



ISTITUTO DI ANALISI DEI SISTEMI ED INFORMATICA
"Antonio Ruberti"

CONSIGLIO NAZIONALE DELLE RICERCHE

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OPTIMAL SMOOTHING FOR FINITE STATE HIDDEN
RECIPROCAL PROCESSES

R. 11-02 2011

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ISSN: 1128–3378

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Abstract

This paper addresses modelling and estimation of a class of finite state random processes called hidden reciprocal chains (HRC). A hidden reciprocal chain consists of a finite state reciprocal process, together with an observation process conditioned on the reciprocal process much as in the case of a hidden Markov model (HMM). The key difference between Markov models and reciprocal models is that reciprocal models are non-causal. The paper presents a characterisation of a HRC by a finite set of hidden Markov bridges, which are HMMs with the final state fixed. The paper then uses this characterisation to derive the optimal fixed interval smoother for a HRC. Performance of linear and optimal smoothers derived for both HMM and HRC are compared (using simulations) for a class of HRC derived from underlying Markov transitions. These experiments suggest that, not surprisingly, the performance of the optimal HMM and HRC smoothers are significantly better than their linear counterparts, and that some performance improvement is obtained using the HRC smoothers compared to the HMM smoothers. The paper concludes by mentioning some ongoing and future work which exploits this new Markov bridge characterisation of a HRC.

Optimal Smoothing for Finite State Hidden Reciprocal Processes

Langford B White, *Senior Member, IEEE*, and Francesco Carravetta

I. INTRODUCTION

Reciprocal processes (RP) are one-dimensional Markov random fields (MRF), although they are not Markov processes in the usual sense. However, any Markov process is reciprocal. Reciprocal processes share the fundamental property of MRFs in that their statistical properties may be specified by a *nearest neighbour* condition. Reciprocal processes were originally studied in detail in the 1930s, in particular by S. Bernstein [1] who formulated in concept of a RP, and also by E. Schrödinger [2], who studied the quantum mechanical behaviour of an electron on a finite interval of the real line. The study of reciprocal processes became prominent in the 1960s and 1970s, when several researchers such as D. Slepian [3] and B. Jamison [4] studied Gaussian RPs. The general theory of RPs was presented by B. Jamison in [5] where important relationships between RPs and Markov processes were established.

In the last 2 decades of the twentieth century, a considerable amount of related work was published by A. Krener [6], A. Krener, R. Frezza and B. Levy [7] and M. Adams, A. Willsky and B. Levy [8] in particular. These papers dealt specifically with continuous time reciprocal processes with the underlying models being constructed from stochastic differential equations. Optimal linear estimation was addressed in [8]. An interesting example which demonstrates one significant application of RPs is described in [9] which showed how tracking with predictive information may be dealt with in the RP framework. This paper also deals with the continuous time, Gaussian case. We also note related work in physics also by B. Levy and A.

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Krener [10], [11] where relationships between reciprocal processes and quantum mechanics are explored.

In [12], a complete description of discrete time Gaussian reciprocal processes from a modelling and optimal estimation perspective is given by B. Levy, R. Frezza and A. Krener. In particular, it is shown that any Gaussian reciprocal process can be represented by a second order difference equation driven by an Gaussian moving average (MA) noise process of order one. A global matrix-vector equation describing the state of the process is derived and a forward-backward procedure for generating realisations of the process in a causal manner is obtained from the global equation via LU factorisation. Using a similar approach, the optimal fixed-interval smoother is obtained, and is realised by a similar forward-backward method, not unlike the familiar fixed-interval smoother for Gauss-Markov processes.

More recently, D. Vats and J. Moura [13] also considered the Gaussian reciprocal case. In their approach, they also used an LU decomposition of a global model to derive a forward-backward representation which, unlike [12], incorporated the boundary conditions explicitly into these recursions. Optimal smoothing was also addressed in [13].

