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ON THE SOLUTION OF MARKOV-SWITCHING RATIONAL
EXPECTATIONS MODELS

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Abstract

Forward-looking stochastic systems under rational expectations have emerged as the reference framework in contemporary macroeconomic theory. This paper describes a method for solving a class of rational expectations models under noisy measurement and Markov jump parameters, by specifying the expectations component as a general-measurable function of the observable states of the system, to be determined optimally via stochastic control and filtering theory. Solution existence is proved by setting this function to the regime-dependent feedback control minimizing the mean-square deviation of the equilibrium path from the corresponding perfect-foresight autoregressive Markov jump state motion. As the exact expression of the conditional (rational) expectations term is derived both in finite and infinite horizon model formulations, no (asymptotic) stationarity assumptions are needed to solve forward the system, for only initial values knowledge is required. A simple sufficient condition for the mean-square stability of the obtained rational expectations equilibrium is also provided.

Key words: Dynamic programming; Kalman filters; Markov parameters; Rational Expectations

1. Introduction

A central issue in contemporary macroeconomics is the identification of the driving forces of aggregate economic phenomena (e.g. economic growth, inflation, short-term income fluctuations). From a theoretical viewpoint, the well-established methodology advocated for this type of investigation builds upon the use of dynamic stochastic general equilibrium models, whose behaviour largely depends on the expectations of economic agents - households, private and public firms, governments and other policy making authorities - on the future evolution of all the economic variables of interest (e.g. prices, interest rates, public policies).

Since the early work of [26] and [22], the concept of rational expectations (RE) has emerged as the reference tool for modelling expectations in macroeconomic systems. It essentially reduces to the assumption that agents collect and make optimal use of all available (pertinent) information as to the economic environment when formulating their forecasts of economic variables of interest (e.g. [28]). A nontrivial example of RE model is given by the following system of n equations:

$$x_{t+1} = A_{s(t)}x_t + B_{s(t)}\mathbf{E}_t[x_{t+h}] + C_{s(t)}v_t, \quad h \geq 1 \quad (1)$$

where $x_t \in \mathfrak{R}^n$ collects economic variables (e.g. private consumption and productive capital stock), $A_{s(t)}$, $B_{s(t)}$ and $C_{s(t)}$ are (conformable) matrix functions of the time-varying (possibly stochastic) system parameters $s_t \in \mathfrak{R}^m$, and $v_t \in \mathfrak{R}^n$ is a sequence of independent zero-mean random variables with finite variance.

RE systems in the form of (1) typically represent the first-order conditions of a complex optimization framework involving different types of economic agents, aiming at optimizing their intertemporal behaviour (e.g. consumption, investment and/or production decisions, choice of monetary and fiscal policies) while holding beliefs on the uncertain evolution of the variables involved¹. This peculiar forward-looking feature is represented by the presence of the conditional expectations component $\mathbf{E}_t[x_{t+h}] := \mathbf{E}[x_{t+h}|\mathcal{F}_t]$, which is based upon some information \mathcal{F}_t available to the agents at time t , where generally $\mathcal{F}_t = \sigma\{x_k, k \leq t\}$, i.e. the σ -algebra generated by the state process up to the current time (perfect measurement case). A *Rational Expectations Equilibrium* (REE) is any locally integrable process $\{x_t\}$ which, for given initial conditions and in both finite and infinite model horizon, satisfies the law of motion (1). Since dynamic RE systems fail to comply with the causality hypothesis - i.e. expected future events do affect current and actual future ones -, no (uniquely defined) REE can be achieved from initial conditions knowledge solely, and additional constraints are typically needed.

The aim of this paper is to extend the model reference adaptive technique developed in [7] to solution of noisily observed forward-looking Markov-switching Rational Expectations (MSRE) models, that is dynamic stochastic frameworks like (1) in which the system parameters s_t are taken to be functions of a discrete-state Markov chain. From a technical viewpoint, regime dependency engenders structural nonlinearities which prevent from employing standard solution tools for linear RE systems, such as [5]'s, [19]'s and [29]'s. In this respect, a rapidly increasing branch of macroeconomic literature has been interested in deriving conditions for existence and uniqueness of RE equilibria to MSRE models. As a leading example, [12] study how the possibility of Markov jump model structure alters uniqueness properties of the REE and provide analytical restrictions to ensure (local) uniqueness of the equilibrium path. The nonlinearity problem is addressed by introducing a two-step solution method that consists in studying an augmented system which is linear in fictitious variables, the latter coinciding with the actual ones in some of the modes of the Markov chain, and then using the solution to the linear representation in order to construct solutions for the original nonlinear system. In a more general perspective, [16] and [17] have provided a series of characterization results for the full set of solutions to MSRE frameworks, which satisfy a suitable stability concept. Their approach rests on expanding the state-space of the underlying stochastic system and to focus on an equivalent model in the expanded space that features

¹As an example, consider a case where the 'agents' are individuals and/or households, who are deciding whether to take a loan or not at a certain time-varying rate, i.e. the cost at which the loan is provided. Their decision then will certainly depend on their beliefs about the future evolution of the interest rate. On the other hand, if most of the individuals and/or households take the same decision (for example, to engage in the loan), the future evolution itself will be affected by such decision (in the same example, the rate will increase).

state-invariant parameters. Furthermore, [17] demonstrate an equivalence property between solution uniqueness in MSRE models and mean-square stability in a class of Markov jump autoregressive systems.

