cdot \chi^{[k]} \end{aligned} \quad (70)$$

Proof of Lemma 1. By applying the Itô formula, according to the Kronecker formalism (cfr. [18] and references therein), it is:

$$\begin{aligned} dz_t &= \left((\nabla_x \otimes \Theta_{\bar{s}}) \Big|_{x_t} (\phi(x_t) + g(x_t)u_t) \right. \\ &\quad \left. + \frac{1}{2} (\nabla_x^{[2]} \otimes \Theta_{\bar{s}}) \Big|_{x_t} \tilde{g}_2(x_t) \right) dt \\ &\quad + (\nabla_x \otimes \Theta_{\bar{s}}) \Big|_{x_t} g(x_t) F dW_t^1 \end{aligned} \quad (71)$$

where, with $F_0 = \sum_{i=1}^p F_i^{[2]}$,

$$\tilde{g}_2(x_t) = \sum_{i=1}^p (g(x_t) F_i)^{[2]} = g^{[2]}(x_t) F_0, \quad (72)$$

and the differential operator $\nabla_x^{[i]} \otimes$ applied to a generic function $f : \mathbb{R}^n \mapsto \mathbb{R}^p$ is defined as follows:

$$\nabla_x^{[0]} \otimes f = f, \quad \nabla_x^{[i+1]} \otimes f = \nabla_x \otimes (\nabla_x^{[i]} \otimes f), \quad i \geq 1, \quad (73)$$

with $\nabla_x = [\partial/\partial x_1 \ \cdots \ \partial/\partial x_n]$ and $\nabla_x \otimes f$ the Jacobian of the vector function f (see [18] for more details). The square brackets at the exponents of vectors and matrices denote the Kronecker power (see [9]). Note that, by suitably exploiting the observability map definition, it is:

$$\begin{aligned} (\nabla_x \otimes \Theta_{\bar{s}}) \Big|_{x_t} \phi(x_t) &= \left[\frac{dh_1}{dx} \phi \ \cdots \ \frac{dL_\phi^{s_1-1} h_1}{dx} \phi \ \cdots \ \frac{dh_q}{dx} \phi \ \cdots \ \frac{dL_\phi^{s_q-1} h_q}{dx} \phi \right]^T \\ &= [L_\phi h_1 \ \cdots \ L_\phi^{s_1} h_1 \ \cdots \ L_\phi h_q \ \cdots \ L_\phi^{s_q} h_q]^T \\ &= \mathcal{A}_{\bar{s}} z_t + \mathcal{B}_{\bar{s}} L_\phi^{\bar{s}} h(x_t), \end{aligned} \quad (74)$$

and, accordingly, by hypothesis **H0**), it is:

$$\begin{aligned} (\nabla_x \otimes \Theta_{\bar{s}}) \Big|_{x_t} g(x_t) &= \begin{bmatrix} \frac{dh_1}{dx} g \\ \vdots \\ \frac{dL_\phi^{s_1-1} h_1}{dx} g \\ \vdots \\ \frac{dh_q}{dx} g \\ \vdots \\ \frac{dL_\phi^{s_q-1} h_q}{dx} g \end{bmatrix} = \begin{bmatrix} B_b^{s_1} L_g L_\phi^{s_1-1} h_1 \\ \vdots \\ B_b^{s_q} L_g L_\phi^{s_q-1} h_q \end{bmatrix} = \mathcal{B}_{\bar{s}} L_g L_\phi^{\bar{s}-1} h(x_t). \end{aligned} \quad (75)$$

It has to be stressed that equation (75) is achieved without using the *strong* relative degree property, but just the standard one. The strong relative degree property will be required in the following. Indeed, rewrite the second order derivative term in (71) as follows:

$$\begin{aligned} (\nabla_x^{[2]} \otimes \Theta_{\bar{s}}) \Big|_{x_t} \cdot g^{[2]}(x_t) &= (\nabla_x \otimes (\nabla_x \otimes \Theta_{\bar{s}})) \Big|_{x_t} \cdot (g(x_t) \otimes g(x_t)) \\ &= \left[(\nabla_x \otimes (\nabla_x \otimes \Theta_{\bar{s}}(x))) \cdot (g(\xi) \otimes g(\xi)) \right]_{\substack{x=x_t \\ \xi=x_t}} \end{aligned} \quad (76)$$

If we formally apply the Kronecker product property (68) to eq. (76) we have

$$\begin{aligned} (\nabla_x^{[2]} \otimes \Theta_{\bar{s}}) \Big|_{x_t} \cdot g^{[2]}(x_t) &= \left[(\nabla_x \cdot g(\xi)) \otimes \left((\nabla_x \otimes \Theta_{\bar{s}}(x)) \cdot g(\xi) \right) \right]_{\substack{x=x_t \\ \xi=x_t}} \end{aligned} \quad (77)$$

where $(\nabla_x \cdot g(\xi)) \otimes$ the differential operator which applied to a function $\eta(x) : \mathbb{R}^n \mapsto \mathbb{R}^{n \times s}$ is such that:

$$\begin{aligned} (\nabla_x \cdot g(\xi)) \otimes \eta(x) &= [(\nabla_x \cdot g_1(\xi)) \otimes \eta(x) \ \cdots \ (\nabla_x \cdot g_s(\xi)) \otimes \eta(x)] \end{aligned} \quad (78)$$

with:

$$(\nabla_x \cdot g_i(\xi)) \otimes \eta(x) = \sum_{j=1}^n g_{ji}(\xi) \frac{\partial \eta(x)}{\partial x_j}. \quad (79)$$