This paper is the first to be concerned with *finite state* discrete time reciprocal processes, although non-recursive models for finite state MRFs in two dimensions were considered in [14]. In the sequel, we shall sometimes refer to a finite state reciprocal process as a *reciprocal chain*. Analogously to a hidden Markov chain (HMC), a HRC consists of a reciprocal chain, and an observation process statistically dependent on the RC. The main contributions of this paper are twofold. Firstly we show that any RC can be completely and uniquely specified by a finite set of discrete state *Markov bridges* (MB). A Markov bridge is a MC with one end fixed to a specified value with probability one. Secondly, we use this representation and a Bayesian approach to derive the optimal smoother for a hidden reciprocal chain. In particular, we show that the optimal smoother for a HRC can be constructed from a finite set of optimal smoothers, one corresponding to each (hidden) Markov bridge (HMB). We note that there is a close relationship with the methodology and characterisation of reciprocal processes used by Jamison in [5]. Optimal *filtering* (in continuous time) for a hidden Markov bridge was considered in [15], where the optimal filter for a hidden Markov model was modified to take into account the known information about the final state. Smoothing was not addressed.

Optimal smoothing for HRC was also considered in a workshop paper [16], but a less general case corresponding to those HRC generated from a specified stationary Markov chain was considered there. This paper extends [16] to the general HRC case.

One of the main motivations for the use of reciprocal processes is their suitability for *non-causal* processes. Indeed, many of the papers we have referred to herein contain examples of such processes, in particular, processes where the index set is not time, but a spatial variable, or a combination of temporal and spatial variables. Importantly, in problems where the index set of the process has dimension greater than one, the artificial imposition of causality often used to permit standard causal processing, can be overcome by using nearest neighbour models, these being higher dimensional versions of HRC. We don't address higher dimensional models in this paper but refer readers to [17], [18], [19] as important examples of the two dimensional MRF case.

The layout of the paper is as follows. In section II, we describe the basic models for RC and HRC, stating the various statistical assumptions inherent in their definitions. This section also proves an important and fundamental characterisation of reciprocal chains - that they can be uniquely represented by a finite set of Markov bridges. In section III, we derive the optimal fixed-interval smoother for a HRC using a Bayesian framework not dissimilar to the approach used for hidden Markov chains. Simulations which illustrate the potential benefits for the use of both a reciprocal model, and a finite state model are then presented in section IV. Finally we draw some conclusions and refer to ongoing and future related work in some detail to further illustrate the significance of the finite HMB characterisation of a HRC in related identification and control problems.

II. MODELLING AND CHARACTERISATION OF HIDDEN RECIPROCAL CHAINS

Consider a random process $\mathcal{X} = \{X_0, \dots, X_T\}$ indexed on the set $\{0, 1, \dots, T\} \subset \mathbb{Z}$ for some integer $T \geq 2$. We assume that the process takes values in some finite set \mathcal{S} with $N > 1$ elements. Without loss of generality, we can take the state space to be the set $\mathcal{S} = \{1, \dots, N\} \subset \mathbb{Z}$. The process X_t is said to be *reciprocal* if

$$\mathbf{P}\{X_t | X_s, s \neq t\} = \mathbf{P}\{X_t | X_{t-1}, X_{t+1}\}, \quad (1)$$

for each $t = 1, \dots, T - 1$. Thus X_t is conditionally independent of $X_0, \dots, X_{t-2}, X_{t+2}, \dots, X_T$ given its neighbours X_{t-1} and X_{t+1} . The RC model is specified by the set

of 3 point transition functions (1) together with a given joint distribution on the end points $\mathbf{P}\{X_0, X_T\}$.

A reciprocal process (RP) is a one-dimensional version of a Markov random field (MRF), although it is not necessarily a Markov process. However, any Markov process is reciprocal ([5], Lemma 1.2).

It has been shown ([5], Lemma 1.4) that fixing the end point of a RP generates a Markov bridge. So for the finite (N) state case, a RP can be regarded as N Markov bridges, one corresponding to each of the possible final states taken by X_T . We will derive the models for each of these Markov bridges from the RC model utilising the following important property of a reciprocal process ([5], property (a3), p. 80). Let $0 \leq s < t < u < v \leq T$. Then

$$\mathbf{P}\{X_u|X_s, X_v\} \mathbf{P}\{X_t|X_s, X_u\} = \mathbf{P}\{X_t|X_s, X_v\} \mathbf{P}\{X_u|X_t, X_v\} . \quad (2)$$

