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EDGE-COLOURING OF JOINS OF REGULAR GRAPHS, II

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Abstract

We prove that the edges of every even graph $G = G_1 + G_2$ that is the join of two regular graphs G_1 and G_2 can be coloured with $\Delta(G)$ colours, whenever $\Delta(G) = \Delta(G_1) + |V_2|$. The proof of this result together with the results in [5] states that every even graph G that is the join of two regular graphs is Class 1.

The proof yields an efficient combinatorial algorithm to find a $\Delta(G)$-edge-colouring of this type of graphs.

Key words: Edge-colouring, Join, Regular graphs

1. Introduction

An *edge-colouring* of a graph G is an assignment of colours to its edges so that no two edges incident to the same vertex receive the same colour. A k-edge-colouring of G is then a partition of its edge set into k matchings. The *chromatic index* of G, denoted by $\chi'(G)$, is the least k for which G has a k-edge-colouring. A well known theorem of Vizing [12] states that $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$. A graph G is said to be Class 1 if $\chi'(G)=\Delta(G)$.

A graph $G=(V,E)$ is the *join* of two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ (with $V_1 \cap V_2 = \emptyset$), if $V=V_1 \cup V_2$ and $E=E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$; the graph G is called a *join graph* and is denoted as $G=G_1+G_2$. If $n_i=|V_i|$ and $\Delta(G_i)=\Delta_i$ then $\Delta(G)=\max\{n_1+\Delta_2, n_2+\Delta_1\}$.

For an extensive introduction of the edge-colouring problem we refer to [1, 2, 8, 10, 11, 12]; for the particular case of the edge-colouring problem of join graphs we refer the readers to [3] and to the first paper of this series [5] whose notation will also be kept here.

In this paper we consider the case when G has an even number of vertices and it is the join of two regular graphs. In [5] it is proved that:

Theorem 1.1. *([5]) Let $G=G_1+G_2$ be a join graph with an even number of vertices such that G_i is a non-complete k_i-regular graph with n_i vertices, $i=1,2$, and $n_1 \leq n_2$. If $\Delta(G)=k_2+n_1$ then G is Class 1.*

The arguments used to prove Theorem 1.1 cannot be extended to solve the case $\Delta(G)=k_1+n_2$. In this paper we extend some of the crucial results in [5] in order to solve this case.

Notice that the proofs in this paper are of a combinatorial nature and the arguments used can be easily translated into an algorithm that in polynomial time finds a $\Delta(G)$-edge-colouring of any even graph that is the join of two regular graphs.

2. Comparable edge-colourings

In this section we provide some preliminary results and technical lemmas. To this purpose, we recall a few notions that will be often used throughout the paper.
Let $\mathcal{C}=\{c_1,\ldots,c_k\}$ be an edge-colouring of a graph $G=(V,E)$. We shall denote by $X(c_i)$ the subset of vertices of G that are missed by colour c_i; clearly, $|X(c_i)|=|V|-2|c_i|$. The edge-colouring $\mathcal{C}$ is said to be *equitable* if each c_i has size equal to either $\lfloor |E|/k \rfloor$ or $\lceil |E|/k \rceil$. Equitable edge-colourings always exist and can be found efficiently (for details we refer the reader to [6],[9]).
We start with a theorem whose proof is an adaptation of the proof of Theorem 3 in [5]:

Theorem 2.1. *Let $G=G_1+G_2$ be an even join graph where G_1 is the empty graph on n_1 vertices and $G_2=(V_2,E_2)$ is a multigraph such that $\chi'(G_2) \leq n_2-1$. Let T be a set of n_1 edges of G such that each edge in T is incident with exactly one vertex of G_1. If $G-T$ is (n_2-1)-regular, then $\chi'(G-T)=n_2-1$.*

Proof. First of all note that:

$$|E_2| = |E(G-T)| - (n_2-1)n_1 = (n_2-1)\frac{n_2+n_1}{2} - (n_2-1)n_1.$$

Let $\mathcal{C}=\{c_1,\ldots,c_{n_2-1}\}$ be an equitable (n_2-1)-edge-colouring of G_2. Clearly, $|c_i|=(n_2-n_1)/2$, for every $i=1,\ldots,n_2-1$, and so

$$|X(c_i)| = n_2 - 2|c_i| = n_1, \quad i=1,\ldots,n_2-1. \tag{1}$$

Let Q denote the set of the $(n_2-1)n_1$ edges of $G-T$ that join the vertices in V_1 to the vertices in V_2; for every $v_j \in V_2$, let $t(v_j)$ denote the number of vertices in V_1 that are adjacent to v_j in $G-T$. To prove the theorem we only need show how to extend the colouring $\mathcal{C}$ to all the edges in Q. To this purpose, let B denote the bipartite graph with bipartition $\mathcal{C}$ and V_2, and edge set $\{c_iv_j : c_i \in \mathcal{C}, v_j \in V_2,$ and v_j missed by colour $c_i\}$ (this graph was introduced by Hoffman and Rodger in [7]).
We claim that $\Delta(B) = n_1$. To see this, first observe that, by (1), $d_B(c_i) = n_1$ for every $c_i \in \mathcal{C}$ and that $d_B(v_j) = (n_2-1) - d_{G_2}(v_j)$ for every $v_j \in V_2$. Now, since $d_{G_2}(v_j) = d_{G-T}(v_j) - t(v_j) = (n_2-1) - t(v_j)$, it follows that $d_B(v_j) = t(v_j) \le n_1$. Thus $\Delta(B) = n_1$.
Let $\mathcal{D} = \{d_1, \ldots, d_{n_1}\}$ be an equitable edge-colouring of B. Since B has precisely $2n_2-1$ vertices and $(n_2-1)n_1$ edges, it follows that $|d_i| = n_2-1$, and so $|X(d_i)| = 1$, for every $i = 1, \ldots, n_1$. Hence, for every $i = 1, \ldots, n_1$, the matching d_i misses precisely one vertex of B and this vertex is a vertex in V_2 (because $d_B(c_k) = n_1$ for every k). Now, every vertex u_i in V_1 ($i = 1, \ldots, n_1$) is adjacent in $G-T$ to every vertex in V_2, but a unique vertex, say v_{u_i}; without loss of generality, we may assume that the matching d_i of B misses precisely vertex v_{u_i}, for every $i = 1, \ldots, n_1$.
Finally, let c_iv_j be an arbitrary edge of the bipartite graph B (with $c_i \in \mathcal{C}$ and $v_j \in V_2$) and let d_r be its colour. We claim that we can colour edge v_ju_r with colour c_i. (Note that v_ju_r is an edge of $G-T$ because $v_j \neq v_{u_r}$.) To verify that the colouring so obtained is admissible, assume the contrary: there exist in Q two adjacent edges e and f that have the same colour c_i. Let $e = v_ju_r$. If $f = v_hu_r$ then in B both edges c_iv_j and c_iv_h would be coloured d_r, which is impossible; if $f = v_ju_t$ then in B the edge c_iv_j would be coloured both d_r and d_t, which again is impossible. Thus, $\chi'(G-T) = n_2-1$, and the theorem follows. ■

