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**QUADRATIC OPTIMAL CONTROL OF LINEAR
SYSTEMS WITH TIME-VARYING INPUT
DELAY**

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Abstract

In this paper the optimal control problem for linear time-varying systems with delay in the control input is investigated, in the time-continuous framework. This note aims to propose a different functional approach, whose versatility could result successful in applying it to the case of time-varying delays. The optimal solution at the current time t is proven to be a feedback from the state of the system at time t , provided that a Riccati-like integral differential equation admits solutions.

1. Introduction

The optimal control problem for linear systems with delayed input is a challenging research topic which has received much attention in the literature in both the discrete and the continuous time cases. A time-delay in the input is frequently encountered in many engineering frameworks, such as network control systems and process control, for instance, due to communication of the input signals (see, e.g. [1, 2]).

For the discrete time case, the reader can refer to the pioneering works [3, 4, 5] or to the more recent [6] where the optimal control problem is set in the general framework of time-varying multiple input delays.

As far as the continuous time case, the optimal control problem of linear systems with input delays has been treated, among the others, by [7, 8] for time-invariant systems with constant input delays, and by [9, 10, 11] for time-varying systems with single input multiple delays. In [6] the time-varying multiple input delays case has been treated for both discrete and continuous time framework.

In [12] the H_2 -optimal control of time invariant linear systems with multiple constant input/output delays is studied. In [13] the same problem is investigated in an H_∞ control setting.

In the above mentioned articles a solution to the problem at issue has been found for constant delays and in particular in [6] the optimal control input is given as the sum of a feedback of the state variable and of an integral in the delay interval of the control input itself. The question whether a (single) state-feedback form exists for the optimal solution, and the extent to which such solution holds even for time-varying delays has remained up to now unsolved.

In this paper we investigate the problem of finding the optimal control law for linear time-varying systems with time-varying delay in the input. For such a problem, we show that a state-feedback optimal solution indeed exists, provided a Riccati-like integral differential system of equations, which generalizes the well-known Linear-Quadratic Optimal control scheme for standard ordinary differential systems, admits a solution.

The paper is organized as follows: in §2 the notation used in the paper is set, whereas in §3 the precise statement of the problem is given. The main result is presented in §3 as Theorem 1. In the conclusive §4 some comments about the obtained results are given and further developments are pointed out.

2. Notation

The following notation will be used. For a given interval $I \subset \mathbb{R}$, $\mathbf{L}^2(I, \mathbb{R}^\alpha)$ will denote the (Hilbert) space of the square Lebesgue-integrable \mathbb{R}^α -valued functions defined over I . H^* will denote the dual space of an Hilbert space H . We use the symbol (x, y) , $x, y \in \mathbb{R}^n$, to denote the standard inner product between real-valued vectors, i.e. $(x, y) = y^T x$ whereas $\langle f, g \rangle = \int_I f^T(t)g(t)dt$, for $f, g \in \mathbf{L}^2(I, \mathbb{R}^\alpha)$, is the standard L_2 -inner product. If T is a linear operator between linear spaces, T^* will denote the adjoint. Moreover, consider the family of square-integrable matrices $\{M(t)\}$ mapping \mathbb{R}^α into \mathbb{R}^β . Then, the following operator $[M]$ may be defined:

$$\begin{aligned} [M] : \mathbf{L}^2(I, \mathbb{R}^\alpha) &\mapsto \mathbf{L}^2(I, \mathbb{R}^\beta), \quad \text{with:} \\ ([M]f)(\tau) &= M(\tau)f(\tau). \end{aligned} \tag{1}$$

In the following M will be sometimes written in place of $[M]$ (with a little abuse of notation). Notice that $[\cdot]$ define a linear transformation so $[T] + [S] = [T + S]$, for any matrix-functions

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T, S . Also, given a family of (suitably dimensioned) matrices $A(t)$, $A[M] = [AM]$.