Then, according to the *strong* relative degree property **H0**), it is:

$$(\nabla_x \otimes \Theta_{\bar{s}}(x)) \cdot g(\xi) = \begin{bmatrix} B_b^{s_1} \left(\frac{d}{dx} L_\phi^{s_1-1} h_1(x) \right) g(\xi) \\ \vdots \\ B_b^{s_q} \left(\frac{d}{dx} L_\phi^{s_q-1} h_q(x) \right) g(\xi) \end{bmatrix} \quad (80)$$

so that, by taking into account that $\nabla_x \otimes$ does not differentiate $g(\xi)$, it is:

$$\begin{aligned} & \left[(\nabla_x \cdot g_i(\xi)) \otimes \left((\nabla_x \otimes \Theta_{\bar{s}}(x)) \cdot g(\xi) \right) \right]_{\substack{x=x_t \\ \xi=x_t}} \\ &= \begin{bmatrix} B_b^{s_1} \sum_{j=1}^n g_{ji}(x_t) \left[\frac{\partial}{\partial x_j} \frac{d}{dx} L_\phi^{s_1-1} h_1(x) \right]_{x=x_t} g(x_t) \\ \vdots \\ B_b^{s_q} \sum_{j=1}^n g_{ji}(x_t) \left[\frac{\partial}{\partial x_j} \frac{d}{dx} L_\phi^{s_q-1} h_q(x) \right]_{x=x_t} g(x_t) \end{bmatrix} \\ &= \begin{bmatrix} B_b^{s_1} g_i^T(x_t) H_{\phi h_1}(x_t) g(x_t) \\ \vdots \\ B_b^{s_q} g_i^T(x_t) H_{\phi h_q}(x_t) g(x_t) \end{bmatrix}. \quad (81) \end{aligned}$$

By exploiting the following property of the stack of suitably dimensioned matrices (recall that the stack of a matrix $A \in \mathbb{R}^{r \times c}$ is the vector in $\mathbb{R}^{r \cdot c}$ that piles up all the columns of matrix A , and is denoted by $\text{st}(A)$, [9]):

$$\text{st}(A \cdot B \cdot C) = (C^T \otimes A) \cdot \text{st}(B) \quad (82)$$

we have

$$\begin{aligned} & (\nabla_x^{[2]} \otimes \Theta_{\bar{s}}) \Big|_{x_t} \tilde{g}_2(x_t) \\ &= \begin{bmatrix} B_b^{s_1} [g_1^T H_{\phi h_1} g \quad \cdots \quad g_s^T H_{\phi h_1} g] \\ \vdots \\ B_b^{s_q} [g_1^T H_{\phi h_q} g \quad \cdots \quad g_s^T H_{\phi h_q} g] \end{bmatrix} F_0 \\ &= \mathcal{B}_{\bar{s}} \begin{bmatrix} \left(\text{st}(g^T H_{\phi h_1} g) \right)^T F_0 \\ \vdots \\ \left(\text{st}(g^T H_{\phi h_q} g) \right)^T F_0 \end{bmatrix} = \mathcal{B}_{\bar{s}} \begin{bmatrix} F_0^T \text{st}(g^T H_{\phi h_1} g) \\ \vdots \\ F_0^T \text{st}(g^T H_{\phi h_q} g) \end{bmatrix} \\ &= \mathcal{B}_{\bar{s}} \begin{bmatrix} \sum_{i=1}^p (F_i^T \otimes F_i^T) (g^T \otimes g^T) \text{st}(H_{\phi h_1}) \\ \vdots \\ \sum_{i=1}^p (F_i^T \otimes F_i^T) (g^T \otimes g^T) \text{st}(H_{\phi h_q}) \end{bmatrix} \end{aligned}$$

$$= \mathcal{B}_{\bar{s}} \begin{bmatrix} \sum_{i=1}^p (F_i^T g^T)^{[2]} \text{st}(H_{\phi h_1}) \\ \vdots \\ \sum_{i=1}^p (F_i^T g^T)^{[2]} \text{st}(H_{\phi h_q}) \end{bmatrix}. \quad (83)$$