Let $s = t - 1, u = t + 1$, and $v = T$, then, formally, for $t = 1, \dots, T - 1$,

$$\mathbf{P}\{X_t|X_{t-1}, X_T\} = \frac{\mathbf{P}\{X_t|X_{t-1}, X_{t+1}\}}{\mathbf{P}\{X_{t+1}|X_t, X_T\}} \mathbf{P}\{X_{t+1}|X_{t-1}, X_T\} . \quad (3)$$

For $1 \leq t \leq T - 1$, let

$$Q_{i,j,\ell}(t) = \mathbf{P}\{X_t = e_j | X_{t-1} = e_i, X_{t+1} = e_\ell\} ,$$

denote the RC three point transition functions. We assume for simplicity that all such three point transitions are strictly positive, although this assumption may not be necessary. Then (3) yields a backwards recursion which fully specifies the set of N MB transitions for $t = T - 2, T - 2, \dots, 0$:

$$B_{i,j}^k(t) = \mathbf{P}\{X_{t+1} = j | X_t = i, X_T = k\} = \frac{Q_{i,j,\ell}(t)}{B_{j,\ell}^k(t+1)} \left(\sum_{m=0}^{N-1} \frac{Q_{i,m,\ell}(t)}{B_{m,\ell}^k(t+1)} \right)^{-1} , \quad (4)$$

the last term on the right being the normalisation constant. Initialisation is with $B_{i,j}^k(T-1) = 1$ if $j = k$ and zero otherwise. Observe that the quantity on the rhs of (4) is independent of the index ℓ . As mentioned in [5] (p. 80), this important property is the RC analogue to the Chapman-Kolmogorov equation for Markov processes. We have also found this property particularly useful in the context of stochastic optimal control [22] for hidden reciprocal chains. Now since the terms $B_{i,\cdot}^k(t)$ form a probability distribution (ie they are non-negative and sum to unity) for each i, k and t , there is at least one index ℓ for which $B_{i,\ell}^k(t+1)$ is

non-zero. So this index may be selected on the rhs of (4), and thus (4) is well defined.

The Markov bridge with final state k is given an initial probability distribution π_i^k given by the conditional distribution

$$\pi_i^k = \mathbf{P} \{X_0 = i | X_T = k\} = \frac{\Pi_{i,k}}{\sum_{j=1}^N \Pi_{j,k}}, \quad (5)$$

where $\Pi_{i,k} = \mathbf{P} \{X_0 = i, X_T = k\}$ is the specified RC end points distribution. Thus we have demonstrated that any RC may be uniquely specified by a finite set of Markov bridges with probability transition matrices given by (4) and initial distributions (5).

We now assume there is an observation process $\mathcal{Y} = \{Y_0, \dots, Y_T\}$, conditionally dependent on the RC \mathcal{X} with the conditional independence property that

$$\mathbf{P} \{Y_0, \dots, Y_t | X_0, \dots, X_T\} = \prod_{t=0}^T \mathbf{P} \{Y_t | X_t\}. \quad (6)$$

The process \mathcal{Y} is called a *hidden reciprocal process* (HRP) or hidden reciprocal chain (HRC) because the property (6) is analogous to the usual assumption made for hidden Markov chains (see eg [14]). Let

$$C_i(t) = \mathbf{P} \{Y_t | X_t = i\}, \quad (7)$$

denote the (conditional) observation likelihoods of the HRC. The observations may be either discrete or continuous random variables defined on an appropriate probability space, however, for reasons of simplicity, we don't introduce any mathematical technicalities associated with the observation space. It suffices for our purposes that the conditional observation likelihoods (7) are well defined and may be evaluated.

III. OPTIMAL SMOOTHING FOR HRC

Optimal smoothers for HRC were first addressed in a less general setting in a workshop paper [16], where only those RC determined by a stationary Markov transition function were considered. The approach of [16] is made general, and more precise by this paper. We define the quantities

$$\begin{aligned} \alpha_i^k(t) &= \mathbf{P} \{X_t = i, Y_0, \dots, Y_t | X_T = k\} \\ \beta_i^k(t) &= \mathbf{P} \{Y_{t+1}, \dots, Y_T | X_t = i, X_T = k\}, \end{aligned}$$

with initialisations

$$\begin{aligned}\alpha_i^k(0) &= \mathbf{P}\{X_0 = i, Y_0 | X_T = k\} \\ &= \pi_i^k C_i(0),\end{aligned}$$

the first term coming from the RP end-points distribution, and

$$\begin{aligned}\beta_i^k(T-1) &= \mathbf{P}\{Y_T | X_{T-1} = i, X_T = k\} \\ &= C_k(T),\end{aligned}$$

which is identical for each $i = 1, \dots, N$. We formally define $\beta_i^k(T) = 1$ for all i, k .