Definition 2.2. *Let G_1 be a graph with n_1 vertices and maximum degree k_1 and let G_2 be a k_2-regular graph with n_2 vertices such that $n_1 \le n_2$, n_1+n_2 even, and $k_1 \le k_2$. Let $\mathcal{C}_2 = \{h_1, \ldots, h_{k_2+1}\}$ be an equitable edge-colouring of G_2. An edge-colouring $\mathcal{C}_1 = \{f_1, \ldots, f_{k_1+1}\}$ of G_1 is said to be* comparable *to $\mathcal{C}_2$ if it satisfies the following two properties:*

i) there exists a partition of the set of vertices of G_1, say $\{X_1, X_2, \ldots, X_{k_1+1}\}$, such that for every vertex u of G_1, $u \in X_i$ implies that u is missed by colour f_i (i.e. $u \in X(f_i)$),

ii) there exists an ordering of the elements of $\mathcal{C}_2$ such that $|X(h_i)| \le |X_i|$ and $|X_i| - |X(h_i)|$ even ($i = 1, \ldots, k_1+1$),

The following lemma generalizes Lemma 1 in [5].

Lemma 2.3. *Let $G = G_1 + G_2$ be an even join graph such that G_i has n_i vertices with $n_1 \le n_2$, $\Delta(G_1) = k_1$, G_2 is k_2-regular with $k_1 \le k_2$, and $k_i < n_i - 1$. Let $\mathcal{C}_2 = \{h_1, \ldots, h_{k_2+1}\}$ be an equitable edge-colouring of G_2. If G_1 admits a (k_1+1)-edge-colouring that is comparable to $\mathcal{C}_2$, then G contains a spanning subgraph H having the following four properties:*

(a) $G_1 \subset H$,

(b) $\chi'(H) = k_1+1$,

(c) $d_{G-H}(u) = n_2-1$ for every vertex u of G_1,

(d) $d_H(v) = k_1+1$ for every vertex v of G_2.

Proof. Let $\mathcal{C}_1 = \{f_1, \ldots, f_{k_1+1}\}$ be an edge-colouring of G_1 that is comparable to $\mathcal{C}_2$ and let $\{X_1, \ldots, X_{k_1+1}\}$ denote a partition of the set of vertices of G_1 satisfying properties i) and ii) of

Definition 2.2.
Let $H_2 = (V_2, F_2)$ be the spanning subgraph of G_2 induced by the matchings h_i ($i = 1, \dots, k_1+1$), i.e., $F_2 = \cup_{i=1}^{k_1+1} h_i$. Note that each vertex of H_2 has degree equal to either k_1 or $k_1 + 1$; let A denote the set of vertices having degree k_1 and let B denote the set of vertices having degree $k_1 + 1$. Since $\mathcal{C}_2$ is an equitabel edge-colouring of G_2, it follows that $\{X(h_1), \dots, X_{h_{k_2+1})}\}$ is a partition of V_2 and moreover

$$A = \bigcup_{i=1}^{k_1+1} X(h_i) \quad \text{with} \quad |A| = \sum_{i=1}^{k_1+1} |X(h_i)|.$$

Set

$$\alpha_i = |X_i| - |X(h_i)|, \quad i = 1, \dots, k_1 + 1.$$

By property i) of Definition 2.2, each α_i is a nonnegative even integer. To build the required graph H we shall consider two cases.

Case 1. $\alpha_i = 0$ for every $i = 1, \dots, k_1 + 1$.
This implies that $|X_i| = |X(h_i)|$ for every i. Hence, with every vertex u_j of V_1 associate a vertex v_{u_j} of H_2 so that if $u_j \in X_i$ for some i, then $v_{u_j} \in X(h_i)$. Let $e_j = u_j v_{u_j}$, $j = 1, \dots, n_1$. Then H is the spanning subgraph of G that is formed by G_1, H_2, and the n_1 edges e_j. By property i) of Definition 2.2, if $u_j \in X_i$ then u_j misses colour f_i. Thus, we can identify each colour h_i with colour f_i, and colour edge $e_j = u_j v_{u_j}$ with the colour f_i missing both u_j and v_{u_j}, and we get a $(k_1 + 1)$-edge-colouring of H.