Finally, by exploiting again (82), it is:

$$\begin{aligned} & (\nabla_x^{[2]} \otimes \Theta) \Big|_{x_t} \tilde{g}_2(x_t) = \mathcal{B}_{\bar{s}} \begin{bmatrix} \sum_{i=1}^p \text{st}(F_i^T g^T H_{\phi h_1} g F_i) \\ \vdots \\ \sum_{i=1}^p \text{st}(F_i^T g^T H_{\phi h_q} g F_i) \end{bmatrix} \\ &= \mathcal{B}_{\bar{s}} \begin{bmatrix} \sum_{i=1}^p F_i^T g^T H_{\phi h_1} g F_i \\ \vdots \\ \sum_{i=1}^p F_i^T g^T H_{\phi h_q} g F_i \end{bmatrix} \quad (84) \end{aligned}$$

which completes the proof. \square

Proof of Theorem 2. According to **H1**), $\Theta_{\bar{s}}(\cdot)$ is an observability map, and we write the high-gain estimator for z_t as in (18). According to hypothesis **H0**), Lemma 1 holds and for the error $\epsilon_t = z_t - \hat{z}_t$ we have

$$\begin{aligned} d\epsilon_t &= (\mathcal{A}_{\bar{s}} - \mathcal{K}\mathcal{C}_{\bar{s}}) \epsilon_t dt + \mathcal{B}_{\bar{s}} \Delta_\epsilon(z_t, \hat{z}_t, u_t) dt \\ &\quad + \mathcal{B}_{\bar{s}} L_g L_\phi^{\bar{s}-1} h(\Theta_{\bar{s}}^{-1}(z_t)) F dW_t^1 - \mathcal{K} G dW_t^2, \end{aligned} \quad (85)$$

where the following notation has been adopted:

$$\begin{aligned} \Delta_\epsilon(z_t, \hat{z}_t, u_t) &= L_\phi^{\bar{s}} h(\Theta_{\bar{s}}^{-1}(z_t)) + L_g L_\phi^{\bar{s}-1} h(\Theta_{\bar{s}}^{-1}(z_t)) u_t \\ &\quad - L_\phi^{\bar{s}} h(\Theta_{\bar{s}}^{-1}(\hat{z}_t)) - L_g L_\phi^{\bar{s}-1} h(\Theta_{\bar{s}}^{-1}(\hat{z}_t)) u_t \\ &\quad + \frac{1}{2} \mathcal{H}_{\phi h}(\Theta_{\bar{s}}^{-1}(z_t)) - \frac{1}{2} \mathcal{H}_{\phi h}(\Theta_{\bar{s}}^{-1}(\hat{z}_t)). \end{aligned} \quad (86)$$

According to the Brunowski matrices definition, each pair $(A_b^{s_j}, C_b^{s_j})$, $j = 1, \dots, q$, constitutes an observable pair, so that matrices K_j can be set in order to choose negative real and distinct eigenvalues for $A_b^{s_j} - K_j C_b^{s_j}$. Denote such spectra as $\sigma_j = \{\lambda_{ji}, i = 1, \dots, s_j\}$, according to which the Vandermonde matrices are defined:

$$V_j(\sigma_j) = \begin{bmatrix} \lambda_{j1}^{s_j-1} & \cdots & \lambda_{j1} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \lambda_{js_j}^{s_j-1} & \cdots & \lambda_{js_j} & 1 \end{bmatrix}. \quad (87)$$

By definition, $V_j(\sigma_j)$ are nonsingular matrices, which diagonalize $A_b^{s_j} - K_j C_b^{s_j}$, that means:

$$\Lambda_j = \text{diag}_{i=1}^{s_j} \{\lambda_{ji}\} = V_j(\sigma_j)(A_b^{s_j} - K_j C_b^{s_j})V_j^{-1}(\sigma_j). \quad (88)$$

By setting the observer gain \mathcal{K} as in (14), taking into account the block-diagonal form of the pair $(\mathcal{A}_{\bar{s}}, \mathcal{C}_{\bar{s}})$, we can define the following coordinate transformation:

$$\tilde{\epsilon}_t = \mathcal{V}(\lambda)\epsilon_t, \quad \mathcal{V}(\lambda) = \text{diag}_{j=1}^q \{V_j(\sigma_j)\} \quad (89)$$

which diagonalizes $\mathcal{A}_{\bar{s}} - \mathcal{K}\mathcal{C}_{\bar{s}}$, that is

$$\Lambda = \text{diag}_{j=1}^q \{\Lambda_j\} = \mathcal{V}(\lambda)(\mathcal{A}_{\bar{s}} - \mathcal{K}\mathcal{C}_{\bar{s}})\mathcal{V}^{-1}(\lambda), \quad (90)$$

so that, according to (85) the $\tilde{\epsilon}_t$ dynamics is given by:

$$\begin{aligned} d\tilde{\epsilon}_t &= \Lambda\tilde{\epsilon}_t dt + \mathbf{1}\Delta_\epsilon(z_t, \hat{z}_t, u_t)dt \\ &+ \mathbf{1}L_g L_\phi^{\bar{s}-1} h(\Theta_{\bar{s}}^{-1}(z_t)) F dW_t^1 - \mathcal{V}\mathcal{K}G dW_t^2, \end{aligned} \quad (91)$$