These quantities are computed recursively via

$$\begin{aligned}\alpha_i^k(t+1) &= \sum_{j=1}^N \mathbf{P}\{X_{t+1} = i, X_t = j, Y_0, \dots, Y_{t+1} | X_T = k\} \\ &= C_i(t+1) \sum_{j=1}^N \mathbf{P}\{X_{t+1} = i | X_t = j, X_T = k\} \mathbf{P}\{X_t = j, Y_0, \dots, Y_t | X_T = k\} \\ &= C_i(t+1) \sum_{j=1}^N B_{j,i}^k(t) \alpha_j^k(t),\end{aligned}$$

for $t = 0, \dots, T-2$. Here we have applied the property that the RC pinned at $X_T = k$ is a Markov bridge with initial state distribution π_i^k and transition probability matrices $B_{i,j}^k(t)$.

Similarly,

$$\begin{aligned}\beta_i^k(t) &= \sum_{j=1}^N \mathbf{P}\{Y_{t+1}, \dots, Y_T, X_{t+1} = j | X_t = i, X_T = k\} \\ &= \sum_{j=1}^N C_j(t+1) \mathbf{P}\{Y_{t+2}, \dots, Y_T, X_{t+1} = j | X_t = i, X_T = k\} \\ &= \sum_{j=1}^N C_j(t+1) \mathbf{P}\{Y_{t+2}, \dots, Y_T | X_t = j, X_T = k\} \mathbf{P}\{X_{t+1} = j | X_t = i, X_T = k\} \\ &= \sum_{j=1}^N C_j(t+1) B_{i,j}^k(t) \beta_j^k(t+1),\end{aligned}$$

for $t = T-2, \dots, 0$.

Thus we can compute the un-normalised hidden Markov bridge *a posteriori* probabilities for state X_t given $X_T = k$ by

$$\gamma_i^k(t) = \mathbf{P}\{Y_0, \dots, Y_T, X_t = i | X_T = k\} = \alpha_i^k(t) \beta_i^k(t),$$

and thus the un-normalised HRC *a posteriori* probabilities

$$\gamma_i(t) = \mathbf{P} \{Y_0, \dots, Y_T, X_t = i\} = \sum_{k=1}^N \alpha_i^k(t) \beta_i^k(t) \mathbf{P} \{X_T = k\} ,$$

the last term being the marginal obtained from the RP end-points distribution. These quantities may then be used to determine maximum *a posteriori* probability (MAP), or conditional mean (minimum variance) state estimates in the usual manner. Overall computational complexity is $O(N^3T)$ compared to $O(N^2T)$ for the HMC optimal smoother.

We thus observe that the optimal smoother consists of N optimal Markov bridge smoothers, each corresponding to a fixed final state. Bayes' rule then shows how these may be combined in a natural way by averaging over the *a priori* final state distribution which is the marginal obtained from the specified RC end points distribution.

IV. SIMULATIONS

In this section, we perform simulation experiments on a HRC where the underlying RC is derived from a stationary Markov process. These simulations are in no way meant to be an exhaustive study of the relative performance of the optimal HRC smoother and other estimators, which would constitute another paper in itself. However, this rather simple experiment does provide some anecdotal evidence that the use of HRC models may be beneficial in some circumstances. We firstly describe the construction of a subclass of RCs arising from an underlying stationary Markov chain.