Case 2. $\alpha_i > 0$ for some $i \le k_1 + 1$.
This implies that there are some matchings h_i, $i \le k_1 + 1$, such that $|X_i| > |X(h_i)|$. In this case, we shall show how to remove precisely $\alpha_i/2$ edges from every matching h_i with $|X_i| > |X(h_i)|$, in order to obtain a new matching h'_i such that $|X(h'_i)| = |X(h_i)| + \alpha_i = |X_i|$.
To this purpose, set $H_2^0 = H_2 = (A^0 \cup B^0, F^0)$ where $A^0 = A$, $B^0 = B$, and $F^0 = F_2$. For each $i = 1, \dots, k_1 + 1$, we apply the following procedure:

1. If $\alpha_i = 0$, then set $h'_i = h_i$ and $H_2^i = H_2^{i-1}$.
2. If $\alpha_i \neq 0$, then select $\alpha_i/2$ arbitrary edges of h_i such that each of these edges is incident to some vertex in B^{i-1}; denote the set of these edges as T^i.
   Set $h'_i = h_i - T^i$, thus $|X(h'_i)| = |X(h_i)| + \alpha_i = |X_i|$. Note that $X(h'_i)$ could contain vertices in $X(h'_j)$ for some $j < i$; to put it differently, some vertex missed by the matching h'_i could be missed by other matchings h'_j with $j < i$.

   Let S^i denote the set of vertices in B^{i-1} that are incident to some edge in T^i. Set $H_2^i = (A^i \cup B^i, F^i)$ with $A^i = A^{i-1} \cup S^i$, $B^i = B^{i-1} - S^i$ and $F^i = F^{i-1} - T^i$.

To prove the correctness of the above procedure, it suffices to show that at each step i such that $\alpha_i \neq 0$, there always exist $\alpha_i/2$ edges of the matching h_i that are incident to some vertex in B^{i-1}. To this purpose, first note that each vertex v in B has degree $k_1 + 1$ and so every matching h_i has (exactly) one edge incident to v. Since $\{X_1, \dots, X_{k_1+1}\}$ is a partition of the set of vertices of G_1, it follows that $n_1 = \sum_{i=1}^{k_1+1} |X_i|$; since, by assumption, $n_1 \le n_2$, it follows that $n_2 \ge \sum_{i=1}^{k_1+1} |X_i|$ and so,

$$|B^0| = |B| = n_2 - |A| = n_2 - \sum_{i=1}^{k_1+1} |X(h_i)| \ge \sum_{i=1}^{k_1+1} |X_i| - \sum_{i=1}^{k_1+1} |X(h_i)| = \sum_{i=1}^{k_1+1} \alpha_i.$$

Then the set B^0 contains at least $\sum_{i=1}^{k_1+1} \alpha_i$ vertices, each incident to exactly one matching h_i. Since at each step i we remove at most α_i vertices from B^{i-1} ($|S_i| \leq \alpha_i$), it follows that $|B^i| \geq \sum_{j>i} \alpha_j$.
Next, note that each graph H_2^i ($i \geq 1$) (built at the end of step i) has the following four properties:

- every vertex in A^i has degree at most k_1,
- every vertex in B^i has degree equal to $k_1 + 1$,
- the edges of H_2^i are coloured $h'_1, \ldots, h'_i, h_{i+1}, \ldots, h_{k_1+1}$ with $|X(h'_j)| = |X_j|$ for every $j \leq i$,
- $A^i = X(h'_1) \cup \ldots \cup X(h'_i) \cup X(h_{i+1}) \cup \ldots \cup X(h_{k_1+1})$ where the sets are not necessarily disjoint.

Now, let $G'_2 = H_2^{k_1+1}$. The edges of G'_2 are coloured $h'_1, \ldots h'_{k_1+1}$ with $|X(h'_i)| = |X_i|$ ($i = 1, \ldots, k_1+1$). Since A^{k_1+1} is the union of the (not necessarily disjoint) sets $X(h'_1), \ldots, X(h'_{k_1+1})$, it follows that with every vertex u_j of G_1 we can associate a vertex v_{u_j} of G'_2, so that if $u_j \in X_i$ for some i, then $v_{u_j} \in X(h'_i)$ ($j = 1, \ldots, n_1$).
Finally, consider the spanning subgraph H of G formed by G_1, G'_2, and the n_1 edges $e_j = u_j v_{u_j}$ ($j = 1, \ldots, n_1$). To show that $\chi'(H) = k_1 + 1$, we only need to identify each colour h'_i with colour f_i ($i = 1, \ldots, k_1 + 1$), and colour edge $e_j = u_j v_{u_j}$ with the colour f_i, missing both v_{u_j} and u_j, for every $j = 1, \ldots, n_1$. Thus, the lemma follows. ∎

Observation 1. *From the proof of Lemma 2.3, it follows that the graph H is formed by G_1, a subgraph of G_2, and by a subset T of n_1 edges such that each edge in T is incident with exactly one vertex of G_1.*

Now we are ready to prove:

Theorem 2.4. *Let $G = G_1 + G_2$ be an even join graph such that G_i has n_i vertices with $n_1 \leq n_2$, $\Delta(G_1) = k_1$, G_2 is k_2-regular with $k_1 \leq k_2$, $k_i < n_i - 1$ ($i = 1, 2$), $k_1 + n_2 \geq k_2 + n_1 \geq n_2 - 1$, and k_1 is even when n_1 is odd. Let $\mathcal{C}_2 = \{h_1, \ldots, h_{k_2+1}\}$ be an equitable edge-colouring of G_2. If G_1 admits a $(k_1 + 1)$-edge-colouring that is comparable to $\mathcal{C}_2$ then G is Class 1.*