3. Problem Setting

Consider a system described by the following linear differential equation with input delay

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t - h(t)), & t \geq t_i - \bar{h}, \\ x(t_i - \bar{h}) &= \bar{x}, \end{aligned} \quad (2)$$

where: $t_i, t_f \in \mathbb{R}$, $t_f > t_i$, h is the (non negative) delay function, upper bounded by a (known) positive real \bar{h} :

$$\sup_{t \in \mathbb{R}} \{h(t)\} = \bar{h} < +\infty, \quad (3)$$

$x(t) \in \mathbb{R}^n$ is the state of the system and $u \in \mathcal{U}$ with \mathcal{U} – a subspace of $\mathbf{L}_2([t_i - 2\bar{h}, t_f]; \mathbb{R}^p)$ – is the delayed control input. Moreover, $h(t)$ is supposed to satisfy the following hypothesis:

$$\exists h^* : \mathbb{R} \rightarrow \mathbb{R} : \quad h^*(t - h(t)) = t. \quad (4)$$

Hypothesis (4) is a technical one, assuring the existence of certain adjoint operators in the following. The above model describes the physical situation where the controller starts sending the input at time $t_i - 2\bar{h}$, and the system starts at time $t_i - \bar{h}$, with a given initial state \bar{x} . The following condition :

$$u(t) = 0, \quad t < t_i - \underline{h}, \quad (5)$$

where

$$\underline{h} = \inf_{t \in \mathbb{R}} \{h(t)\}, \quad (6)$$

which corresponds to the physical situation of system unaffected by the input before t_i , endows the set of basic assumptions for the control problem we are going to introduce.

We consider here the optimal control problem: find

$$\inf_{u \in \mathcal{U}} J(u), \quad (7)$$

under the differential constraint (2), where J is the quadratic functional:

$$\begin{aligned} J(u) &= x^T(t_f)Fx(t_f) \\ &+ \int_{t_i}^{t_f} \left(x^T(t)Q(t)x(t) + u^T(t - h(t))R(t)u(t - h(t)) \right) dt \end{aligned} \quad (8)$$

with $F, \{Q(t)\}$ families of symmetric positive semi-definite matrices, and $\{R(t)\}$ is a family of symmetric positive definite matrices.

4. Main result

Let $I_k = [t_i - k\bar{h}, t_f]$, $k = 0, 1, 2, \dots$. For a given function $f \in \mathbf{L}^2(I_1, \mathbb{R}^\alpha)$ we define $f^h \in \mathbf{L}^2(I, \mathbb{R}^\alpha)$ as:

$$f^h(t) = f(t - h(t)). \quad (9)$$

Let us rewrite the input-delayed system (2), according to (9):

$$\dot{x}(t) = A(t)x(t) + B(t)u^h(t). \quad (10)$$

The explicit solution of (10) may be written, for $t \in [t_i, t_f]$, as:

$$x(t) = \Phi_h^t x^h(t) + \Psi_h^t u^h, \quad (11)$$

where $\Phi_h^t = \Phi(t, t - h(t)) : \mathbb{R}^n \mapsto \mathbb{R}^n$ is the state-transition matrix and Ψ_h^t is the linear map

$$\Psi_h^t : \mathbf{L}^2(I; \mathbb{R}^p) \mapsto \mathbb{R}^n, \quad (12)$$

defined by

$$\Psi_h^t f = \int_{t_i - \bar{h}}^{t_f} \Phi(t, \tau) B(\tau) \chi_{[t-h(t), t]}(\tau) f(\tau) d\tau. \quad (13)$$

with $\chi_I(\cdot)$ denoting the characteristic function over an interval $I \subset \mathbb{R}$.

Remark 1. Define the following operators:

$$\begin{aligned} \phi_h &: \mathbf{L}^2(I_0; \mathbb{R}^n) \rightarrow \mathbf{L}^2(I_0; \mathbb{R}^n), \\ \psi_h &: \mathbf{L}^2(I_1; \mathbb{R}^n) \rightarrow \mathbf{L}^2(I_0; \mathbb{R}^n), \\ (\phi_h f)(t) &= \Phi_h^t f(t), \quad (\psi_h g)(t) = \Psi_h^t g. \end{aligned} \quad (14)$$