Suppose A is the transition probability matrix for a N state stationary Markov chain, then a reciprocal process can be constructed with three point transitions given by [5]

$$Q_{i,j,k}(t) = \frac{A_{i,j} A_{j,k}}{\sum_{\ell=1}^N A_{i,\ell} A_{\ell,k}} . \quad (8)$$

for $t = 1, \dots, T - 2$, together with a specified boundary points distribution $\Pi_{i,j} = \mathbf{P} \{X_0 = i, X_T = j\}$. Indeed, Jamison [5] shows that such a process is in general non-Markovian if X_0 and X_T are not statistically independent. Importantly, (8) doesn't hold in general at the final transition $t = T - 1$. The process for generating sample paths of this RC is to draw the initial and final points X_0 and X_T from the boundary points distribution Π , and then construct the Markov bridge transitions corresponding to the chosen final point

say $X_T = k$. These transitions are given by ¹

$$\begin{aligned} B_{i,j}^k(t) &= \mathbf{P}\{X_{t+1} = j | X_t = i, X_T = k\} \\ &= \frac{\mathbf{P}\{X_T = k | X_{t+1} = j\} \mathbf{P}\{X_{t+1} = j | X_t = i\}}{\mathbf{P}\{X_T = k | X_t = i\}} \\ &= \frac{A_{i,j} (A^{T-t+1})_{j,k}}{(A^{T-t})_{i,k}}, \end{aligned}$$

for $t = 0, \dots, T - 2$. The RC is then constructed in the usual way starting from X_0 and necessarily terminating at $X_T = k$. Although the underlying Markov and three point reciprocal transitions are stationary (apart from at $t = T - 1$), the Markov bridge transitions are not, although some reflection on the Perron-Frobenius theory for Markov processes will show that when $t \ll T$, the Markov bridge transitions $B^k(t)$ are close to the Markov chain transitions A , and so the RC behaves in a Markovian manner. Thus for the class of RC derived from Markov chains as above, the ‘‘reciprocal’’ behaviour only becomes evident as $t \rightarrow T$. This has important ramifications for RC modelling which are discussed in detail in work as yet to be published [23]. Thus, for the following simulations, the length of the HRC sequences are somewhat small so that the reciprocal property is made more apparent.

The observations were chosen to be independent Gaussian random variables with mean given by the state value (ie $1, 2, \dots, N$), and constant variance σ_s^2 . The Markov transition matrix was given by a ‘‘discretised’’ Gaussian

$$A_{i,j} = \frac{e^{-(i-j)^2/(2\sigma_s^2)}}{\sum_{k=1}^N e^{-(i-k)^2/(2\sigma_s^2)}},$$

where the variance σ_s^2 controls the variability of the RC.

The initial distribution π_0 was chosen to be uniform. The RC boundary distribution was specified to emphasis the source-destination pairs $(i, N - i + 1)$ for $i = 1, \dots, N$. Thus the conditional distribution $\mathbf{P}\{X_T = i | X_0 = j\}$ was zero unless $j = N - i + 1$ in which case it was unity. The boundary distribution was thus specified by Bayes’ rule.

We implemented the following four estimators :

- Optimal HRC smoother,

¹We thank Robert Elliott for pointing out these calculations to us.

- Optimal HMC smoother,
- Linear optimal HRC smoother (see [12]), and
- Linear optimal HMC smoother.

In order to sensibly compare the linear and non-linear smoothers, mean-square state estimation error (MSE) was used as the performance metric. although maximum *a posteriori* probability (MAP) estimates would be generally used if the states were known to be discrete, and not have arisen from the discretisation of a continuous state problem. We assume that readers are familiar with the construction of the HMM based smoothers (see eg [20]), but for reasons of completeness, we include a derivation of the linear (or more correctly, the affine) HRC smoother in the appendix.

For the simulations, we chose the number of states $N = 20$, and varied both the additive observation noise variance σ^2 , and the signal sequence length T . In all cases, 10,000 independent realisations were averaged to obtain the MSE estimates. Figure ?? shows the MSE for each of the 4 smoothers for $\sigma^2 = 0.5, 1.0, 1.5, 2.0$. Here we chose $\sigma_s^2 = 0.75$, and $T = 10$. It is evident that (i) the optimal estimators both perform better than the corresponding linear estimators, and (ii) in both the linear and optimal case, the HRC based estimators perform better than their HMM counterparts, with the performance difference increasing as the additive noise increases.