Proof. Let $\mathcal{C}_1 = \{f_1, \ldots, f_{k_1+1}\}$ be an edge-colouring of G_1 that is comparable to $\mathcal{C}_2$ and let $\{X_1, \ldots, X_{k_1+1}\}$ denote a partition of the set of vertices of G_1 satisfying properties i) and ii) of Definition 2.2. Then, for every $i = 1, \ldots, k_1 + 1$, $|X(h_i)| \leq |X(f_i)|$ (because $X_i \subseteq X(f_i)$).
Let $r = (n_2 - n_1) - (k_2 - k_1)$, and set

$$t = \begin{cases} r & \text{if } r \text{ is even} \\ r - 1 & \text{otherwise} \end{cases}$$

Consider the join graph $G^t = (G_1 \cup 0^t) + G_2$ where 0^t denotes the empty graph with t vertices; in other words, G^t is the graph obtained from G by adding t new pairwise disjoint vertices and by joining each new vertex to every vertex of G_2. Note that $G = G^t$ when $r = 0$. Now, since t is even and since $t \leq r$, it follows that G^t has an even number $n_1 + t + n_2$ of vertices with $n_1 + t \leq n_2$ and $\Delta(G^t) = k_1 + n_2$. To prove the theorem, we only need show that G^t is Class 1: indeed, as soon as this is accomplished, it follows that also G is Class 1 (because $G \subseteq G^t$ and $\Delta(G^t) = \Delta(G)$).
To this purpose, let $w_1, \ldots, w_t$ denote the vertex set of 0^t and set

$$X_i^t = \begin{cases} X_i \cup \{w_{2i-1}, w_{2i}\} & \textit{if } i = 1, \dots, t/2 \\ X_i & \textit{if } i = t/2+1, \dots, k_1+1 \end{cases}$$

(note that $t/2 \leq k_1 + 1$: if $t/2 > k_1 + 1$ then $(n_2 - n_1) - (k_2 - k_1) > 2k_1 + 2$ (because $(n_2 - n_1) - (k_2 - k_1) = r \geq t$), and so $n_2 - n_1 - k_2 > k_1 + 2$, contradicting the assumption that $n_2 - n_1 - k_2 \leq 1$). Now, $\mathcal{C}_1$ along with the partition $\{X_1^t, \dots, X_{k_1+1}^t\}$, is an edge-colouring of $G_1 \cup 0^t$ that is comparable to $\mathcal{C}_2$ (because $|X(h_i)| \leq |X_i|$ implies that $|X(h_i)| \leq |X_i^t|$). Thus, the graph G^t satisfies the hypotheses of Lemma 2.3, and so there exists a spanning subgraph H of G^t such that: $(G_1 \cup 0^t) \subset H$, $\chi'(H) = k_1 + 1$, $d_{G^t - H}(u) = n_2 - 1$ for every vertex u of G_1, $d_{G^t - H}(w) = n_2 - 1$ for every vertex w of 0^t, and $d_H(v) = k_1 + 1$ for every vertex v of G_2.
Let $\tilde{G}_2$ be the subgraph of $G^t - H$ induced by the vertices of G_2 and let T denote the set of $n_1 + t$ edges of H that are incident with exactly one vertex of $G_1 \cup 0^t$ (such set T exists by Observation 1). Note that $G^t - H = (0^{n_1+t} + \tilde{G}_2) - T$.
Now, for every vertex v of $\tilde{G}_2$,

$$d_{G^t - H}(v) = d_{G^t}(v) - d_H(v) = (k_2 + n_1 + t) - (k_1 + 1).$$

If r is even then $t = r$, and so the graph $(0^{n_1+t} + \tilde{G}_2) - T$ is $(n_2 - 1)$-regular. Moreover, since $\Delta(\tilde{G}_2) \leq k_2$ and since, by assumption, $k_2 < n_2 - 1$, it follows that $\chi'(\tilde{G}_2) \leq n_2 - 1$. In this case, the graph $0^{n_1+t} + \tilde{G}_2$, along with the set T, satisfies the hypotheses of Theorem 2.1, thus implying that $(0^{n_1+t} + \tilde{G}_2 - T$ is $(n_2 - 1)$-edge-colourable. Since $G^t - H = (0^{n_1+t} + \tilde{G}_2) - T$, it follows that $\chi'(G^t - H) = n_2 - 1$, and we are done.
Hence, we shall assume that r is odd. In this case, $t = r - 1$, and so $d_{G^t - H}(v) = n_2 - 2$ for every vertex v of $\tilde{G}_2$; moreover, $k_2 < n_2 - 2$ (because $n_2 - k_2 = r + (n_1 - k_1)$, $r \geq 1$, and (by hypothesis) $n_1 - k_1 > 1$). Since $\Delta(\tilde{G}_2) \leq k_2$, there exists an $(n_2 - 2)$-edge-colouring $\mathcal{C}$ of $\tilde{G}_2$. Note that n_2 must be even: if n_2 were odd then n_1 would be odd (because $n_1 + n_2$ is even), and so, by assumption, k_1 would be even; but then k_2 would be even (because G_2 is k_2-regular), contradicting the assumption that r is odd. So, the vertices of $\tilde{G}_2$ can be numbered $\{v_1, v_2, \dots, v_{n_2}\}$.

Let M denote the following set of $n_2/2$ pairwise nonadjacent edges $\{v_{2i-1}v_{2i} : i = 1, \dots, n_2/2\}$ and consider the graph $\tilde{G}_2'$ obtained from $\tilde{G}_2$ by adding all the edges in M (the graph $\tilde{G}_2'$ could have edges of multiplicity two). It follows that the graph $(0^{n_1+t} + \tilde{G}_2') - T$ is $(n_2 - 1)$-regular and that $\mathcal{C} \cup M$ is an $(n_2 - 1)$-edge-colouring of $\tilde{G}_2'$. Thus, the graph $0^{n_1+t} + \tilde{G}_2'$, along with the set T, satisfies the hypotheses of Theorem 2.1, and so $(0^{n_1+t} + \tilde{G}_2') - T$ is $(n_2 - 1)$-edge-colourable. Since $G^t - H = (0^{n_1+t} + \tilde{G}_2) - T$ is a subgraph of $(0^{n_1+t} + \tilde{G}_2') - T$, it follows that $\chi'(G^t - H) \leq n_2 - 1$, and the theorem follows. ■

3. Join of regular graphs

In this section, we study the edge-colouring properties of even graphs that are the join of two regular graphs G_i of degree k_i, $i = 1, 2$. Without loss of generality we assume that $n_1 \leq n_2$.

