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ON THE EQUIVALENCE BETWEEN NONLINEAR- AND FRACTIONAL BILINEAR-CONTROL SYSTEMS

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Abstract

Stated in a few word, the aim of this paper is to give evidence to the following author's conjecture: 'any' nonlinear control system is equivalent to a (larger dimensioned) bilinear fractional differential system. The main purpose is to motivate a new approach in nonlinear control, and for this reason, in this paper, some simple examples, nevertheless yet meaningful, are given, where the above conjecture holds. Starting with a simple scalar example, in order to present the basic feature of the method, the paper is endowed with a case consisting in a two dimensional control system, which is nevertheless amenable to be readily generalized to a general state-space dimension. A sub-result of this paper is interesting by itself: for classical polynomial systems, where just positive integers powers are involved, the result holds always, and the equivalent system result in an ordinary (non-fractional) bilinear system.

1. Introduction

In this work my aim is, first of all, to formulate a conjecture which in my opinion has a big importance in Nonlinear Control, from both a theoretical than a practical point of view, and second, to give evidence that the conjecture is actually true in some meaningful cases. I express the conjecture with the following, somewhat rough, statement: 'any' nonlinear control system in the form:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \tag{1}$$

is equivalent to a *bilinear* system, possibly *fractionary*, of the following kind:

$$\mathbf{D}z(t) = L(z(t))z(t) + B(z(t), u(t)), \tag{2}$$

where, $x(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^N$, (with $N \geq n$ in general), $L(\cdot)$ is a matrix-valued linear map, $B(\cdot,\cdot)$ is bilinear, $z(t) = \Phi(x(t))$, Φ an invertible map, and $\mathbf{D}$ is a row-vector of fractional-differential operators, namely $D^{r_1}, ..., D^{r_N}$, where $-1 \leq r_i \leq 1$ are real numbers for $i = 1, ..., N$, denoting the *order* of the derivative, with the noticeable particular cases of $r_i = 1, -1$ which correspond to a ordinary first derivative, and an ordinary Riemann integral, respectively.
By saying 'any' I actually mean 'to a large extent', and in particular to such an extent covering all cases of interest from an engineering application point of view. The system functions f, g are supposed to be infinitely differentiable in an open set of the state space, and possibly additional conditions are supposed verified in order to guarantee the existence of an unique solution for any initial condition in such open set.
In the present paper the above conjecture will be proven true in some meaningful particular cases: for f polynomial, g constant, and for a second order system, and in a simple scalar case with $f = x^\alpha$, with α any positive real. These cases are in some sense prototypical, in that the method I introduce for building up the above mentioned transformation Φ is easily explained in these two simple cases. Also, we will see that the bilinear system which such classes of nonlinear system are reduced to, does reveal an even more regular, and simple, structure than one can infer from (2).
Reducing a nonlinear system such as (1), which indeed is an huge class, to an unique class of structurally simpler, and allegedly yet studied, systems, is a common approach in Control Theory. To this purpose peraphs the most meaningful work is the textbook of Isidori [1], which includes all the main results in Nonlinear *geometric* Control. The first chapter of this book is devoted to 'establishing a number of interesting analogies with ... *linear* control systems.' and indeed the characterisation of a nonlinear system is carried thoughout out successfully by extending from linear-algebraic classical concepts/characterisations for linear systems (controllability, observability etc.) to analogous differential-geometric concepts for nonlinear systems. More relevant for us is the topic of 'exact linearisation', aiming to find conditions under which a nonlinear system is *equivalent* to a linear one in another state-space. Such conditions (necessary and sufficient) are indeed found, but the class of nonlinear systems undergoing that appears to be quite small for many application purposes.
In this perpective, the main idea, and the proposal, of the present paper, which also could open a new way in Nonlinear Control, is to slightly weaken the 'goal'-class of systems which have to be reduced to from nonlinear systems, in the sense of accepting bilinear, besides linear, system's functions, as well as the *order of differentiation*, by accepting to be faced with *fractional* integral/differential systems.
Fractional systems have been studied by many authors in the engineering area and have found a direct application as for electrochemical processes, viscoelasticity, colored noise, long distributed lines, and for studying chaos [2]–[10]. Fractional systems are in particular used in control, when the controlled systems and/or the controller are described by fractional differential equations. A lot of work has indeed been devoted in recent years to control and identification of fractional systems, which has becoming a growing-interest research area. As for the problem of stability/stabilisation, I point out the survey paper of Petras [11], which also includes a large set of references. Much of the nice features of ordinary linear systems, such as undergoing to frequency-response based analysis/syntesis, have resulted applyable to the fractional case as well, see for instance [14] as for the root-locus method, as well as [12], [13], [15], as for using PI and PID fractional-order controller, and [15] for stabilisation of fractional delay systems. In [2] a large account of PID controller for fractional systems, as well as of other related topics is given,