In the second set of simulations, we investigated the effect of varying the HRC sequence length T . Figure ?? shows the MSE performance of the 4 estimators for $T = 5, 10, 15, 20$. We observe the general trend that the HMC and HRC estimator performance becomes similar (in both the linear and optimal cases) as T increases. This is because of the behaviour inherent in the underlying RC, which has been derived from a stationary Markov transition function. As previously alluded to, this is because the reciprocal behaviour becomes apparent only as $t \rightarrow T$, and thus the per-sample MSE for the HMC and HRC will be similar for a large proportion of the signal as T becomes larger. It is thus of interest to examine the more general case where the MB transitions are determined from specified RC three point transitions. Such an application is considered in [23].

V. CONCLUSION

This paper has addressed the problem of optimal fixed interval smoothing for finite state hidden reciprocal processes (also termed a hidden reciprocal chain or HRC). A HRC is a natural *non-causal* generalisation of a hidden Markov model (HMM). A HRC is specified by a set of three point state transition functions, a boundary probability distribution, and a set of conditional (on the state) observation distributions. A unique representation of such a process in terms of a finite set of hidden Markov bridges is obtained. This representation is used to derive the optimal fixed interval smoother for a HRC. This smoother is constructed as a set of hidden Markov bridge smoothers, each similar to the familiar forward-backward smoother for HMMs. They are combined in a natural way using a Bayesian approach, to yield the optimal HRC smoother. A simple simulation is presented to illustrate potential utility of the model and associated smoother.

The Markov bridge model of a HRC appears to also be particularly useful in related problems which we have completed or are currently working on. For example, in [21], the authors show how to derive a descriptor state space model for a HRC using an approach motivated by Markov bridges. The significance of this work is that it allows the derivation of many of the results from the Gaussian reciprocal case presented in [12], but applied to finite state reciprocal chains. This highlights the fact that the Gaussianity of the processes considered in [12], and the associated linearity of models is not something inherent to the Gaussian nature of the signals, but truly reflective in a more general sense, of the reciprocal nature of the process. System identification for HRC is addressed in [24] which presents a generalisation of the EM algorithm as applied to the identification of HMMs (see eg [14]).

Finally, we mention two other interesting applications. In [22], we consider the reciprocal generalisation of partially observed Markov decision processes (POMDPs) which have found application in many areas. Not surprisingly, we have termed these processes *Partially Observed Reciprocal Decision Processes* (PORDPs). We have derived a Bellman type optimality equation which is second order rather than first order in nature as for the POMDPs. This is not surprising given the generic second order nature of HRC models. One particularly simple class of such control problems is the determination of the *maximum likelihood state sequence* - the RC generalisation of the familiar Viterbi algorithm for maximum likelihood

state estimation of HRCs. It is suggested that this approach might be generalised to obtain a reciprocal generalisation of the continuous-time, continuous state Hamilton-Jacobi-Bellman optimality equation. The RP models of [6], [7] would prove useful in this context.

The second idea of significance is explored in [23], and is connected with the fact mentioned above, that for those reciprocal chains generated from Markov transitions, there is a “forgetting” property which means that the effect of the imposed probability distribution on the final state becomes less significant at the start of the process. Indeed, as the start, the reciprocal process may behave in a Markovian manner, depending on the form of the transition dynamics. Thus one is motivated to “glue” together sets of “short” HRC, appropriately matching boundary conditions at each end so that the reciprocal property can be maintained despite underlying Markovian dynamics. We call these *reciprocal waypoint models* and they lead in a very natural way to the idea of a *multiscale* reciprocal process. Navigation applications of such models are also addressed in [23].

ACKNOWLEDGEMENTS

The authors acknowledge helpful communications with Bernard Levy and Robert Elliott.

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APPENDIX

DERIVATION OF THE OPTIMAL LINEAR SMOOTHER FOR HRC

Consider the RC \mathcal{X} and observations process \mathcal{Y} . The optimal linear (or more correctly, affine) estimator of \mathcal{X} given the observations \mathcal{Y} is given by the familiar formula

$$\hat{\mathcal{X}} = \mathbf{E}\{\mathcal{X}\} + \text{Cov}(\mathcal{X}, \mathcal{Y}) \text{Cov}(\mathcal{Y}, \mathcal{Y})^{-1} (\mathcal{Y} - \mathbf{E}\{\mathcal{Y}\}) . \quad (9)$$