where $\mathbf{1} = \text{diag}_{j=1}^q \{\mathbf{1}^{s_j}\}$ and $\mathbf{1}^{s_j} = [1 \ 1 \ \dots \ 1]^T$ is the s_j -dimensional column vector of ones. Then, the integral equation associated to (91) is:

$$\begin{aligned} \tilde{\epsilon}_t &= e^{\Lambda t} \tilde{\epsilon}_0 + \int_0^t e^{\Lambda(t-\tau)} \mathbf{1}\Delta_\epsilon(z_\tau, \hat{z}_\tau, u_\tau) d\tau \\ &+ \sum_{i=1}^p \int_0^t e^{\Lambda(t-\tau)} \mathbf{1}L_g L_\phi^{\bar{s}-1} h(\Theta_{\bar{s}}^{-1}(z_\tau)) F_i dW_{\tau,i}^1 \\ &- \sum_{j=1}^q \int_0^t e^{\Lambda(t-\tau)} \mathcal{V}(\lambda)\mathcal{K}G_j dW_{\tau,j}^2 \end{aligned} \quad (92)$$

where $W_{t,i}^1$, $i = 1, \dots, p$, is the i -th component of vector W_t^1 , $W_{t,j}^2$, $j = 1, \dots, q$, is the j -th component of vector W_t^2 and G_j is the j th column of matrix G . Since dW_t^1 and dW_t^2 are uncorrelated, the expected value of the

square norm of $\tilde{\epsilon}_t$ can be written as

$$\begin{aligned} \mathbb{E}[\|\tilde{\epsilon}_t\|^2] &= \mathbb{E}[\tilde{\epsilon}_t^T \tilde{\epsilon}_t] \leq \mathbb{E}[\|e^{\Lambda t} \tilde{\epsilon}_0\|^2] \\ &+ 2\mathbb{E}\left[\|e^{\Lambda t} \tilde{\epsilon}_0\| \cdot \int_0^t \|e^{\Lambda(t-\tau)} \mathbf{1}\Delta_\epsilon(z_\tau, \hat{z}_\tau, u_\tau)\| d\tau\right] \\ &+ \mathbb{E}\left[\int_0^t \int_0^t \|e^{\Lambda(t-\tau)} \mathbf{1}\Delta_\epsilon(z_\tau, \hat{z}_\tau, u_\tau)\| \right. \\ &\quad \left. \cdot \|e^{\Lambda(t-\theta)} \mathbf{1}\Delta_\epsilon(z_\theta, \hat{z}_\theta, u_\theta)\| d\tau d\theta\right] \\ &+ \mathbb{E}\left[\sum_{i=1}^p \int_0^t \|e^{\Lambda(t-\tau)} \mathbf{1}L_g L_\phi^{\bar{s}-1} h(\Theta_{\bar{s}}^{-1}(z_\tau)) F_i\|^2 d\tau\right] \\ &+ \sum_{j=1}^q \int_0^t \|e^{\Lambda(t-\tau)} \mathcal{V}(\lambda)\mathcal{K}G_j\|^2 d\tau. \end{aligned} \quad (93)$$

Let us take into account the terms in the right hand side of (93). As far as the first term, it is:

$$\mathbb{E}[\|e^{\Lambda t} \tilde{\epsilon}_0\|^2] \leq e^{2\lambda_M t} \mathbb{E}[\|\tilde{\epsilon}_0\|^2], \quad (94)$$

where λ_M denotes the largest eigenvalue of Λ .

As far as the second term in the right hand side of eq.(93), note that hypotheses **H1**) and **H3**) imply that $L_\phi^{\bar{s}} h(\Theta_{\bar{s}}^{-1}(z))$, $L_g L_\phi^{\bar{s}-1} h(\Theta_{\bar{s}}^{-1}(z))$ and $\mathcal{H}_{\phi h}(\Theta_{\bar{s}}^{-1}(z))$ are Lipschitz with Lipschitz constants $\gamma_2\gamma_\theta$, $\gamma_3\gamma_\theta$ and $\gamma_4\gamma_\theta$, respectively. Indeed:

$$\begin{aligned} &\|L_\phi^{\bar{s}}(\Theta_{\bar{s}}^{-1}(z_1)) - L_\phi^{\bar{s}}(\Theta_{\bar{s}}^{-1}(z_2))\| \\ &\leq \gamma_2 \|\Theta_{\bar{s}}^{-1}(z_1) - \Theta_{\bar{s}}^{-1}(z_2)\| \leq \gamma_2\gamma_\theta \|z_1 - z_2\| \end{aligned} \quad (95)$$

and analogously for the other two functions. Therefore, according also to hypothesis **H4**), it is

$$\|\Delta_\epsilon(z_t, \hat{z}_t, u_t)\| \leq \gamma_\Delta \|\epsilon_t\|^2 \quad (96)$$

with

$$\gamma_\Delta = \gamma_\theta \left(\gamma_2 + \gamma_3 U_M + \frac{\gamma_4}{2} \right). \quad (97)$$

Moreover, note that, by setting $s_M = \max\{s_1, \dots, s_q\}$ and by exploiting the $\mathbf{1}$ definition, it is $\|\mathbf{1}\| = \sqrt{s_M}$.