especially from an application point of view. The problem of identification has been recently studied as well to some extent. To this purpose it should be stressed that the identification issue results often in more complex problem than for integer-order case, in that even the number of frational operators, as well as the order of derivation is to be estimated from data [16]. As for the interpretation and physical meaning of fractional differentiation/integration, the article [17] can be pointed out for more insight.
As for (integer-order) bilinear systems, their feature of being, in a structural-complexity scale, the simpler class of dynamic systems after the linear ones, is a well known and long-established fact in systems and control theory. For the more classical result on control,stability, identification and structural properties of bilinear systems, I point out the paper [18] and the reference therein, for a more recent account see [19].
The article is organized as follows. In §I a brief account is given about the basic definition and properties of the concept of fractional derivatives and fractional-order differenial equation, just enough for make self-contained the paper. In §II the basic idea of the *reduction method* is explained, in simple cases. Section III is devoted to present a first generalizion of the reduction method to a more meaningful vector non-linear system. Section IV includes some conclusive remarks.

2. Some preliminaries about fractional derivatives

In this section some basic definitions and properties from fractional differential equations theory – which will be used in the following – are given. The main reference for our purposes is the textbook of Podlubny [3]. Besides the earlier attempts dating back to 18-th century, performed successfully to some extent, the issue of generalizing to real numbers the order of a derivative has undergone considerable advance in particular in past decades, as for giving a very general definition of *fractional*[1] derivative, i.e. an α-order derivative of a function $f(z)$, covering even the cases of a complex α, and for $f(z)$ being an analitic complex function of the complex variable z. Many approaches have been developed, differing from each other as for the basic definition of fractional derivative, nevertheless mostly equivalent to each other. The brief account here presented is limited to the case of real number α, and a real function, namely $f(t)$, $t \in \mathbb{R}$, having all positive-integer-order derivatives in some open interval of $\mathbb{R}$. The Gruenwald-Letnikov definition of fractional derivative will now given just below, by showing the way it is actually built up. Consider the general formula of the p-th order derivative of $f(t)$, for $p \in \mathbb{N}$:

$$f^{(p)}(t) = \lim_{h\to 0} \frac{1}{h^p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t - rh), \tag{3}$$

which is defined for any $n \geq p$, because when $r > p$, the binomial coefficient vanishes. We assume that the binomial coefficient is defined as

$$\binom{p}{r} = \frac{p(p-1)\cdots(p-r+1)}{r!}, \tag{4}$$

which is the ordinary definition, but one avoids using $p!$ therein. The advantage is that formula (4) can be still used even if p is a negative integer. By this setting the following problem gets well posed: calculate, for p negative integer, the limit in the r.h.s. of (3) as the limit for $n \to +\infty$, with $h = t/n$. It can be proven that such limit does exist, and is equal to the well known Cauchy expression for the $|p|$-fold integral:

$$f^{(p)}(t) = \frac{1}{(|p|-1)!} \int_0^t (t-\tau)^{(|p|-1)} f(\tau) d\tau, \tag{5}$$

which is exactly what a 'negative'-order derivative has to be (the folded integral). Now, consider the Gamma function:

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt, \tag{6}$$

[1] The word 'fractional' is somewhat misleading, because the concept is not actually restricted to a *rational* order, however, as Podlubny commented in his textbook, giving misleading names is a very common and recurrent fact in mathematics.

which is defined and analitic for $Re\{z\} > 0$. Also, it has the important property of being an extension of the factorial of a positive integer n, in that $\Gamma(n+1) = n!$. By using the Gamma function one can extend as well the definition of the binomial coefficient in (3) for any *real* number $p > 0$. Thus we define, for $p > 0$ real, or $p < 0$ integer, the p-order Gruenwall-Letnikov derivative of the function f, namely $D_t^p f(t) = f^{(p)}(t)$ as the r.h.s. of eq. (3), where the limit is taken in such a way (above described) that $n \to +\infty$ as $h \to 0$. It can be shown that the Gruenwall-Letnikov derivative indeed exists (under the assumptions been made, i.e. infinite differentiability of the function f) for any positive real value of p, and satisfies the following formula (see [3], p. 55):

$$D_t^p f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(0)t^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+m+1)} \int_0^t (t-\tau)^{m-p} f^{(m+1)}(\tau) d\tau, \quad (7)$$

where[2] $m = [p]$. Besides the Gruenwall-Letnikov, another definition of fractional derivative, and overall the most widely known, is the following, which we denote $\mathcal{D}_t^p$:

$$\mathcal{D}_t^p f(t) = \left(\frac{d}{dt}\right)^{m+1} \int_0^t (t-\tau)^{m-p} f(\tau) d\tau. \quad m \le p \le m+1, \quad (8)$$

which is called the *Riemann-Liouville* p-th order fractional derivative. By performing repeatedly integration by parts and differentiation in the l.h.s. of (8) one obtains the l.h.s. of (7), thus showing the equivalence between the Gruenwall-Letnikov and the Riemann-Liouville definitions.

The definition of pth order fractional derivative with p real and negative (thus actually the fractional integral), is obtained in a similar way by further extension of the definition above given in formula (5) for p negative integer, and replacing as usual $\Gamma(|p|)$ for $(|p|-1)!$.

With the p-th order derivative being defined for all real p, one can consider fractional (integral)-differential (F(I)D) equations as well, and raising the issue of existence/unicity for their solution. Important for our purposes is the so called *one-term* equation:

$$D_t^\alpha y(t) = f(t) \quad (9)$$

where α is a real positive number, and $f(t)$ is another (known) funtion. As is well known (see [3] pp. 153-154) the solution of eq. (9), for $y(0) = 0$ is given by [3]

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau) d\tau}{(t-\tau)^{1-\alpha}}. \quad (10)$$

3. A simple scalar case

In order to make clear how the reduction method works, in this section I show first the very simple case of a system described by a single, scalar, differential equation:

$$\dot{x} = x^p, \qquad x(0) = 0. \quad (11)$$

where $p \in \mathbb{N}$. Let us set :

$$Z_0 = x^p, \quad (12)$$

and

$$Z_1 = \frac{Z_0}{x}. \quad (13)$$

[2] $[\cdot]$ denotes integer part.

[3] This directly follows by formula (5) generalized to negative real numbers as described just before.

Then, taking the derivatives in (12), one has:

$$\dot{Z}_0 = px^{p-1}Z_0 = p\frac{{Z_0}^2}{x} = pZ_1Z_0. \tag{14}$$

Now, let's define

$$Z_2 = \frac{Z_0}{x^2}, \tag{15}$$

and take the derivatives in eq. (13):

$$\begin{aligned}\dot{Z}_1 &= \frac{\dot{Z}_0}{x} - \frac{{Z_0}^2}{x^2} = p\frac{Z_1Z_0}{x} - {Z_1}^2 \\ &= p{Z_1}^2 - Z_1^2 = (p-1)Z_1^2 = (p-1)\frac{Z_0^2}{x^2} \\ &= (p-1)Z_2Z_0.\end{aligned} \tag{16}$$