Thus we need to evaluate the following probability distributions : $\mathbf{P}\{X_t = i\}$, $\mathbf{P}\{Y_t\}$, $\mathbf{P}\{X_t = i, Y_s\}$, and $\mathbf{P}\{Y_t, Y_s\}$ for $s, t = 0, \dots, T$. Again, we don't want to get involved in mathematical technicalities regarding the observation probability space, but we assume that the conditional likelihoods $C_i(t) = \mathbf{P}\{Y_t | X_t = i\}$ are well defined and can be evaluated. The Markov bridge (MB) approach can be used to evaluate these quantities. Firstly, consider

the MB *a priori* probabilities

$$\begin{aligned}
\pi_i^k(t+1) &= \mathbf{P}\{X_{t+1} = i | X_T = k\} \\
&= \sum_{j=1}^N \mathbf{P}\{X_{t+1} = i, X_t = j | X_T = k\} \\
&= \sum_{j=1}^N \mathbf{P}\{X_{t+1} = i | X_t = j, X_T = k\} \mathbf{P}\{X_t = j | X_T = k\} \\
&= \sum_{j=1}^N B_{j,i}^k(t) \pi_j^k(t),
\end{aligned}$$

for $t = 0, \dots, T-1$. Initialisation is with $\pi_i^k(0) = \mathbf{P}\{X_0 = i | X_T = k\}$ which is obtained from the RC boundary points distribution. The RC *a priori* probabilities are then given by

$$\begin{aligned}
\pi_i(t) &= \mathbf{P}\{X_t = i\} \\
&= \sum_{k=1}^N \pi_i^k(t) \mathbf{P}\{X_T = k\},
\end{aligned}$$

the last term being the marginal distribution on X_T obtained from the RC boundary points distribution. We also require the joint MB state distributions for $0 \leq t < s < T$,

$$\begin{aligned}
\Phi_{i,j}^k(t,s) &= \mathbf{P}\{X_t = i, X_s = j | X_T = k\} \\
&= \frac{\mathbf{P}\{X_s = j | X_t = i, X_T = k\}}{\mathbf{P}\{X_t = i | X_T = k\}} \\
&= \frac{(B^k(s-1) B^k(s-2) \cdots B^k(t))_{i,j}}{\pi_i^k(t)}.
\end{aligned}$$

From these quantities, we can derive the joint RC state distributions for $0 \leq t < s < T$

$$\Phi_{i,j}(t,s) = \mathbf{P}\{X_T = i, X_s = j\} = \sum_{k=1}^N \Phi_{i,j}^k(t) \mathbf{P}\{X_T = k\}.$$

Now we can derive expressions for the several mean and covariance expressions appearing in the optimal affine estimation formula (9) :

$$\begin{aligned}
\mathbf{E}\{X_t\} &= \sum_{i=1}^N \pi_i(t) S(i) \\
\mathbf{E}\{Y_t\} &= \mathbf{E}\{\mathbf{E}\{Y_t | X_t\}\} = \mathbf{E}\{X_t\} \\
\mathbf{E}\{X_t^2\} &= \sum_{i=1}^N \pi_i(t) S(i)^2 \\
\mathbf{E}\{Y_t^2\} &= \mathbf{E}\{\mathbf{E}\{Y_t^2 | X_t\}\} = \mathbf{E}\{X_t^2\} + \sigma^2
\end{aligned}$$

for $0 \leq t \leq T$, and

$$\begin{aligned} \mathbf{E}\{X_t Y_s\} &= \mathbf{E}\{X_t \mathbf{E}\{Y_s|X_s\}\} = \mathbf{E}\{X_t X_s\} \\ &= \sum_{i=1}^n S(i) \sum_{j=1}^N S(j) \Phi_{i,j}(t, s) \\ \mathbf{E}\{Y_t Y_s\} &= \mathbf{E}\{\mathbf{E}\{Y_t|X_t\} \mathbf{E}\{Y_s|X_s\}\} = \mathbf{E}\{X_t X_s\} , \end{aligned}$$

for $0 \leq t < s \leq T$. with the usual symmetry properties applied for $s < t$. The mean vectors and covariance matrices can then be constructed by suitably arranging these terms in the usual manner. Here $S(i)$ is the value taken by the RC when in state i , assumed to be real scalars.