Then, the solution $x \in \mathbf{L}^2(I_0; \mathbb{R}^n)$ may be written as:

$$x = \phi_h x^h + \psi_h u^h. \quad (15)$$

Also, note that:

$$(\phi_h^* f)(t) = \Phi^T(t, t - h(t)) f(t), \quad (16)$$

and

$$\begin{aligned} (\psi_h^* f)(t) &= B^T(t) \int_{t_i - \bar{h}}^{t_f} \Phi^T(\tau, t) f(\tau) \\ &\quad \cdot \chi_{[\max\{t_i, t\}, \min\{h^*(t), t_f\}]}(\tau) d\tau, \end{aligned} \quad (17)$$

where h^* is defined in (4). This last computation is achieved by suitably exploiting the inner product properties:

$$\begin{aligned} \langle \psi_h f, g \rangle &= \int_{t_i}^{t_f} g^T(t) \int_{t_i - \bar{h}}^{t_f} \Phi(t, \tau) B(\tau) f(\tau) \\ &\quad \cdot \chi_{[t-h(t), t]}(\tau) d\tau dt \\ &= \int_{t_i}^{t_f} \int_{t-h(t)}^t g^T(t) \Phi(t, \tau) B(\tau) d\tau dt \\ &= \int_{t_i - \bar{h}}^{t_f} \left(\int_{\max\{t_i, \tau\}}^{\min\{h^*(\tau), t_f\}} g^T(t) \Phi(t, \tau) dt \right) \\ &\quad \chi_{[t_i - h(t_i), t_f]} B(\tau) f(\tau) d\tau \\ &= \langle f, \psi_h^* g \rangle. \end{aligned} \quad (18)$$

6.

We are now in a position to show our main result, which is stated in the following theorem.

Theorem 1. *Suppose there exist a solution, $(\mathbf{L}, \mathbf{V}, K)$, for the following set of operator-valued equations:*

$$K = [R] + \psi_h^* [L^T] K [L] \psi_h + B^T S \psi_h + \psi_h^* S^* B \quad (19)$$

$$S = [V] + V^h, \quad (20)$$

$$[L^T] K = -[L^T] K [L] \psi_h - S^* B, \quad (21)$$

$$\begin{aligned} & [\dot{V} + A^T V + V A + Q + 2V] - (1 - \dot{h})(V_h + V_h^*) \\ & + \partial V^h + (\partial V^h)^* + A^T V^h + V^h A = [L^T] K [L] \end{aligned} \quad (22)$$

$$\begin{aligned} V(t_f, t_f) &= F; \quad V(t_f, t) = 0, \quad V(t, t_f) = 0, \\ & (t_f - h(t_f) \leq t < t_f) \end{aligned} \quad (23)$$

where $K : \mathbf{L}^2(I_1; \mathbb{R}^p) \mapsto \mathbf{L}^2(I_1; \mathbb{R}^p)$ is a self-adjoint, non-negative operator, with kernel $\mathbf{K}(\cdot, \cdot)$:

$$(Kf)(t) = \int_{t-h(t)}^t \mathbf{K}(t, \tau) f(\tau) d\tau, \quad (24)$$

\mathbf{L} is a family of matrices $\{L(t) \in \mathbb{R}^{n \times n}, t \in \mathbb{R}\}$, defining the operator $[L]$ accordingly to definition given in (1); \mathbf{V} a family of differentiable matrices $\{V(t, t) \in \mathbb{R}^{n \times n}, t \in \mathbb{R}\}$, defining (with some abuse of notation) the operators $[V], [\dot{V}], V_h, \partial V^h, V^h : \mathbf{L}^2(I_1; \mathbb{R}^n) \mapsto \mathbf{L}^2(I_1; \mathbb{R}^n)$ as follows:

$$([V]f)(t) = V(t, t)f(t), \quad (25)$$

$$([\dot{V}]f)(t) = \dot{V}(t, t)f(t), \quad (26)$$

$$(V_h f)(t) = V(t, t-h(t))f(t-h(t)) \quad (27)$$

$$(\partial V^h f)(t) = \int_{t-h(t)}^t \frac{\partial V}{\partial t}(t, \tau) f(\tau) d\tau \quad (28)$$

$$(V^h f)(t) = \int_{t-h(t)}^t V(t, \tau) f(\tau) d\tau. \quad (29)$$

Then at each time t the optimal control problem (7) has a feedback solution, namely $u^o(t)$, given by:

$$u^o(t) = L(h^*(t)) \Phi_h^{h^*(t)} x(t) \quad (30)$$

where h^* is the function defined in (4).