First we solve the case when $k_1 \leq k_2$, $k_i < n_i - 1$ $(i = 1, 2)$ and $k_1 + n_2 > k_2 + n_1$.

Set

$$x = \frac{n_1 k_1}{2(k_1 + 1)}, \qquad y = \frac{n_2 k_2}{2(k_2 + 1)}. \tag{2}$$

Let $\mathcal{C}_1 = \{f_1, \dots, f_{k_1+1}\}$ be an equitable edge-colouring of G_1 and let $\mathcal{C}_2 = \{h_1, \dots, h_{k_2+1}\}$ be an equitable edge-colouring of G_2. Clearly, each f_i has size equal to either $\lfloor x \rfloor$ or $\lceil x \rceil$, and so either $|X(f_i)| = n_1 - 2\lfloor x \rfloor$ or $|X(f_i)| = n_1 - 2\lceil x \rceil$. Similarly, each h_i has size equal to either $\lfloor y \rfloor$ or $\lceil y \rceil$. Note that

$$n_1 - 2\lceil x \rceil \geq 0. \tag{3}$$

Set

$$p = (k_1 + 1)(\lceil x \rceil - x) \tag{4}$$

and

$$q = (k_2 + 1)(\lceil y \rceil - y). \tag{5}$$

Then

$$q - p = (k_2 - k_1)(\lceil y \rceil - y) + (k_1 + 1)(\lceil y \rceil - \lceil x \rceil - y + x). \tag{6}$$

Note that

$$\text{when } x \text{ is not integer, } p \text{ is the number of } f_i \text{ having size } \lfloor x \rfloor. \tag{7}$$

To show the validity of (7), it is sufficient to note that, when x is not integer, we can write

$$p\lfloor x \rfloor + (k_1 + 1 - p)\lceil x \rceil = -p + (k_1 + 1)\lceil x \rceil;$$

and so, by (4),

$$p\lfloor x \rfloor + (k_1 + 1 - p)\lceil x \rceil = x(k_1 + 1);$$

where $x(k_1 + 1)$ is, by (2), nothing but the number of all edges of G_1. Similarly,

$$\text{when } y \text{ is not integer, } q \text{ is the number of } h_i \text{ having size } \lfloor y \rfloor. \tag{8}$$

Theorem 1 in [5] shows a sufficient condition for two equitable edge-colourings to be comparable. In the current terminology that result can be restated as follows:

Theorem 3.1. *([5]) For $i = 1, 2$, let G_i be a graph with n_i vertices and maximum degree k_i; let $\mathcal{C}_1 = \{f_1, \ldots, f_{k_1+1}\}$ and $\mathcal{C}_2 = \{h_1, \ldots, h_{k_2+1}\}$ be equitable edge-colourings of G_1 and G_2, respectively. Assume that $n_1 \leq n_2$, $n_1 + n_2$ even, and $k_1 \leq k_2$.*
If $y - x \geq (n_2 - n_1)/2$ then $\mathcal{C}_1$ is comparable with $\mathcal{C}_2$.

The above result together with Theorem 2.4 implies that:

Corollary 3.2. *Let $G = G_1 + G_2$ be an even join graph such that G_i are regular, $i = 1, 2$, $n_1 \leq n_2$, $k_1 \leq k_2$ and $k_1 + n_2 > k_2 + n_1$. If $k_2 + n_1 \geq n_2 - 1$ and $y - x \geq (n_2 - n_1)/2$ then G is Class 1.*

Proposition 3.3. *Let $G = G_1 + G_2$ be an even join graph such that G_i are regular, $i = 1, 2$, $n_1 \leq n_2$, $k_1 \leq k_2$ and $k_1 + n_2 > k_2 + n_1$. If*

$$y - x \leq \frac{n_2 - n_1}{2} - 1 \tag{9}$$

then $(n_2 - n_1) - (k_2 - k_1) \geq 2(k_2 + 1)$.

Proof. To prove the validity of Proposition 3.3, it is sufficient to note that (9) and (2) imply

$$(n_2 - n_1)k_1k_2 + n_2k_2 - n_1k_1 \leq (n_2 - n_1)(k_1k_2 + k_1 + k_2 + 1) - 2(k_1 + 1)(k_2 + 1),$$

that is

$$0 \leq n_2(k_1 + 1) - n_1(k_2 + 1) - 2(k_1 + 1)(k_2 + 1).$$

Now, it is easy to verify that the above inequality can be written as

$$(k_2 - k_1)(n_1 - k_1 - 1) \leq (k_1 + 1)(n_2 - n_1 - k_2 + k_1 - 2k_2 - 2).$$

Then $n_2 - n_1 - k_2 + k_1 - 2k_2 - 2$ is non negative, and so $(n_2 - n_1) - (k_2 - k_1) \geq 2(k_2 + 1)$, and we are done. ■