Then:

$$\begin{aligned}
& 2\mathbb{E} \left[\left\| e^{\Lambda t} \tilde{\epsilon}_0 \right\| \cdot \int_0^t \left\| e^{\Lambda(t-\tau)} \mathbf{I} \Delta_\epsilon(z_\tau, \hat{z}_\tau, u_\tau) \right\| d\tau \right] \\
& \leq 2\mathbb{E} \left[e^{\lambda_M t} \|\tilde{\epsilon}_0\| \int_0^t e^{\lambda_M(t-\tau)} \sqrt{s_M} \gamma_\Delta \|\mathcal{V}^{-1}(\lambda)\| \|\tilde{\epsilon}_\tau\| d\tau \right] \\
& \leq \sqrt{s_M} \gamma_\Delta \|\mathcal{V}^{-1}(\lambda)\| e^{\lambda_M t} \int_0^t e^{\lambda_M(t-\tau)} \\
& \quad \cdot \left(\mathbb{E} [\|\tilde{\epsilon}_0\|^2] + \mathbb{E} [\|\tilde{\epsilon}_\tau\|^2] \right) d\tau \\
& \leq \sqrt{s_M} \gamma_\Delta \|\mathcal{V}^{-1}(\lambda)\| e^{\lambda_M t} \left(\frac{1-e^{\lambda_M t}}{|\lambda_M|} \mathbb{E} [\|\tilde{\epsilon}_0\|^2] \right. \\
& \quad \left. + \int_0^t e^{\lambda_M(t-\tau)} \mathbb{E} [\|\tilde{\epsilon}_\tau\|^2] d\tau \right) \\
& \leq \sqrt{s_M} \gamma_\Delta \|\mathcal{V}^{-1}(\lambda)\| e^{\lambda_M t} \left(\frac{\mathbb{E} [\|\tilde{\epsilon}_0\|^2]}{|\lambda_M|} \right. \\
& \quad \left. + \int_0^t e^{\lambda_M(t-\tau)} \mathbb{E} [\|\tilde{\epsilon}_\tau\|^2] d\tau \right), \tag{98}
\end{aligned}$$

where the property $2\|a\| \cdot \|b\| \leq \|a\|^2 + \|b\|^2$ has been used. As for the third term in the right hand side of eq.(93), it is:

$$\begin{aligned}
& \mathbb{E} \left[\int_0^t \int_0^t \left\| e^{\Lambda(t-\tau)} \mathbf{I} \Delta_\epsilon(z_\tau, \hat{z}_\tau, u_\tau) \right\| \right. \\
& \quad \left. \cdot \left\| e^{\Lambda(t-\theta)} \mathbf{I} \Delta_\epsilon(z_\theta, \hat{z}_\theta, u_\theta) \right\| d\tau d\theta \right] \\
& \leq \int_0^t \int_0^t e^{\lambda_M(t-\tau)} e^{\lambda_M(t-\theta)} s_M \gamma_\Delta^2 \|\mathcal{V}^{-1}(\lambda)\|^2 \\
& \quad \cdot \mathbb{E} [\|\tilde{\epsilon}_\tau\| \cdot \|\tilde{\epsilon}_\theta\|] d\theta d\tau \\
& \leq \frac{s_M \gamma_\Delta^2 \|\mathcal{V}^{-1}(\lambda)\|^2}{2} \int_0^t \int_0^t e^{\lambda_M(t-\tau)} e^{\lambda_M(t-\theta)} \\
& \quad \cdot \left(\mathbb{E} [\|\tilde{\epsilon}_\tau\|^2] + \mathbb{E} [\|\tilde{\epsilon}_\theta\|^2] \right) d\theta d\tau \\
& \leq s_M \gamma_\Delta^2 \|\mathcal{V}^{-1}(\lambda)\|^2 \frac{1-e^{\lambda_M t}}{|\lambda_M|} \int_0^t e^{\lambda_M(t-\tau)} \mathbb{E} [\|\tilde{\epsilon}_\tau\|^2] d\tau \\
& \leq \frac{s_M \gamma_\Delta^2 \|\mathcal{V}^{-1}(\lambda)\|^2}{|\lambda_M|} \int_0^t e^{\lambda_M(t-\tau)} \mathbb{E} [\|\tilde{\epsilon}_\tau\|^2] d\tau. \tag{99}
\end{aligned}$$