Proof. According to the notation stated in the previous section, the functional can be rewritten as:

$$J(u) = (Fx(t_f), x(t_f)) + \langle Qx, x \rangle + \langle Ru^h, u^h \rangle \quad (31)$$

Let us define:

$$\begin{aligned} \xi(t) &= (V(t)x(t), x(t)) + \int_{t-h(t)}^t (V(t, \tau)x(\tau), x(t)) d\tau, \\ &+ \int_{t-h(t)}^t (V(\tau, t)x(t), x(\tau)) d\tau \end{aligned} \quad (32)$$

where it is $V(t) = V(t, t)$, and $V(\cdot, \cdot)$ is any matrix function solving (22), and as such satisfies $V(t, t) = V^T(t, t) \geq 0$, $V(t, \tau) = V^T(\tau, t)$, as (22) is an auto-adjoint equation. By derivating

(32), one obtains

$$\begin{aligned}
\dot{\xi}(t) &= (\dot{V}(t)x(t), x(t)) \\
&\quad + (V(t)\dot{x}(t), x(t)) + (V(t)x(t), \dot{x}(t)) \\
&\quad + \frac{d}{dt} \int_{t-h(t)}^t (V(t, \tau)x(\tau), x(t)) d\tau \\
&\quad + \frac{d}{dt} \int_{t-h(t)}^t (V(\tau, t)x(t), x(\tau)) d\tau \\
&= (\dot{V}(t)x(t), x(t)) \\
&\quad + (V(t)A(t)x(t) + V(t)B(t)u^h(t), x(t)) \\
&\quad + (V(t)x(t), A(t)x(t) + B(t)u^h(t)) \\
&\quad + 2(V(t, t)x(t), x(t)) \\
&\quad - (V(t, t-h(t))x(t-h(t)), x(t))(1 - \dot{h}(t)) \\
&\quad - (V(t-h(t), t)x(t), x(t-h(t)))(1 - \dot{h}(t)) \\
&\quad + \int_{t-h(t)}^t \left(\frac{\partial V}{\partial t}(t, \tau)x(\tau), x(t) \right) d\tau \\
&\quad + \int_{t-h(t)}^t \left(\frac{\partial V}{\partial t}(\tau, t)x(t), x(\tau) \right) d\tau \\
&\quad + \int_{t-h(t)}^t (V(t, \tau)x(\tau), \dot{x}(t)) d\tau \\
&\quad + \int_{t-h(t)}^t (V(\tau, t)\dot{x}(t), x(\tau)) d\tau \\
&= (\dot{V}(t)x(t), x(t)) \\
&\quad + ((A(t)^T V(t) + V(t)A(t))x(t), x(t)) \\
&\quad + (x(t), V(t)B(t)u^h(t)) + (x(t), V(t)B(t)u^h(t)) \\
&\quad + 2(V(t, t)x(t), x(t)) \\
&\quad - 2(V(t, t-h(t))x(t-h(t)), x(t))(1 - \dot{h}(t)) \\
&\quad + \left(\int_{t-h(t)}^t \frac{\partial V}{\partial t}(t, \tau)x(\tau) d\tau, x(t) \right) \\
&\quad + \left(x(t), \int_{t-h(t)}^t \frac{\partial V}{\partial t}(t, \tau)x(\tau) d\tau \right) \\
&\quad + \left(\int_{t-h(t)}^t V(t, \tau)x(\tau) d\tau, A(t)x(t) \right) \\
&\quad + \left(A(t)x(t), \int_{t-h(t)}^t V(t, \tau)x(\tau) d\tau \right) \\
&\quad + \left(\int_{t-h(t)}^t V(t, \tau)x(\tau) d\tau, B(t)u^h(t) \right) \\
&\quad + \left(B(t)u^h(t), \int_{t-h(t)}^t V(t, \tau)x(\tau) d\tau \right)
\end{aligned}$$

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Thus, on account of the final condition (23), since $\int_{t_i}^{t_f} \dot{\xi}(t)dt = \xi(t_f) - \xi(t_i)$, and by using the definition of ξ , J can be rewritten as

$$\begin{aligned} J(u) &= \int_{t_i}^{t_f} \dot{\xi}(t)dt + \langle Qx, x \rangle + \langle Ru^h, u^h \rangle \\ &\quad + (V(t_i)x(t_i), x(t_i)) + \int_{t_i-h(t_i)}^{t_i} (V(t_i, \tau)x(\tau), x(t_i))d\tau \\ &\quad + \int_{t_i-h(t_i)}^{t_i} (V(\tau, t_i)x(t_i), x(\tau))d\tau \end{aligned} \quad (34)$$