Lemma 3.4. *Let $G = G_1 + G_2$ be an even join graph such that G_i are regular, $i = 1, 2$, $n_1 \leq n_2$, $k_1 \leq k_2$ and $k_1 + n_2 > k_2 + n_1$. If*

$$\frac{n_2 - n_1}{2} - 1 < y - x < \frac{n_2 - n_1}{2} \tag{10}$$

then x or y is not integer; moreover either

$$\lceil y \rceil - \lceil x \rceil = \frac{n_2 - n_1}{2} \quad \text{and} \quad 0 < q - p \leq \frac{n_2 - n_1}{2} \tag{11}$$

or

$$\lceil y \rceil - \lceil x \rceil = \frac{n_2 - n_1}{2} - 1, \; n_2 - n_1 - k_2 + k_1 > 1, \quad \text{and} \quad q - p \leq \frac{n_2 - n_1}{2} - (k_2 + 1). \tag{12}$$

Proof. Set $N = (n_2 - n_1)/2$. Since N is integer, by (10), at least one of x and y is not integer. Since $\lceil y \rceil \geq y$ and since $\lceil x \rceil < x + 1$, it follows that $\lceil y \rceil - \lceil x \rceil > y - x - 1$, and so, by (10), $\lceil y \rceil - \lceil x \rceil > N - 2$; since $\lceil y \rceil < y + 1$ and since $\lceil x \rceil \geq x$, it follows that $\lceil y \rceil - \lceil x \rceil < y - x + 1$, and so, by (10), $\lceil y \rceil - \lceil x \rceil < N + 1$. Then, by integrality, either $\lceil y \rceil - \lceil x \rceil = N$ or $\lceil y \rceil - \lceil x \rceil = N - 1$. Now, by (2), $x(k_1 + 1) = n_1k_1/2$ and $y(k_2 + 1) = n_2k_2/2$; hence, by (4) and (5), we can write

$$p = (k_1 + 1) \lceil x \rceil - \frac{n_1k}{2} \quad \text{and} \quad q = (k_2 + 1) \lceil y \rceil - \frac{n_2k}{2}.$$

First, assume that $\lceil y \rceil - \lceil x \rceil = N$. By (6), we can write

$$q - p = (k_2 - k_1)(\lceil y \rceil - y) + (k_1 + 1)(N - y + x) \tag{13}$$

and so, by (10),

$$q - p > (k_2 - k_1)(\lceil y \rceil - y).$$

Thus, $q > p$. Now, it is easy to verify that (13) can be rewritten as

$$q - p = \frac{k_2 - k_1}{2}(2 \lceil x \rceil - n_1) + N,$$

and so, by (3), $q - p \leq N$.
Next, assume that $\lceil y \rceil - \lceil x \rceil = N - 1$. Then, by (4) and (5), we can write

$$q-p=\frac{k_2-k_1}{2}(2\lceil x\rceil-n_1)+N-(k_2+1),$$

and so, by (3),

$$q-p\leq N-(k_2+1) \tag{14}$$

(because $2\lceil x\rceil\leq n_1$).
Thus, to prove the lemma we only need show that $n_2-n_1-k_2+k_1>1$. To this purpose, assume the contrary: $n_2-n_1-k_2+k_1\leq 1$. Since $k_1+n_2-k_2-n_1>0$ by hypothesis, it follws that $n_2-n_1-k_2+k_1=1$ and so,

$$N-\frac{k_2-k_1+1}{2}=0. \tag{15}$$

Since $(k_1+1-p)+q>0$, by (15), we can write

$$k_1+1-p+q>N-\frac{k_2-k_1+1}{2}.$$

Then, $q-p-N>-(k_1+1)-(k_2-k_1+1)/2$, and so (by (14)) $-(k_2+1)>-(k_1+1)-(k_2-k_1+1)/2$, that is $k_2-k_1<1$. But then $k_2=k_1$, contradicting the assumption that $n_2-n_1-k_2+k_1=1$.
■

Theorem 3.5. *Let $G=G_1+G_2$ be a join graph with an even number of vertices such that G_i is a non-complete k_i-regular graph with n_i vertices, $i=1,2$, and $n_1\leq n_2$. If $\Delta(G)>k_2+n_1$ then G is Class 1.*

Proof. Let $\mathcal{C}_1=\{f_1,\dots,f_{k_1+1}\}$ and $\mathcal{C}_2=\{h_1,\dots,h_{k_2+1}\}$ be equitable edge-colourings of G_1 and G_2, respectively.
Clearly, $\{X(f_1),\dots,X(f_{k_1+1})\}$ satisfies property i) of Definition 2.2. Thus, $\mathcal{C}_1$ is comparable to $\mathcal{C}_2$ if and only if there exists an ordering of the elements of $\mathcal{C}_2$ such that $|X(h_i)|-|X(f_i)|\leq 0$ for $i=1,\dots,k_1+1$.

Since $\Delta(G)=k_1+n_2>k_2+n_1$, we set $r=(n_2-n_1)-(k_2-k_1)>0$ and

$$t=\begin{cases} r & \textit{if } r \textit{ is even}\\ r-1 & \textit{if } r \textit{ is odd.}\end{cases}$$

Note that t is even and that

$$\text{when } r \text{ is odd, the parity of } k_1 \text{ and } k_2 \text{ is different,} \tag{16}$$

and so both n_1 and n_2 must be even (because G_i is k_i-regular, $i=1,2$); moreover if n_i is odd then k_i must be even, $i=1,2$. We shall distinguish among two cases: $r\leq k_1+1$ and $r>k_1+1$.
Case 1 $r\leq k_1+1$.
Since $r\leq k_1+1$, it follows that

$$k_2+n_1\geq n_2-1, \tag{17}$$

and that $(n_2-n_1)-(k_2-k_1)<2(k_2+1)$. Thus, by Proposition 3.3, $y-x>(n_2-n_1)/2-1$. If $y-x\geq(n_2-n_1)/2$ then G satisfies the hypotheses of Corollary 3.2, thus implying that G is

Class 1.
Hence we shall assume that $y - x < (n_2 - n_1)/2$, and so

$$\frac{n_2 - n_1}{2} - 1 < y - x < \frac{n_2 - n_1}{2}. \tag{18}$$

By Lemma 3.4, x and y cannot be both integer and either (11) holds or (12) holds.