Once more, for more clearness: let's define

$$Z_3 = \frac{Z_0}{x^3}, \tag{17}$$

and take the derivatives in eq. (15):

$$\dot{Z}_2 = \frac{\dot{Z}_0}{x^2} - 2\frac{Z_0^2}{x^3} = p\frac{Z_0^2}{x^3} - 2\frac{Z_0^2}{x^3} = (p-2)\frac{Z_0^2}{x^3} = (p-2)Z_3Z_0. \tag{18}$$

It's clear that the above procedure can be iterated, if p is integer the last two steps are:

$$\dot{Z}_{p-1} = Z_pZ_0 \tag{19}$$

and

$$Z_p = \frac{Z_0}{x^p} = 1, \tag{20}$$

which follows by (12). Thus last differential equation, endowing the scheme, is

$$\dot{Z}_p = 0. \tag{21}$$

As a result, a $p+2$-dimensioned vector $Z^T = [x, Z_0, Z_1, ..., Z_p]^T$ gets defined, such that

$$\dot{Z} = A(Z_0)Z, \tag{22}$$

where $A(Z_0)$ is a matrix depending linearly by the 2th entry of Z, i.e. Z_0, and has the following structure:

$$A(Z_0) = \begin{bmatrix} 0 & 1 & 0 & \dots & & 0 \\ 0 & 0 & pZ_0 & \ddots & \vdots & \\ 0 & \ddots & \ddots & & & \\ & & & & 2Z_0 & 0 \\ 0 & & & \dots & 0 & Z_0 \\ 0 & & & & & 0 \end{bmatrix}. \tag{23}$$

Thus, when p is a positive integer, the simple nonlinear system (11) can indeed be reduced to a system of ordinary differential bilinear equations.
Now consider the case of a scalar system

$$\dot{x} = x^\alpha, \qquad x(0) = \bar{x} \tag{24}$$

where α is a positive *real* number. In such case we can define $p = [\alpha]$, thus $\alpha = p + r$, where $r \in [0, 1)$ is the real number denoting the remainder of the division α/p. In the following theorem it's shown that the result above proven for integer powers can be extended as well to any $\alpha > 0$, provided that the bilinear differential system which the system is reduced to is intended in the fractional sense.

Theorem 1. *The ordinary differential system (24) is equivalent to the following couple of bilinear differential systems, the first of which is an ordinary-sense one:*

$$\dot{\xi} = A'(Z_0)\xi + b(Z_0)Z_p, \qquad \xi(0) = \bar{\xi}, \tag{25}$$

where

$$A'(Z_0) = \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & (\alpha)Z_0 & \ddots & \vdots & \\ 0 & 0 & 0 & (\alpha-1)Z_0 & & \\ 0 & \ddots & \ddots & & & \\ & & & & 0 & (r+2)Z_0 \\ 0 & & & & & 0 \end{bmatrix}; \tag{26}$$

$$b(Z_0) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ (r+1)Z_0 \end{bmatrix}. \tag{27}$$

ξ is the $p-1$-dimensioned vector

$$\xi^T = [x, Z_0, ..., Z_{p-1}]^T, \tag{28}$$

and the vector $\bar{\xi}$ of initial conditions is given by:

$$\bar{\xi}^T = [\bar{x}, \bar{x}^{\alpha}, \bar{x}^{\alpha-1}, ..., \bar{x}^{r+1}]^T, \tag{29}$$

the second one is fractional, described by the following two differential equations:

$$D_t^{1-r} W = Z_p Z_1, \qquad W(0) = 0, \tag{30}$$

$$D_t^r Z_p = rW + \phi(t), \qquad Z_p(0) = \bar{x}^r, \tag{31}$$

with

$$\phi(t) = \frac{\bar{x}^r t^{-r}}{\Gamma(1-r)}, \tag{32}$$

Proof. Let us apply a similar procedure as in (12)–(20), where Z_i, for $i = 1, 2, ..., p$ has the same definition as before, that is:

$$Z_i = \frac{Z_0}{x^i}, \tag{33}$$

but now we define:

$$Z_0 = x^{\alpha}. \tag{34}$$

Thus taking the derivatives in (34), one has:

$$\dot{Z}_0 = \alpha x^{\alpha-1} Z_0 = \alpha \frac{{Z_0}^2}{x} = \alpha Z_1 Z_0. \tag{35}$$