In the nonstochastic (deterministically time-varying) parameters setup, [7] show that, for an important class of discrete-time RE systems (1), a solution can be obtained via a *causal* (controllable) system forced by a general-measurable function of the available information, estimated via a Kalman filter technique. More specifically, an exact solution of the RE system is determined by forcing it with the optimal minimum variance estimate of the future state, recursively computed on the autoregressive equation describing the perfect-foresight dynamics of the economy. In this work, we describe a method for solving MSRE models under noisy measurement, as the outcome of a Kalman filter-based mechanism of optimal state filtering. This approach, which can be traced back to [3], complies with a broader definition of rationally formed expectations, comparable to some form of dynamic optimizing behaviour, which does not require - as typically done in the RE macroeconomic literature - that economic agents possess a priori knowledge of the structure of the model's solution itself. The RE hypothesis indeed requires that the prediction made by the forecaster be consistent with the conditional expectations structure of the equilibrium process (see 1). We depart from this hypothesis and specify the expectations component as a (square-integrable) random function adapted to the filtration - the actually available information - generated by the Markov-switching system itself.

Formally, the novelty of our solution method is based on using dynamic programming arguments and optimal filtering techniques applied to a causal Markov jump system, where RE are replaced by a specific control sequence, the latter being measurable with respect to the observable states of the system. Such input is chosen optimally as the feedback regime-dependent control law minimizing the mean-square deviation of the equilibrium path from the corresponding autoregressive Markov jump state motion (the reference model). We present a recursive algorithm, based upon optimal stochastic control and Kalman filtering theory, for the design of the control law and show that the latter has the same structure of the conditional expectation operator featuring in the canonical (possibly regime switching) RE systems.

It is well known that the dimension of the solution set for RE models is closely related the stability properties of the latter, and that stability restrictions can be advocated in order to weaken the multiplicity issue (e.g. [5], [27]). However, as the agents' expectations in RE frameworks are typically obtained by recursively iterating the system into the future, (asymptotic) stationarity is needed for this process to be well-defined (e.g. [17]). While equilibrium stability is usually enforced by the existence of terminal conditions in the underlying (infinite horizon) dynamic economic frameworks, there exist models for which no such conditions arise or rather, though present, they do not serve as necessary optimality requirements (e.g. [18], [14]). In this respect, we emphasize that, by providing a readily computable expression of the RE component both in finite and infinite horizon model representations, our method need not invoke approximation hypotheses or stability concepts to solve forward the system, for only initial conditions knowledge is required. We also provide an easy-to-check sufficient condition for the mean-square stability of the obtained RE equilibrium.

While concerned with computational issues in MSRE models, our analysis also relates to studies on the process of expectations formation in macroeconomic modelling. In fact, it should be stressed that the RE hypothesis does not imply that the way in which the agents form their expectations is *known*, what indeed it supposes is that *in some way* - no matter *how* - the economic agents are able to undertake a 'rational' decision, which is mathematically translated into an 'optimal prediction' (i.e. the conditional expectation of some future state value). Even in its strongest forms, the RE framework thus fails to shed light on the process by which economic agents translate current information - what they truly observe - into optimal forecasts ([23]). To address this issue, the recursive learning literature (e.g. [24], [15]) only allows agents to take decisions based on an arbitrary forecasting function, which need not coincide with the RE and thereby is to be optimally parameterized over time based on new data and observable (past) forecasting errors. Conditional expectations are thus regarded as asymptotic outcomes of this updating process, where the convergence of the latter towards RE has been shown to be equivalent to the local stability of a certain ordinary differential equation (ODE). Though methodologically related, our method also differs from the Bayesian learning literature (e.g. [25], [6]), as these studies typically assume that agents employ filtering techniques to update estimates of (possibly time-varying) parameters within not fully rational forecasting functions. Rather, our approach posits that rational agents may be thought of

as revising their (best) estimate of the (hidden) variables governing the dynamics of the economic system as new observations are generated, when only a reduced information set - the measurement process - is available to them.

The paper is organized as follows. In Section 2 the class of stochastic MSRE models we deal with is introduced. In Section 3, we develop the solution algorithm, whereas Section 4 is devoted to the stability analysis. Section 5 concludes.