Subcase 1.1. (11) holds.

In this case, y cannot be integer: if y is integer then (by (5)) $q = 0$, and so (by 11)) $p < 0$, which is impossible.
First, assume that $k_2 + 1 - q \geq k_1 + 1 - p$. If x is integer, then (by 4)) $p = 0$, and so $k_2 + 1 - q \geq k_1 + 1$; moreover $|X(f_i)| = n_1 - 2x$ for every $i = 1, \ldots, k_1 + 1$. In this case, we can always order the elements of $\mathcal{C}_2$ so that $\mathcal{C}_1$ is comparable to $\mathcal{C}_2$: order the h_i so that $|X(h_i)| = n_2 - 2\lceil y \rceil$ for $i = 1, \ldots, k_1 + 1$. Hence G satisfies the hypotheses of Theorem 2.4, which implies that G is Class 1. If x is not integer, then again we can always order the elements of $\mathcal{C}_2$ so that $\mathcal{C}_1$ is comparable to $\mathcal{C}_2$: order the h_i so that $|X(h_i)| = n_2 - 2\lfloor y \rfloor$ for $i = 1, \ldots, p$, and $|X(h_i)| = n_2 - 2\lceil y \rceil$ for $i = p+1, \ldots, k_1 + 1$. Hence G again satisfies the hypotheses of Theorem 2.4, and so G is Class 1.
Hence we can assume that $k_2 + 1 - q < k_1 + 1 - p$ and so $k_2 - k_1 < q - p$. By (11), $k_2 - k_1 < (n_2 - n_1)/2$. In this case, it is easy to verify that it is impossible to find an ordering of the elements of $\mathcal{C}_2$ so that $\mathcal{C}_1$ is comparable to $\mathcal{C}_2$. Then, let $N = 2[(q-p) - (k_2 - k_1)]$ and consider the even join graph $G^N = (G_1 \cup 0^N) + G_2$ where 0^N denotes the empty graph with N vertices. Clearly, $N \leq (n_2 - n_1) - (k_2 - k_1)$. As a consequence, $k_1 + n_2 \geq k_2 + n_1 + N$ and $n_2 \geq n_1 + N$; moreover (by (17)) $k_2 + n_1 + N > n_2 - 1$. Since $G \subset G^N$ and $\Delta(G) = \Delta(G^N)$, to show that G is Class 1, it is sufficient to show that G^N is Class 1. To this purpose, we shall show that there exists an edge-colouring of $G_1 \cup 0^N$, say $\{f'_1, \ldots, f'_{k_1+1}\}$ that is comparable to $\mathcal{C}_2$. Indeed, as soon as this is accomplished, we can apply Theorem 2.4 to the join graph G^N and deduce that G^N is Class 1.
Let $w_1, \ldots, w_N$ denote the vertex set of 0^N; since $\mathcal{C}_1$ is also an edge-colouring of $G_1 \cup 0^N$, we can choose $f'_i = f_i$ for all i. To partition the vertex set of $G_1 \cup 0^N$, set

$$X_i = \begin{cases} X(f_i) & \text{\textit{if} } i = 1, \ldots, p \\ X(f_i) \cup \{w_{2i-1}, w_{2i}\} & \text{\textit{if} } i = p+1, \ldots, p + N/2 \\ X(f_i) & \text{\textit{if} } i = p + N/2 + 1, \ldots, k_1 + 1 \end{cases}$$

whenever x is not integer (note that $p + N/2 = q - (k_2 - k_1) < k_1 + 1$, because $q < k_2 + 1$); and set

$$X_i = \begin{cases} X(f_i) \cup \{w_{2i-1}, w_{2i}\} & \text{\textit{if} } i = 1, \ldots, N/2 \\ X(f_i) & \text{\textit{if} } i = N/2 + 1, \ldots, k_1 + 1 \end{cases}$$

whenever x is integer. But then we can always order the elements of $\mathcal{C}_2$ so that $\{f'_1, \ldots, f'_{k_1+1}\}$ is comparable to $\mathcal{C}_2$: order the h_i so that if x is not integer then $|X(h_i)| = n_2 - 2\lfloor y \rfloor$ for $i = 1, \ldots, p$ (because $q > p$) and $|X(h_i)| = n_2 - 2\lceil y \rceil$ for $i = p+1, \ldots, k_1 + 1$; if x is integer then $|X(h_i)| = n_2 - 2\lfloor y \rfloor$ for $i = 1, \ldots, N/2$ and $|X(h_i)| = n_2 - 2\lceil y \rceil$ for $i = N/2 + 1, \ldots, k_1 + 1$. Hence G^N satisfies the hypotheses of Theorem 2.4, and so G^N is Class 1 and we are done.

Subcase 1.2. (12) holds.

In this case, x cannot be integer (for otherwise, $(n_2 - n_1)/2 - 1 = \lceil y \rceil - \lceil x \rceil \geq y - x$, contradicting (18)); moreover, it is easy to verify that it is impossible to find an ordering of the elements of $\mathcal{C}_2$ so that $\mathcal{C}_1$ is comparable to $\mathcal{C}_2$. Then we shall proceed as in the previous case.