Again, take the derivatives of Z_1:

$$\dot{Z}_1 = \frac{\dot{Z}_0}{x} - \frac{{Z_0}^2}{x^2} = \alpha \frac{Z_1 Z_0}{x} - \frac{{Z_0}^2}{x^2}$$

$$= (\alpha - 1)\frac{Z_0^2}{x^2} = (\alpha - 1) Z_2 Z_0. \tag{36}$$

Thus we have for $0 \leq i \leq p-1$:

$$\dot{Z}_i = (\alpha - i) Z_{i+1} Z_0, \tag{37}$$

but now Z_p is no more equal to the unity, and in fact, by recalling that $\alpha = p + r$, by (33) it is:

$$Z_p = x^r. \tag{38}$$

Equations (37) can be aggregated by defining the vector ξ as in (28), and eq. (25) ensues. The initial conditions follows by $Z_i = x^{\alpha - i}$, the latter coming directly from definition.
Let us calculate the r-th order derivative of Z_p in (38). This can be done using formula (7) where, on account that $0 < r < 1$, we can set $m = 0$:

$$\begin{aligned} D_t^r Z_p - \frac{\bar{x}^r t^{-r}}{\Gamma(1-r)} & \\ &= \frac{1}{\Gamma(1-r)} \int_0^t (t-r)^{-r} \frac{d}{d\tau} \left(x^r(\tau) \right) d\tau \\ &= \frac{1}{\Gamma(1-r)} \int_0^t (t-r)^{-r} r x^{r-1}(\tau) \dot{x}(\tau) d\tau \\ &= \frac{r}{\Gamma(1-r)} \int_0^t \frac{x^r(\tau) Z_0(\tau)}{(t-\tau)^r x} d\tau \\ &= \frac{r}{\Gamma(1-r)} \int_0^t \frac{Z_p(\tau) Z_1(\tau)}{(t-\tau)^r} d\tau. \end{aligned} \tag{39}$$

Now let us set:

$$W(t) = \frac{1}{\Gamma(1-r)} \int_0^t \frac{Z_p(\tau) Z_1(\tau)}{(t-\tau)^r} d\tau, \tag{40}$$

then by (10) and (9) one has (30), whereas eq. (39) becomes (31). •

It has thus proven that the scalar nonlinear system (24) is equivalent to a couple of mutually interconnected bilinear system, the first one (integer order) having state-space dimension $p+1$, and given by (25), the second one two-dimensional, fractional, and given by (30), (31. Note in particular that (31) is linear, however has a forcement funtion $\phi(t)$ which is unbounded, in that its limit for $t \to 0^+$ goes to infinity. Whether this actually constitute or not a problem is too early to say at this stage (initial) of the present research. What has to be stressed is that the original conjecture stated in the introduction, at least in the simple case of a scalar system is indeed verified in the prototypical case of a polynomial system with a positive power. More, if the power is integer, the equivalent bilinear system will be ordinary, whereas for a non-integer power – which case hugely increases the class of tested systems – such reduction does work as well, provided the addition of a couple of fractional auxiliary equations.

4. Towards the general case: a two dimensional nonlinear control system

A scalar system is of course too much simple for applications in control, thus the reduction method described in the previous section has to be developed for general systems in $\mathbb{R}^n$. To this purpose, a more complicated system function, meaningful for showing the capability in the vector case[4] of the reduction method depicted in the previous section, is now presented for integer powers. Let us consider the system (in $\mathbb{R}^2$):

$$\dot{x} = a x^p y^q + u_1, \tag{41}$$

$$\dot{y} = b x^{p-k} y^{q-m} + u_2, \tag{42}$$

[4]No particular theoretical problems indeed arise in considering a system of the same kind in $\mathbb{R}^n$, but of course this complicates a lot the calculations.

where a, b are real coefficients, u_1, u_2 are two (known) functions and p, q, k, m are integers, with $0 \leq k \leq p$, $0 \leq m \leq q$. The following theorem holds.