2. The class of MSRE models

We study the following class of purely forward-looking MSRE models with noisy observations on the state vector, defined on a properly filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$:

$$x_{t+1} = \Gamma_{s(t)}^{-1} \mathbf{E}[x_{t+2} | \mathcal{F}_t] + \Psi_{s(t)} v_t, \quad x_0 = \bar{x} \quad (2)$$

$$y_t = \Phi_{s(t)} x_t + w_t \quad (3)$$

where x_t is an n -dimensional real vector of random variables of economic interest, y_t is an l -dimensional real vector of observables, and the state error $v_t \in \mathbb{R}^n$, the measurement noise $w_t \in \mathbb{R}^l$ and the initial state $\bar{x} \in \mathbb{R}^n$ are zero-mean white Gaussian processes. With no loss of generality, the covariances of the unobserved structural disturbance and of the measurement noise are normalized to the $I_{n \times n}$ and $I_{l \times l}$ identity matrices respectively, whereas \bar{x} has covariance P_0 . $\Gamma_{s(t)}, \Psi_{s(t)}$ and $\Phi_{s(t)}$ are conformable matrices holding the coefficients of the underlying economic model, with $\Gamma_{s(t)}$ assumed invertible, as in [17].

In (2)-(3), the regime switches are governed by an ergodic discrete-state Markov chain indexed by $s(t)$, with $s(t) \in \mathbb{S} := \{1, \dots, S\}$. Let $\tilde{\mathcal{S}}$ denote the σ -field of all subsets in \mathbb{S} , and $\tilde{\mathcal{F}}_t$ the σ -field of \mathbb{R}^{n+l} in which (x_t, y_t) lie. We define:

$$\Omega := \prod_{t \in \mathcal{T}} (\mathbb{R}_t^{n+l} \times \mathcal{S}_t)$$

where $\mathbb{R}_t^{n+l}, \mathcal{S}_t$ are copies of $\mathbb{R}^{n+l}, \mathcal{S}$ and \mathcal{T} denotes a discrete-time set of interest. Let $\mathcal{T}_t := \{k \in \mathcal{T}; k \leq t\}$ for each $t \in \mathcal{T}$, then:

$$\mathcal{F} := \sigma \left\{ \prod_{t \in \mathcal{T}} (\alpha_t \times \beta_t); \alpha_t \in \tilde{\mathcal{F}}_t, \beta_t \in \tilde{\mathcal{S}}, \forall t \in \mathcal{T} \right\}$$

and for each $t \in \mathcal{T}$:

$$\mathcal{F}_t := \sigma \left\{ \prod_{\iota \in \mathcal{T}_t} \alpha_\iota \times \beta_\iota \times \prod_{\tau \in \mathcal{T} \setminus \mathcal{T}_t} \mathbb{R}_\tau^{n+l} \times \mathcal{S}_\tau; \alpha_\iota \in \tilde{\mathcal{F}}_\iota, \beta_\iota \in \tilde{\mathcal{S}}, \iota \in \mathcal{T}_t \right\}$$

with $\mathcal{F}_t \subset \mathcal{F}$. Then $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$ defines a stochastic basis for (2)-(3), with \mathcal{P} representing a probability measure such that:

$$\mathcal{P} \{s(t+1) = j | \mathcal{F}_t\} = \mathcal{P} \{s(t+1) = j | s(t)\} = p_{s(t)j}$$

with $p_{i,j} \geq 0$ for $i, j \in \mathbb{S}$ and $\sum_{j=1}^S p_{ij} = 1$ for each $i \in \mathbb{S}$. The initial conditions (\bar{x}, s_0) are taken to be independent random variables.

More specifically, the information set available at time t , upon which conditional (rational) expectations $\mathbf{E}[\cdot | \mathcal{F}_t]$ in (2) are built, includes the complete filtrations generated by the output process (3), namely $\{y_k, k \leq t\}$, and by the Markov state realizations $\{s_k, k \leq t\}$. We thus allow for observable shifts in modes solely, as in most of the macroeconomic literature on regime switching RE models (e.g. [12], [17])². Accordingly, while the current values of parameters are known, future ones are uncertain. As a working assumption, we also require that \bar{x}, v_t, w_t and s_t be mutually independent.

²For theoretical work dealing with unobserved current regimes, see, among others, [2], [21] and [11].

Since no expression for the actual two-step ahead variables x_{t+2} in terms of current observables is available from (2)-(3), additional terminal constraints or stability requirements are commonly advocated in the macroeconomic literature (e.g. [5]). The goal of this paper is to develop a model reference adaptive technique in order to solve (2)-(3) for a particular REE by directly computing the conditional expectations term from actually available information, without imposing any a priori stability concept. This will be accomplished by specifying the RE component as a (general-measurable) function of the t -dated filtration \mathcal{F}_t of the stochastic economy, and employing optimal stochastic control and filtering techniques to adapt the actual system evolution to the perfect-foresight Markov jump autoregressive state motion (the reference model), where the RE component $\mathbf{E}[x_{t+2}|\mathcal{F}_t]$ coincides with its actual value x_{t+2} :

$$x_{t+1}^* = \Gamma_{s(t)}^{-1} x_{t+2}^* + \Psi_{s(t)} v_t, \quad x_0^* = \bar{x}, \quad x_{-1}^* = 0 \quad (4)$$

$$y_t^* = \Phi_{s(t)} x_t^* + w_t \quad (5)$$

with (4)-(5) defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$.