To this purpose, first observe that $(n_2 - n_1)/2 - (k_2 + 1) = r/2 - (k_2 + k_1)/2 - 1$; since $k_1 \leq k_2$ and $r \leq k_1 + 1$, it follows that $(n_2 - n_1)/2 - (k_2 + 1) < 0$. Hence (by (12)) $q < p$, and so $k_2 + 1 - q > k_1 + 1 - p$.
Recall that $t = r$ (when r is even) or $t = r - 1$ (when r is odd). We claim that

$$(k_1 + 1) - (p - q) \leq \frac{t}{2}. \tag{19}$$

To prove the claim, suppose the contrary: $(k_1 + 1) - (p - q) > t/2$. But then (by (12)), when $t = r$ we can write

$$-(k_2 + 1) \geq (q - p) - \frac{n_2 - n_1}{2} > -(k_1 + 1) - \frac{k_2 - k_1}{2},$$

and so $k_1 > k_2$, contradicting the hypothesis; if $t = r - 1$ we can write

$$-(k_2 + 1) \geq (q - p) - \frac{n_2 - n_1}{2} > -(k_1 + 1) - \frac{k_2 - k_1 + 1}{2},$$

and so $k_1 = k_2$, which is again impossible (because, when r is odd, k_1 and k_2 have different parity). Thus the claim follows.
Now, let $N = 2[(k_1 + 1 - (p - q)]$ and consider the even join graph $G^N = (G_1 \cup 0^N) + G_2$ where 0^N denotes the empty graph with N vertices. By (19), $N \leq t \leq r = (n_2 - n_1) - (k_2 - k_1)$; and so $k_1 + n_2 \geq k_2 + n_1 + N$ and $n_2 \geq n_1 + N$. ; moreover (by (17)) $k_2 + n_1 + N > n_2 - 1$ Since $G \subset G^N$ and $\Delta(G) = \Delta(G^N)$, to show that G is Class 1, it is sufficient to show that G^N is Class 1. To this purpose, we shall show that there exists an edge-colouring of $G_1 \cup 0^N$, say $\{f'_1, \ldots, f'_{k_1+1}\}$ that is comparable to $\mathcal{C}_2$. Indeed, as soon as this is accomplished, we can apply Theorem 2.4 to the join graph G^N and deduce that G^N is Class 1.
Let $w_1, \ldots, w_N$ denote the vertex set of 0^N; since $\mathcal{C}_1$ is also an edge-colouring of $G_1 \cup 0^N$, we can choose $f'_i = f_i$ for all i. To partition the vertex set of $G_1 \cup 0^N$, set

$$X_i = \begin{cases} X(f_i) \cup \{w_i, w_{i+1}\} & i = 1, \ldots, q \\ X(f_i) & i = q + 1, \ldots, p \\ X(f_i) \cup \{w_i, w_{i+1}\} & i = p + 1, \ldots, k_1 + 1 \end{cases}$$

whenever y is not integer; and set

$$X_i = \begin{cases} X(f_i) & i = 1, \ldots, p \\ X(f_i) \cup \{w_i, w_{i+1}\} & i = p + 1, \ldots, k_1 + 1 \end{cases}$$

whenever y is integer. But then we can always order the elements of $\mathcal{C}_2$ so that $\{f'_1, \ldots, f'_{k_1+1}\}$ is comparable to $\mathcal{C}_2$: order the h_i so that if y is not integer then $|X(h_i)| = n_2 - 2\lfloor y \rfloor$ for $i = 1, \ldots, q$ and $|X(h_i)| = n_2 - 2\lceil y \rceil$ for $i = q + 1, \ldots, k_1 + 1$; if y is integer then $|X(h_i)| = n_2 - 2y$ for $i = 1, \ldots, k_1 + 1$. Hence G^N satisfies the hypotheses of Theorem 2.4, and so G^N is Class 1 and again we are done.

Case 2. $r > k_1 + 1$.

In this case, consider the join graph $G^+ = (G_1 \cup R_t) + G_2$ where R_t denotes a k_1-regular graph with t vertices (R_t exists because t is even and $t \geq k_1 + 1$). Now, G^+ has $n_1 + t + n_2$ vertices with $n_1 + t \leq n_2$ (because $t \leq (n_2 - n_1) - (k_2 - k_1)$); since t is even, it follows that G^+ has an even number of vertices. Moreover, $G \subset G^+$ and $\Delta(G^+) = \Delta(G) = k_1 + n_2$. It follows that if G^+ is Class 1, also G is. If $t = r$, then G^+ is a regular join graph with an even number of vertices, and so G^+ is Class 1 (see Theorem 2 in [4]). If $t = r - 1$ then it easy to verify that we are back to Case 1 with G replaced by G^+: indeed if we set $r' = [n_2 - (n_1 + t)] - (k_2 - k_1)$, we have $r' = r - t = 1$ and so, $r' \leq k_1 + 1$. Hence G^+ is Class 1, and the theorem follows. ■

We can now state the final result:

Corollary 3.6. *Every even graph that is the join of two regular graphs is Class 1.*

Proof. Let $G = G_1 + G_2$ be a graph with an even number of vertices obtained as the join of two regular graphs G_i of degree k_i, $i = 1, 2$. Let n_i denote the number of vertices of G_i, $i = 1, 2$. Without loss of generality we may assume that $n_1 \leq n_2$. In [3] it was shown that if $k_1 > k_2$ then G is Class 1. Hence we may assume that $k_1 \leq k_2$.

If $k_1 = n_1 - 1$ then $\Delta(G) = n_1 + n_2 - 1 = \Delta(K_{n_1+n_2})$, but then G is Class 1 (because $G \subset K_{n_1+n_2}$ and $K_{n_1+n_2}$ is Class 1). Hence, we may assume that $k_1 < n_1 - 1$, and analogously that $k_2 < n_2 - 1$.

Clearly, $\Delta(G) = \max\{k_2 + n_1, k_1 + n_2\}$. Now, if $\Delta(G) = k_2 + n_1$ then G is Class 1 by Theorem 1.1. If $\Delta(G) > k_2 + n_1$ then G is Class 1 by Theorem 3.5, and the thesis follows. ∎

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