Theorem 2. *The two dimensional nonlinear system (41), (42), is equivalent to the following bilinear control system in the new vector state $Z(t)$ of dimension $N = (q+1)(p+1)$:*

$$\dot{Z} = \mathcal{A}(Z)Z + \big(\mathcal{B}_1 u_1 + \mathcal{B}_2 u_2\big)Z, \tag{43}$$

where $A(\cdot)$ is a bilinear map in $\mathbb{R}^{N \times N}$, with

$$\mathcal{A}(Z) = \begin{bmatrix} A(Z_0) & qZ_{k,m}I_{p+1} & 0 & \cdots \\ 0 & A(Z_0) & (q-1)Z_{k,m}I_{p+1} & \ddots \\ & \ddots & \ddots & \\ & & A(Z_0) & Z_{k,m}I_{p+1} \\ 0 & \cdots & 0 & A(Z_0) \end{bmatrix}, \tag{44}$$

with $A(\cdot) : \mathbb{R}^{q+1} \to \mathbb{R}^{(q+1)\times(q+1)}$, bilinear, and given by

$$A(Z_0) = \begin{bmatrix} apZ_{1,0} & 0 & \cdots & 0 \\ a(p-1)Z_{2,0} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ aZ_{p,0} & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{45}$$

I_n denoting the identity in $\mathbb{R}^n$, Z_r, $r = 0, ..., q$ given by

$$Z_r^T = \begin{bmatrix} Z_{0,r} & Z_{1,r} & \ldots & Z_{q,r} \end{bmatrix}^T, \tag{46}$$

and

$$\mathcal{B}_1 = \text{bldiag}\{B, ..., B\}; \tag{47}$$

$$\mathcal{B}_2 = \begin{bmatrix} 0 & q & 0 & \cdots \\ 0 & 0 & (q-1) & \ddots \\ & \ddots & \ddots & \\ & & 0 & 1 \\ 0 & \cdots & & 0 \end{bmatrix}, \tag{48}$$

with the matrix B defined as

$$B = \begin{bmatrix} 0 & p & 0 & \cdots & 0 \\ 0 & 0 & p-1 & \ddots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ & & & & 1 \\ 0 & & \cdots & & 0 \end{bmatrix}. \tag{49}$$

Moreover Z has the following structure

$$Z^T = \begin{bmatrix} Z_0^T & Z_1^T & \ldots & Z_p^T \end{bmatrix}^T \in \mathbb{R}^{(p+1)(q+1)}, \tag{50}$$

where

$$Z_{i,j} = x^{p-i}y^{q-j}. \qquad 0 \leq i \leq p; \quad 0 \leq j \leq q. \tag{51}$$

Proof. From the definition of the variables $Z_{i,j}$, given above in (51), the following relation immediately follows, for any $0 \leq s \leq p$, $0 \leq r \leq q$:

$$Z_{i+s,j} = \frac{Z_{i,j}}{x^s}; \qquad Z_{i,j+r} = \frac{Z_{i,j}}{y^r}. \tag{52}$$

Let's take the derivative of $Z_{0,r}$, with $0 \leq r \leq q$:

$$\begin{aligned}\dot{Z}_{0,r} &= px^{p-1}\dot{x}y^{q-r} + (q-r)x^p y^{q-r-1}\dot{y} \\ &= apx^p \frac{Z_{0,0}}{x} y^{q-r} + b(q-r)x^p y^{q-r-1} Z_{k,m} \\ &\quad + pZ_{1,r}u_1 + (q-r)Z_{0,r+1}u_2 \\ &= apZ_{1,0}Z_{0,r} + b(q-r)Z_{0,r+1}Z_{k,m} \\ &\quad + pZ_{1,r}u_1 + (q-r)Z_{0,r+1}u_2.\end{aligned}$$