We finally demonstrate equivalence to RE equilibrium of the obtained solution, both in finite and infinite horizon representations, by showing that the optimal (regime-dependent) feedback controller has the same structure of the RE component.

3. The solution algorithm

We define on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$ the following Markov jump (controllable) system with linear noise corrupted observations:

$$x_{t+1} = \Gamma_{s(t)}^{-1} u_t + \Psi_{s(t)} v_t, \quad x_0 = \bar{x} \quad (6)$$

$$y_t = \Phi_{s(t)} x_t + w_t \quad (7)$$

where u_t is an \mathcal{F}_t -measurable input process. Let us also introduce:

$$\epsilon_t := x_t - x_t^*, \quad z_t' := \begin{pmatrix} \epsilon_t' & x_t^{*'} & x_{t+1}^{*'} \end{pmatrix}$$

and consider the problem of finding an input sequence $u = \{u_t\}_{t \in \mathbf{T}}$, $\mathbf{T} = [0, T] \subset \mathbb{N}$, $u_t \in U_t$ - with U_t denoting the space of all square-integrable \mathcal{F}_t -measurable random vectors - which minimizes the objective functional:

$$J(u) = \mathbf{E} \sum_{t=0}^{T+1} (z_t' M z_t) \quad (\text{OF})$$

under the following state-space recursive constraints:

$$z_{t+1} = A_{s(t)} z_t + B_{s(t)} u_t + C_{s(t)} v_t, \quad z_0 = \bar{z} \quad (8)$$

$$y_t = \bar{\Phi}_{s(t)} z_t + w_t \quad (9)$$

where:

$$A_{s(t)} = \begin{pmatrix} 0 & 0 & -I \\ 0 & 0 & I \\ 0 & 0 & \Gamma_{s(t)} \end{pmatrix}; \quad B_{s(t)} = \begin{pmatrix} \Gamma_{s(t)}^{-1} \\ 0 \\ 0 \end{pmatrix}$$

$$C_{s(t)} = \begin{pmatrix} \Psi_{s(t)} \\ 0 \\ -\Gamma_{s(t)} \Psi_{s(t)} \end{pmatrix}; \quad \bar{\Phi}_{s(t)} = \begin{pmatrix} \Phi_{s(t)} \\ \Phi_{s(t)} \\ 0 \end{pmatrix}$$

and M consists of the identity matrix $I_{n \times n}$ as first block on the main diagonal and 0's elsewhere. Expression (9) can be properly used as the observation equation for the augmented Markov jump system

(8) in which the first n entries of the state vector z_t describe the evolution of the deviation from the autoregressive behavior of the MSRE model.

The design of an input sequence $\{\hat{u}_t\}$, $t \in \mathbf{T}$ minimizing (OF) subject to (8)-(9) is accomplished by employing an optimal Markov jump feedback controller in conjunction with the minimum mean-square estimate (MMSE) obtained by a time-varying Kalman filter. We indeed show that a separation principle holds for the system at issue - i.e., the optimal input sequence depends on the observed state only through the optimal estimate of the latter. In the classical literature on Markov jump linear quadratic (MJLQ) problems (e.g. [10]), it has been shown that the solution of such problems engenders a twofold set of coupled Riccati equations, each associated to the filtering and control programs respectively. Since these backward-recursive equations cannot be represented as a single higher-dimensional Riccati equation, structural concepts and algorithms from the classical linear theory are not directly applicable to Markov jump systems. While further requirements are generally needed to determine the existence of a steady-state solution for the coupled Riccati equations (e.g. [4], [8], [1]), we prove that, when applied to the solution method for MSRE models we propose in this paper, this issue vanishes for the Riccati gain is shown to admit a simple time-invariant and state-independent representation, both in finite and infinite horizon problems. The following statement clarifies this insight:

Theorem 1. *Given the system (6)-(7), the input sequence $\hat{u}_t := \{\hat{u}_t\}$ which produces for any $t = 0, 1, \dots$ the mean-square minimum deviation from the Markov jump autoregressive state motion (4), is in the form:*

$$\hat{u}_t = \Gamma_{s(t)} \hat{x}_{t+1|t}^* \quad (10)$$

where the optimal estimate $\hat{x}_{t+1|t}^* := (0 \ 0 \ I \ 0)' \mathbf{E}[z_t | \mathcal{F}_t]$ is obtained recursively via a time-varying Kalman filter.