As for the fourth term in the right hand side of eq.(93), according to hypothesis **H2**), it is:

$$\mathbb{E} \left[\sum_{i=1}^p \int_0^t \left\| e^{\Lambda(t-\tau)} \mathbf{I} L_g L_\phi^{\bar{s}-1} h(\Theta_{\bar{s}}^{-1}(z_\tau)) F_i \right\|^2 d\tau \right]$$

$$\begin{aligned}
& \leq \sum_{i=1}^p \int_0^t s_M e^{2\lambda_M(t-\tau)} \mathbb{E} \left[\left\| L_g L_\phi^{\bar{s}-1} h(\Theta_{\bar{s}}^{-1}(z_\tau)) \right\|^2 \right] d\tau \|F_i\|^2 \\
& \leq \gamma_1 s_M \int_0^t e^{2\lambda_M(t-\tau)} d\tau \cdot \sum_{i=1}^p \|F_i\|^2 \\
& = \frac{\gamma_1 s_M (1-e^{2\lambda_M t})}{2|\lambda_M|} \sum_{i=1}^p \|F_i\|^2 \leq \frac{\gamma_1 s_M}{2|\lambda_M|} \sum_{i=1}^p \|F_i\|^2, \tag{100}
\end{aligned}$$

and, finally, for the fifth term in the right hand side of eq.(93) it is:

$$\begin{aligned}
& \sum_{j=1}^q \int_0^t \left\| e^{\Lambda(t-\tau)} \mathcal{V}(\lambda) \mathcal{K} G_j \right\|^2 d\tau \\
& \leq \int_0^t e^{2\lambda_M(t-\tau)} \|\mathcal{V}(\lambda) \mathcal{K}\|^2 d\tau \sum_{j=1}^q \|G_j\|^2 \tag{101} \\
& \leq \frac{\|\mathcal{V}(\lambda) \mathcal{K}\|^2}{2|\lambda_M|} \sum_{j=1}^q \|G_j\|^2.
\end{aligned}$$

By re-arranging the inequalities (94,98-101) we have

$$\begin{aligned}
\mathbb{E} [\|\tilde{\epsilon}_t\|^2] & \leq e^{\lambda_M t} \left(1 + \frac{\sqrt{s_M} \gamma_\Delta \|\mathcal{V}^{-1}(\lambda)\|}{|\lambda_M|} \right) \mathbb{E} [\|\tilde{\epsilon}_0\|^2] \\
& + \left(\sqrt{s_M} \gamma_\Delta \|\mathcal{V}^{-1}(\lambda)\|^2 + \frac{s_M \gamma_\Delta^2 \|\mathcal{V}^{-1}(\lambda)\|^2}{|\lambda_M|} \right) \\
& \cdot \int_0^t e^{\lambda_M(t-\tau)} \mathbb{E} [\|\tilde{\epsilon}_\tau\|^2] d\tau \\
& + \frac{s_M \gamma_1 \sum_{i=1}^p \|F_i\|^2 + \|\mathcal{V}(\lambda) \mathcal{K}\|^2 \sum_{j=1}^q \|G_j\|^2}{2|\lambda_M|}. \tag{102}
\end{aligned}$$

Note that, by denoting $\beta(t) = e^{-\lambda_M t} \mathbb{E} [\|\tilde{\epsilon}_t\|^2]$, it is:

$$\beta(t) \leq \alpha_1 + \alpha_2 e^{-\lambda_M t} + \alpha_3 \int_0^t \beta(\tau) d\tau \tag{103}$$

with

$$\begin{aligned}
\alpha_1 & = \left(1 + \frac{\sqrt{s_M} \gamma_\Delta \|\mathcal{V}^{-1}(\lambda)\|}{|\lambda_M|} \right) \mathbb{E} [\|\tilde{\epsilon}_0\|^2] \\
\alpha_2 & = \frac{s_M \gamma_1 \sum_{i=1}^p \|F_i\|^2 + \|\mathcal{V}(\lambda) \mathcal{K}\|^2 \sum_{j=1}^q \|G_j\|^2}{2|\lambda_M|} \\
\alpha_3 & = \sqrt{s_M} \gamma_\Delta \|\mathcal{V}^{-1}(\lambda)\|^2 + \frac{s_M \gamma_\Delta^2 \|\mathcal{V}^{-1}(\lambda)\|^2}{|\lambda_M|} \tag{104}
\end{aligned}$$

By applying the Gronwall's Lemma we have

$$\begin{aligned}\beta(t) &\leq \alpha_1 + \alpha_2 e^{-\lambda_M t} + \int_0^t \alpha_3 (\alpha_1 + \alpha_2 e^{-\lambda_M s}) e^{\int_s^t \alpha_3 d\tau} ds \\ &= \alpha_1 + \alpha_2 e^{-\lambda_M t} + \alpha_1 (e^{\alpha_3 t} - 1) \\ &\quad + \frac{\alpha_2 \alpha_3}{\alpha_3 + \lambda_M} (e^{(\alpha_3 + \lambda_M)t} - 1) e^{-\lambda_M t},\end{aligned}\quad (105)$$

from which

$$\begin{aligned}\mathbb{E}[\|\tilde{\epsilon}_t\|^2] &\leq \alpha_1 e^{\lambda_M t} + \alpha_2 + \alpha_1 (e^{\alpha_3 t} - 1) e^{\lambda_M t} \\ &\quad + \frac{\alpha_2 \alpha_3}{\alpha_3 + \lambda_M} (e^{(\alpha_3 + \lambda_M)t} - 1).\end{aligned}\quad (106)$$