Differentiating $Z_{1,r}$, on account of (52), gives

$$\begin{aligned}\dot{Z}_{1,r} &= \frac{d}{dt}\left(\frac{Z_{0,r}}{x}\right) = \frac{\dot{Z}_{0,r}}{x} - a\frac{Z_{0,r}Z_{0,0}}{x^2} - \frac{Z_{0,r}}{x^2}u_1 \\ &= ap\frac{Z_{1,0}Z_{0,r}}{x} + b(q-r)\frac{Z_{0,r+1}Z_{k,m}}{x} - a\frac{Z_{0,r}Z_{0,0}}{x^2} \\ &\quad - \frac{Z_{0,r}}{x^2}u_1 + p\frac{Z_{1,r}}{x}u_1 + (q-r)\frac{Z_{0,r+1}}{x}u_2 \\ &= a(p-1)Z_{2,0}Z_{0,r} + b(q-r)Z_{1,r+1}Z_{k,m} \\ &\quad + (p-1)Z_{2,r}u_1 + (q-r)Z_{1,r+1}u_2.\end{aligned}$$

Then, it's clear that the following system of differential equation can be derived:

$$\begin{aligned}\dot{Z}_{s,r} &= a(p-s)Z_{s+1,0}Z_{0,r} + b(q-r)Z_{s,r+1}Z_{k,m}, \\ &\quad + (p-s)Z_{s+1,r}u_1 + (q-r)Z_{s,r+1}u_2, \\ &\text{for} \quad 0 \leq s \leq p.\end{aligned} \tag{53}$$

Now define, for $0 \leq r \leq q$, the vector $Z_r \in \mathbb{R}^{p+1}$ as in (46), then eq.ns (53) can be aggregated as follows:

$$\dot{Z}_r = \big(A(Z_0) + Bu_1\big)Z_r + (q-r)\big(bZ_{k,m} + u_2\big)Z_{r+1}, \quad 0 \leq r \leq q-1, \tag{54}$$

$$\dot{Z}_q = A(Z_0)Z_q, \tag{55}$$

with $A(Z_0)$ given by (23), B given by (49) By defining Z as in (50) we finally obtain the bilinear system (43). •

The solution, $x(t), y(t)$ of the system (41), (42), can be strightly recovered by addressing the entries $Z_{p-1,q}(t)$, and $Z_{p,q-1}(t)$, which by (51) are equal to $x(t)$ and $y(t)$, respectively. Note that $\mathcal{A}(Z)$ depends actually only of $p+1$ components of Z, i.e.: $Z_{i,0}$, $i = 0, ..., p-1$, and $Z_{k,m}$, and is a sparse $N \times N$-matrix, such that the product $A(Z)Z$ is $O(N)$ as for the computational complexity (instead of $O(N^2)$). The same considerations holds for the $\mathcal{B}_1, \mathcal{B}_2$ matrices, so we can say that the equivalent augmented bilinear system has a basic computational complexity (where simulated) of $O(N)$ and does not increase the execution time for simulation with respect to the original nonlinear system. What is gained, and what more important, is obviously the fact of reducing a very large class of nonlinear systems to a *basic* kind of system, the bilinear one, which comes just after the linear in a scale of conceptual complexity.

5. Conclusion

The results of the paper are expressed by Theorems 1,2. Theorem 1 shows that a scalar ordinary differential equation with a system function being any positive real power of the function $x(t)$ can be always expressed as the ordinary bilinear system (25) endowed by the couple of fractional bilinear differential equations (30), (31). In Theorem 2 is proven that any two dimensional polynomial system (with integer powers) is equivalent to an ordinary bilinear system. This result is new in my knowledge, and interesting by itself. Nonetheless it represent a first step towards the desired result, proving the conjecture formulated in the introduction in a vector case: whatever are the powers (i.e. real) such nonlinear system is equivalent to a set of fractional integral/differential equations. This constitute the subject of future research.

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