Proof. To save notation, let us define $u_t^+ = [u_t', u_{t+1}', \dots, u_T']'$, and let $\xi^t = [\xi_0', \dots, \xi_t']'$ denote a sequence of random vectors ξ_0, \dots, ξ_T . The σ -algebra generated by ξ_0, \dots, ξ_t , namely $\sigma(\xi^t)$, will be for simplicity identified with the vector ξ^t .

We first derive the conditional expectations for the augmented state vector z_t . This is accomplished by employing a time-varying Kalman filter for the state-space system (8)-(9). Indeed, the objective is to identify at every time step t , an estimate \hat{z}_t that minimizes the mean-squared error covariance:

$$P_t = \mathbf{E}[(z_t - \hat{z}_t)(z_t - \hat{z}_t)' | s^t]$$

A potential issue lies in that the noise provides information about the state since the regime switching matrices multiplying the two depend on the same underlying Markov state. However, as long as the current realization of the Markov chain is observable, the state variable z_t and the noise v_t become independent. Likewise, though the noise turns correlated, conditioned on the current state estimate and the Markov state, the next period noise remains (conditionally) zero-mean.

Since the estimator at time t has access to observations (y_0, \dots, y_t) and the Markov state values (s_0, \dots, s_t) , the optimal linear MMSE filtering estimate $\mathbf{E}[z_t | \mathcal{F}_t]$ is obtained from a time-varying (sample path) Kalman filter (e.g. [9]). Let $s_t = i \in \mathcal{S}$ be the Markov state observed in time t , then³:

$$\hat{z}_t = \hat{z}_{t|t-1} + \bar{K}_t (y_t - \bar{\Phi}_i \hat{z}_{t|t-1}), \quad \hat{z}_0 = \mathbf{E}\{z_0\} \quad (11)$$

$$\bar{K}_t = P_{t|t-1} \bar{\Phi}_i' (I + \bar{\Phi}_i P_{t|t-1} \bar{\Phi}_i')^\dagger \quad (12)$$

$$\hat{z}_{t+1|t} = A_i \hat{z}_t + B_i u_t$$

$$P_t = P_{t|t-1} - \bar{K}_t C_i P_{t|t-1}$$

$$P_{t+1|t} = A_i P_t A_i' + C_i C_i'$$

where $P_0 = \text{cov}(z_0, z_0 | s_0)$.

Using the measurement equation (9), (11) rewrites:

$$\hat{z}_{t+1} = A_i \hat{z}_t + B_i u_t \bar{K}_t (\bar{\Phi}_i (z_t - \hat{z}_t) + w_t)$$

³Given a matrix Ξ , we denote its pseudoinverse by Ξ^\dagger .

which along with (8) yields the equation of the estimation error $\eta_t := z_t - \hat{z}_t$:

$$\eta_{t+1} = (A_i - \bar{K}_t \bar{\Phi}_i) \eta_t + C_i v_t - \bar{K}_t w_t \quad (13)$$

from which we observe that η_t is independent of u_t .

We turn now to the Markov jump LQG problem described by (OF)-(8)-(9). Let us define the cost-to-go at t :

$$J_t(u_t^+, \mathcal{F}_t) = \mathbf{E} \left\{ \sum_{s=t}^{T+1} z_s^T M z_s \middle| \mathcal{F}_t \right\} \quad (14)$$

and the optimal cost-to-go (at t):

$$J_t^*(\mathcal{F}_t) = \min_{u \in U} J_t(u_t^+, \mathcal{F}_t), \quad (15)$$

where U readily follows from the above defined U_t , and the min is taken samplewise with respect to \mathcal{F}_t . Finally denote:

$$u_t^{+*} = \arg \min_{u \in U} J_t(u_t^+, \mathcal{F}_t) \quad (16)$$

The optimality principle ensures that $(u_t^{+*})_{t+1}^+ = u_{t+1}^{+*}$, i.e. the restriction of the optimal control sequence for the t -th instance of the sequence (15) of optimal control problems, is the optimal control for the $t+1$ -th problem. Straightforward computation yields the following recursive relation between the optimal cost-to-go functionals (15):

$$J_t^*(\mathcal{F}_t) = \mathbf{E} \{ z_t' M z_t | \mathcal{F}_t \} + \min_{u_t} \mathbf{E} \{ J_{t+1}^*(\mathcal{F}_{t+1}) | \mathcal{F}_t \} \quad (17)$$

which is the general equation of the Dynamic Programming Algorithm (DPA). Going backwards, at the last stage one has:

$$u_0^{+*} = \arg \min_{u \in U} J_0(u_0^+, \mathcal{F}_0)$$

hence a fortiori:

$$u_0^{+*} = \arg \min_{u \in U} \mathbf{E} \{ J_0(u_0^+, \mathcal{F}_0) \} = \arg \min_{u \in U} J(u)$$

which delivers the desired solution.