Now, if $\alpha_3 + \lambda_M < 0$, we finally have

$$\mathbb{E}[\|\tilde{\epsilon}_t\|^2] \leq 2\alpha_1 + \alpha_2 + \frac{\alpha_2 \alpha_3}{|\alpha_3 + \lambda_M|}.\quad (107)$$

Notice that the requirement $\alpha_3 + \lambda_M < 0$ is equivalent to

$$\sqrt{s_M} \gamma_\Delta \|\mathcal{V}^{-1}(\lambda)\|^2 \left(1 + \frac{\sqrt{s_M} \gamma_\Delta}{|\lambda_M|}\right) + \lambda_M < 0,\quad (108)$$

and since it is possible (see [11]) to fix λ_M and choose the remaining eigenvalues λ_i in order to have $\|\mathcal{V}^{-1}(\lambda)\|$ arbitrarily close to 1, the left-hand side of (108) can assume any prescribed negative value. This way it is shown that there exists a bound for $\mathbb{E}[\|\tilde{\epsilon}_t\|^2]$, eq.(107), and, therefore, the z_t -observer defined in (18) has a bounded error in the mean square sense, with bound given by:

$$\mathbb{E}[\|\epsilon_t\|^2] \leq \|\mathcal{V}^{-1}(\lambda)\|^2 \left(2\alpha_1 + \alpha_2 + \frac{\alpha_2 \alpha_3}{|\alpha_3 + \lambda_M|}\right).\quad (109)$$

As far as the x_t -observer defined in (9-14) is concerned, let us first show that its equations comes out by coupling eq.(18) with $\hat{x}_t^o = \Theta_{\bar{s}}^{-1}(\hat{z}_t)$. Indeed:

$$d\hat{x}_t^o = \left. \frac{d\Theta_{\bar{s}}^{-1}}{dz} \right|_{z=\Theta_{\bar{s}}(\hat{x}_t^o)} \cdot d\hat{z}_t.\quad (110)$$

Since, by exploiting (18):

$$\left. \frac{d\Theta_{\bar{s}}^{-1}}{dz} \right|_{z=\Theta_{\bar{s}}(\hat{x}_t^o)} = \left[\left. \frac{d\Theta_{\bar{s}}}{dx} \right]^{-1} \right|_{\hat{x}_t^o} = Q^{-1}(\hat{x}_t^o),\quad (111)$$

it is:

$$\begin{aligned}d\hat{x}_t^o &= Q^{-1}(\hat{x}_t^o) \left(\mathcal{A}_{\bar{s}} \Theta_{\bar{s}}(\hat{x}_t^o) + \mathcal{B}_{\bar{s}} L_{\phi}^{\bar{s}} h(\hat{x}_t^o) \right. \\ &\quad \left. + \mathcal{B}_{\bar{s}} L_g L_{\phi}^{\bar{s}-1} h(\hat{x}_t^o) u_t \right) dt + \frac{1}{2} Q^{-1}(\hat{x}_t^o) \mathcal{B}_{\bar{s}} \mathcal{H}_{\phi h}(\hat{x}_t^o) dt \\ &\quad + Q^{-1}(\hat{x}_t^o) \mathcal{K} (dy_t - \mathcal{C}_{\bar{s}} \Theta_{\bar{s}}(\hat{x}_t^o) dt).\end{aligned}\quad (112)$$

By suitably exploiting the full relative degree property, it is from (74-75):

$$Q(x)\phi(x) = \mathcal{A}_{\bar{s}} \Theta_{\bar{s}}(x) + \mathcal{B}_{\bar{s}} L_{\phi}^{\bar{s}} h(x)\quad (113)$$

and

$$Q(x)g(x) = \mathcal{B}_{\bar{s}} L_g L_{\phi}^{\bar{s}-1} h(x)\quad (114)$$

so that the x_t -observer equations (9) are readily obtained.

Finally, since $\Theta_{\bar{s}}^{-1}(\cdot)$ is Lipschitz,

$$\mathbb{E}[\|x_t - \hat{x}_t^o\|^2] \leq \gamma_{\theta}^2 \mathbb{E}[\|z_t - \hat{z}_t\|^2] \leq L\quad (115)$$

with

$$L = \gamma_{\theta}^2 \|\mathcal{V}^{-1}(\lambda)\|^2 \left(2\alpha_1 + \alpha_2 + \frac{\alpha_2 \alpha_3}{|\alpha_3 + \lambda_M|}\right),\quad (116)$$