As to the initial stage, we need $J_T^*(\mathcal{F}_T)$, which requires us to solve for:

$$u_T^* = \arg \min_{u_T} J_T(u_T, \mathcal{F}_T) \quad (18)$$

$$= \arg \min_{u_T} \mathbf{E} \{ z_T' M z_T + z_{T+1}' M z_{T+1} | \mathcal{F}_T \} \quad (19)$$

and then to substitute it into the functional:

$$\begin{aligned} J_T^* \quad (I_T) &= J_T(u_T^*, \mathcal{F}_T) \\ &= \mathbf{E} \left\{ z_T' M z_T + z_{T+1}' M z_{T+1} \middle| \mathcal{F}_T \right\} \\ &= \mathbf{E} \left\{ z_T' M z_T + z_T' A'_{s(T)} M A_{s(T)} z_T \right. \\ &\quad \left. + u_T^{*'} B'_{s(T)} M B_{s(T)} u_T^* + 2 z_T' A'_{s(T)} M B_{s(T)} u_T^* \right. \\ &\quad \left. + v_T' C'_{s(T)} M C_{s(T)} v_T \middle| \mathcal{F}_T \right\} \end{aligned} \quad (20)$$

where it has been used the independence of $z_T, s(T)$ and v_T , which implies:

$$\begin{aligned} &\mathbf{E} \left\{ z_T' \quad A'_{s(T)} M C_{s(T)} v_T \middle| \mathcal{F}_T \right\} \\ &= \mathbf{E} \left\{ z_T' A'_{s(T)} M C_{s(T)} \mathbf{E} \{ v_T \} \middle| \mathcal{F}_T \right\} = 0 \end{aligned} \quad (21)$$

as well as:

$$\begin{aligned} \mathbf{E}\left\{u_T' \quad B'_{s(T)}MC_{s(T)}v_T|\mathcal{F}_T\right\} \\ = \mathbf{E}\left\{u_T'B'_{s(T)}MC_{s(T)}\mathbf{E}\{v_T\}|\mathcal{F}_T\right\} = 0 \end{aligned} \quad (22)$$

by the independence of s^T, y^T , hence of $u_T \equiv u_T(\mathcal{F}_T)$, and v_T .

Noting that u_T only affects the quadratic form of z_{T+1} in (19), thus using the system equation, it holds:

$$u_T^* = \arg \min_{u_T} \mathbf{E}\left\{z'_{T+1}Mz_{T+1}|\mathcal{F}_T\right\} \quad (23)$$

Using (21), (22), and noting that u_T does not affect the quadratic terms in z_T and v_T , we obtain:

$$\begin{aligned} u_T^* &= \arg \min_{u_T} \mathbf{E}\left\{u_T'B'_{s(T)}MB_{s(T)}u_T \right. \\ &\quad \left. + 2z'_T A'_{s(t)}MB_{s(T)}u_T|\mathcal{F}_T\right\} \\ &= \arg \min_{u_T} \left\{u_T'B'_{s(T)}MB_{s(T)}u_T \right. \\ &\quad \left. + 2\hat{z}'_T A'_{s(t)}MB_{s(T)}u_T\right\} \end{aligned}$$

By setting to zero the derivative respect to u_T of the positive quadratic functional in the above equation, and solving with respect to u_T , we get u_T^* :

$$u_T^* = - \left(B'_{s(T)}MB_{s(T)}\right)^{-1} B'_{s(T)}MA_{s(T)}\hat{z}_T \quad (24)$$

and substituting (24) into (20), the following expression of the optimal cost at time T obtains:

$$\begin{aligned} J_T^*(I_T) &= \mathbf{E}\left\{z'_T K_T z_T + (z_T - \hat{z}_T)' L_T (z_T - \hat{z}_T) \right. \\ &\quad \left. + v'_T C'_{s(T)} M C_{s(T)} v_T|\mathcal{F}_T\right\} \end{aligned} \quad (25)$$

where:

$$L_T = A'_{s(T)} M A_{s(T)} \quad (26)$$

$$K_T = M - L_T + A'_{s(T)} M A_{s(T)} = M \quad (27)$$

Now, the DPA (17) for $t = T - 1$ implies:

$$\begin{aligned} u_{T-1}^* &= \arg \min_{u_{T-1}} \mathbf{E}\left\{J_T^*(\mathcal{F}_T)|\mathcal{F}_{T-1}\right\} \\ &= \arg \min_{u_{T-1}} \mathbf{E}\left\{z'_T K_T z_T|\mathcal{F}_{T-1}\right\}, \\ &= \arg \min_{u_{T-1}} \mathbf{E}\left\{z'_T \mathbf{E}\{K_T\} z_T|\mathcal{F}_{T-1}\right\} \end{aligned} \quad (28)$$

where the second equality comes from being the estimation error $(z_t - \hat{z}_t)$ not affected by u_t , and the third one from being z_T, \mathcal{F}_{T-1} independent of $s(T)$. Equations (23) and (28) show the recursive representation of the problem at hand, thereby the following general characterization holds for the optimal control:

$$u_t^* = \arg \min_{u_t} \mathbf{E}\left\{z'_t \mathbf{E}\{K_t\} z_t|\mathcal{F}_{t-1}\right\}$$

whose value is given by:

$$u_t^* = - \left(B'_{s(t)} \mathbf{E}\{K_t\} B_{s(t)}\right)^{-1} B'_{s(t)} \mathbf{E}\{K_t\} A_{s(t)} \hat{z}_t \quad (29)$$

where the gain K_t solves the backward-recursive equations:

$$L_t = A'_{s(t)} \mathbf{E}\{K_{t+1}\} B_{s(t)} \left(B'_{s(t)} \mathbf{E}\{K_{t+1}\} B_{s(t)} \right)^{-1} \times B'_{s(t)} \mathbf{E}\{K_{t+1}\} A_{s(t)} \quad (30)$$

$$\begin{aligned} K_t &= \mathbf{E}\{K_{t+1}\} - L_t + A'_{s(t)} \mathbf{E}\{K_{t+1}\} A_{s(t)} \\ K_{T+1} &= M \end{aligned} \quad (31)$$

As M is a square, idempotent matrix, from (27) it follows that $K_t = M$ for all periods $t = 1, \dots, T$ and states $s(t) \in S$.

Finally, by substitution of K_t in (29) we derive⁴:

$$u_t^* = \Gamma_{s(t)} \hat{x}_{t+1|t}^* \quad (32)$$

Insofar as the expression for the feedback matrices does not depend on the finite horizon T , it yields the optimal control law for all the LQG control problems in the (OF)-(8)-(9) form for any $T = 1, 2, \dots$ \square

The estimator of the one-step ahead perfect-foresight state, $\hat{x}_{t+1|t}^*$, is mean-square optimal with respect to the σ -algebra generated by the *actual* measurement process (3), the only available data⁵. Our final claim rests on showing that, for any t and the input (10), the optimal two-step ahead prediction of the perfect-foresight state x_t^* following the regime switching law of motion (4), given the measurement (y_0, \dots, y_t) and the filtration $\sigma(s^t) = (s_0, \dots, s_t)$, is equal to that relative to the actual state x_t in (6):

Theorem 2. *Let $x = (x_t), y = (y_t)$ be the solution of (6)-(7) under the control law \hat{u}_t . Then, for any t and Markov state $s(t) = i \in \{1, 2, \dots, S\}$, it holds:*

$$\hat{x}_{t+2|t} = \hat{x}_{t+2|t}^* \quad (33)$$

Proof. It readily follows from Theorem 1 and the independence assumption between v_t and s_t . \square

Consider now the perfect-foresight Markov jump state motion (4). It is easily verified that:

$$\Gamma_{s(t)}^{-1} E[x_{t+2}^* | \mathcal{F}_t] = \Gamma_{s(t)}^{-1} \hat{u}_t$$

which shows, in conjunction with the assertion of Theorem 2, that the optimal feedback controller (10) has the same structure of the conditional (rational) expectation term $E[x_{t+2} | \mathcal{F}_t]$, and hence the solution $x = (x_t), y = (y_t)$ of (6)-(7) with $\hat{u}_t \equiv E[x_{t+2} | \mathcal{F}_t]$ is an REE for the Markov-switching model (2)-(3). In other words, both in finite and infinite horizon formulations, there always exists an REE $x = (x_t)$ - that is, a stochastic sequence of (functions of) states and observables in \mathcal{F}_t fulfilling the *noncausal* regime switching RE model (2)-(3) - which is computable via a *causal* Markov jump (controllable) system of the form (6)-(7) forced by the optimal Markov jump feedback control law \hat{u} inducing the minimum variance displacement between the actual x -state and the perfect-foresight one x_t^* .

4. The stability of the RE equilibrium

As in [17], we focus on the concept of mean-square stability for RE equilibria under regime switching. We set $u_t = \hat{u}_t$ in (6)-(7), and consider the evolution equation for the RE solution as derived in Section 3:

$$x_{t+1} = \hat{x}_{t+1|t}^* + \Psi_{s(t)} v_t \quad (34)$$

⁴The third entry of \hat{z}_t is $\mathbf{E}[x_{t+1}^* | \mathcal{F}_t]$.

⁵This result, presented in [7], improves upon [13]'s filtering approach to solution of linear RE models, as their MMSE estimator rather exploits the *fictitious* observations y_t^* according to (5), which are clearly not available. Moreover, [13] only find an approximate solution to the original (linear) RE model.

with:

$$\hat{x}_{t+1|t}^* = \Gamma_{s(t-1)} \hat{x}_{t|t-1}^* + \bar{K}_{t-1} \bar{\Phi}_{s(t-1)} \eta_{t-1} + \bar{K}_{t-1} w_{t-1} \quad (35)$$

We study the first two moments of the equilibrium process x_t , i.e. $m_t = \mathbf{E}[x_t]$ and $\gamma_t = \mathbf{E}[x_t x_t']$, which characterize its mean-square stability. Indeed, system (34) is mean-square stable if its first and second moments converge to finite (possibly zero) values in the limit for $t \rightarrow \infty$. From (34) we have $m_t \rightarrow 0$ if and only if $m_t^* = \mathbf{E}[\hat{x}_{t+1|t}^*] \rightarrow 0$. Moreover, provided that the noise covariance is uniformly bounded with respect to t , i.e. there exists $L \in \mathfrak{R}$ such that:

$$\sum_{i=1}^S \|\Psi_i \Psi_i'\| \mathcal{P}\{s(t) = i\} \leq L < +\infty, \quad \forall t \quad (36)$$

then $\gamma_t \rightarrow 0$ if and only if $\gamma_t^* = \mathbf{E}[\hat{x}_{t+1|t}^* \hat{x}_{t+1|t}^{*'}] \rightarrow 0$.

Taking expectations in (35) yields:

$$\mathbf{E}[\hat{x}_{t+1|t}^*] = \mathbf{E}[\Gamma_{s(t-1)}] \mathbf{E}[\hat{x}_{t|t-1}^*]$$

from which $m_t^* \rightarrow 0$ obtains if:

$$\max_i \max_j |\lambda_j(\Gamma_i)| < 1 \quad (37)$$

where $\lambda_j(\Xi)$ denotes the j -th eigenvalue of a matrix Ξ .

As for the second moment, since η_t is orthogonal to $\hat{x}_{t+1|t}^*$ and the measurement noise w_t proves independent of x_t and the σ -algebra $\{y_k, k \leq t\}$, we readily derive:

$$\begin{aligned} \gamma_t^* &= \mathbf{E}[\Gamma_{s(t-1)} \gamma_{t-1}^* \Gamma_{s(t-1)}'] + \mathbf{E}[\bar{K}_{t-1} \bar{K}_{t-1}'] \\ &\quad + \mathbf{E}[\bar{K}_{t-1} \bar{\Phi}_{s(t-1)} \bar{P}_{t-1} \bar{\Phi}_{s(t-1)}' \bar{K}_{t-1}'] \end{aligned} \quad (38)$$

Thus, $\gamma_t^* \rightarrow 0$ for $t \rightarrow \infty$ obtains if:

$$\max_i \max_j |\lambda_j(\Gamma_i \Gamma_i')| < 1 \quad (39)$$

and:

$$\sum_{i=1}^S \|\Phi_i \Phi_i'\| \mathcal{P}\{s(t) = i\} \leq L < +\infty, \quad \forall t \quad (40)$$

$$\|\bar{P}_t\| \leq L < +\infty, \quad \forall t \quad (41)$$

In fact, (40) is always verified as \mathcal{P} is a probability measure, and from (41) it follows that $\bar{K}_t \bar{K}_t'$ is bounded as well⁶. As for \bar{P}_t , its evolution is described by the following recursive equation:

$$\begin{aligned} \bar{P}_{t+1} &= \mathbf{E}[A_{s(t)} \bar{P}_t A_{s(t)}'] + \mathbf{E}[C_{s(t)} C_{s(t)}'] \\ &\quad - \mathbf{E}\left[A_{s(t)} \bar{P}_t \bar{\Phi}_{s(t)}' \left(I + \bar{\Phi}_{s(t)} \bar{P}_t \bar{\Phi}_{s(t)}'\right)^\dagger \bar{\Phi}_{s(t)} \bar{P}_t A_{s(t)}'\right] \end{aligned} \quad (42)$$

where $\bar{P}_0 = \text{cov}(z_0, z_0)$. Riccati equations with Markov jump coefficients have been extensively studied in the engineering literature (e.g. [8], [9], [1], [10]). According to well-known results established in the mentioned references, condition (37) entails condition (41). It follows that requirement (37) - which also implies (39) - is sufficient for the mean-square stability of the obtained RE equilibrium.

⁶Note that $\mathbf{E}[A_{s(t)} A_{s(t)}']$ has the same eigenvalues as $\mathbf{E}[\Gamma_{s(t)} \Gamma_{s(t)}']$ and in addition zero eigenvalues.

5. Conclusion

In this paper, we describe a model reference adaptive approach to solution of RE models under noisy measurement and Markov jump parameters, where the expectations component is replaced by a (general-measurable) function of the actually available information. We define nearly perfect-foresight equilibrium dynamics by adapting the evolution of a *causal* system to the corresponding Markov jump perfect-foresight state behaviour (the reference model); the resulting state motion is shown to be an RE equilibrium for the original *noncausal* regime switching model for the optimal feedback control features the same structure of the conditional expectations component. As a consequence, contrary to the economic literature, our equilibrium existence result does not rest on any stochastic stability conditions or approximation hypotheses, for only initial conditions knowledge is required.

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