which completes the proof. \square

References

- [1] J.H. Ahrens and H.K. Khalil, High gain observers in the presence of measurement noise: a switched-gain approach. *Automatica*, 45 (4), 936–943, 2009.
- [2] V. Andrieu, L. Praly and A. Astolfi, High gain observers with updated gain and homogeneous correction term. *Automatica*, 45 (2), 422–428, 2009.
- [3] M. Basin, D. Calderon-Alvarez, Optimal filtering for incompletely measured polynomial systems with multiplicative noise *Circuits Syst Signal Process*, 28, 223–239, 2009.
- [4] N. Boizot, E. Busvelle, and J.-P. Gauthier, An adaptive high-gain observer for nonlinear systems. *Automatica*, 46 (9), 1483–1488, 2010.
- [5] R. Bucy and E. Jonckheere. Singular filtering problems. *Syst. Contr. Lett.*, 13:339–344, 1989.
- [6] R. Bucy and P. Joseph, *Filtering for Stochastic Processes with Applications to Guidance*, New York: Wiley, 1968.
- [7] F. Cacace, A. Germani, and P. Palumbo, A New Approach to Nonlinear Filtering Via a Mixed State Observer and Polynomial Scheme. *Proceedings of the 18th IFAC World Congress*, Vol. 18, Part 1, 4477–4482, 2011.
- [8] F. Cacace, A. Germani, and P. Palumbo, A state observer approach to filter stochastic nonlinear differential systems. *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, 7917–7922, 2011.
- [9] F. Carravetta, A. Germani, and M. Raimondi, Polynomial filtering of discrete-time stochastic linear systems with multiplicative state noise, *IEEE Trans. Autom. Contr.*, 42, 1106–1126, 1997.
- [10] F. Carravetta, A. Germani, M.K. Shuakayev. A new suboptimal approach to the filtering problem for bilinear stochastic differential systems. *SIAM J. Contr. Optim.*, 38: 1171–1203, 2000.
- [11] G. Ciccarella, M. Dalla Mora, A. Germani. A Luenberger-like observer for nonlinear systems. *Int. J. Contr.*, Vol., 57: 537–556, 1993.

- [12] M. Dalla Mora, A. Germani, C. Manes. Design of state observers from a drift-observability property *IEEE Trans. Autom. Control*, 45:1536–1540, 2000.
- [13] M.P. Di Ciccio, A. Germani, P. Pepe. A filter for nonlinear systems by means of deterministic nonlinear observers *International Journal of Pure and Applied Mathematics*, 49: 317–322, 2008.
- [14] G.B. Di Masi, M. Pratelli, W.J. Runggaldier, An Approximation for the Nonlinear Filtering Problem, with Error Bound, *Stochastics*, 14,247–271, 1985.
- [15] R.J. Elliot and R. Glowinski, Approximations to solutions of the Zakai filtering equation, *Stoch. Anal. Appl.*, 7(2), 145–168, 1989.
- [16] J.P. Gauthier, H. Hammouri, and S. Othman, A simple observer for nonlinear systems: Application to bioreactors, *IEEE Trans. Automat. Contr.*, vol. 37, 875-880, 1992.
- [17] A. Germani, C. Manes, P. Palumbo Polynomial Extended Kalman Filter. *IEEE Trans. Autom. Control*, 50:2059–2064, 2005.
- [18] A. Germani, C. Manes, P. Palumbo. Filtering of stochastic differential systems via a Carleman approximation approach *IEEE Trans. Autom. Control*, 52:2166–2172, 2007.
- [19] D.J. Highman An algorithmic introduction to numerical simulation of stochastic differential equations *SIAM Review*, 43:525–546, 2001.
- [20] K. Ito, Approximation of the Zakai equation for nonlinear filtering theory, *SIAM J. Control Optim.*, 34(2), 620–634, 1996.
- [21] K. Kowalski and W.H. Steeb, *Nonlinear Dynamical Systems and Carleman Linearization*, World Scientific, Singapore, 1991.
- [22] H.J. Kushner, Nonlinear filtering: the exact dynamical equations satisfied by the conditional models, *IEEE Trans. Autom. Contr.*, 12(3),262–267, 1967.
- [23] R. S. Liptser and A. N. Shiriyayev, *Statistics of Random Processes I and II*, Berlin, Springer, 1977.
- [24] E. Pardoux, Stochastic partial differential equations and filtering of diffusion processes, *Stochastics.*, 3,127–167, 1979.
- [25] A. N. Phillips, Reduction of HIV concentration during acute infection: independence from a specific immune response, *Science*, 271 (5248), pp. 497–499, 1996.
- [26] S. Sastry, *Nonlinear systems: Analysis, Stability and Control*, Springer-Verlag, New-York, 1999.
- [27] W.S. Wong and S.T. Yau, The estimation algebra of nonlinear filtering systems, *Mathematical Control Theory*, J. Baillieul and J. C. Willems eds., Springer Verlag, 33–65, 1999.
- [28] M. Zakai, On the optimal filtering of diffusion processes *Z. Wahrschein. Verw. Geb.*, 11,230–243, 1969.