By using the above computation of $\dot{\xi}$, and the property : $2\langle \mathcal{O}x, x \rangle = \langle (\mathcal{O} + \mathcal{O}^*)x, x \rangle$, (\mathcal{O} being any operator), one has

$$\begin{aligned} &\int_{t_i}^{t_f} \dot{\xi}(t) + \langle Qx, x \rangle + \langle Ru^h, u^h \rangle \\ &= \left\langle \left([\dot{V} + A^T V + V A + Q + 2V] - (1 - \dot{h})(V_h + V_h^*) \right. \right. \\ &\quad \left. \left. + \partial V^h + (\partial V^h)^* + A^T V^h + V^{h*} A \right) x, x \right\rangle \\ &\quad + 2\langle x, [VB]u^h \rangle + 2\langle x, V^{h*} B u^h \rangle + \langle Ru^h, u^h \rangle. \end{aligned} \quad (35)$$

Then, since $x = \phi_h x^h + \psi_h u^h$, by using (22), one has

$$\begin{aligned} &\int_{t_i}^{t_f} \dot{\xi}(t)dt + \langle Qx, x \rangle + \langle Ru^h, u^h \rangle \\ &= \langle \phi_h^* ([L^T]K[L])\phi_h x^h, x^h \rangle \\ &\quad + \langle (\psi_h^* ([L^T]K[L])\psi_h + [R])u^h, u^h \rangle \\ &\quad \quad + 2\langle u^h, \psi_h^* ([V] + V^{h*})B u^h \rangle \\ &\quad + 2\langle x^h, (\phi_h^* ([L^T]K[L])\psi_h + \phi_h^* [VB] + \phi_h^* V^{h*} B)u^h \rangle \end{aligned} \quad (36)$$

Substituting the above expression in (34), and using eq. (21) to eliminate $[VB] + V^{h*}B = S^*B$, results in

$$\begin{aligned} J(u) &= (V(t_i)x(t_i), x(t_i)) + \int_{t_i-h(t_i)}^{t_i} (V(t_i, \tau)x(\tau), x(t_i))d\tau \\ &\quad + \int_{t_i-h(t_i)}^{t_i} (V(\tau, t_i)x(t_i), x(\tau))d\tau \\ &\quad + \langle \phi_h^* [L^T]K[L]\phi_h x^h, x^h \rangle + \langle u^h, ([R] + \psi_h^* [L^T]K[L]\psi_h \\ &\quad + B^T([V] + V^h)\psi_h + \psi_h^*([V] + V^{h*})B)u^h \rangle \\ &\quad - 2\langle x^h, \phi_h^* [L^T]K u^h \rangle. \end{aligned}$$

Note that the r.h.s. of eq. (19) does stand in the above expression as well, hence substituting K right there entails

$$J(u) = \langle K(u^h - [L]\phi_h x^h), u^h - [L]\phi_h x^h \rangle$$

$$\begin{aligned}
& + \langle V(t_i)x(t_i), x(t_i) \rangle + \int_{t_i-h(t_i)}^{t_i} (V(t_i, \tau)x(\tau), x(t_i))d\tau \\
& + \int_{t_i-h(t_i)}^{t_i} (V(\tau, t_i)x(t_i), x(\tau))d\tau
\end{aligned} \tag{37}$$

where the quadratic form is non-negative, as by hypothesis K exists, self-adjoint and non-negative, and the last two terms are unaffected by u . In particular, as to the two integrals above, the $x(\tau)$ to be integrated is indeed a free evolution because, as by the basic assumptions given in §3 it is $u(t) = 0$ for $t < t_i - h(t_i)$, one has $u^h(t) = 0$ for $t < t_i$. Thus the minimum of the functional J exists, and is achieved for:

$$u^h = [L]\phi_h x^h, \tag{38}$$

that is

$$u(t - h(t)) = L(t)\Phi_h^t x(t - h(t)). \tag{39}$$

Changing the variable t endows the proof. •

Remark 2. For $h \equiv 0$ (zero delay) we have ψ_h is the null operator, thus, by (19) we have $K = [R]$, i.e. we have for $K(t) = R(t)$, K is the (instantaneous) operator $(Kf)(t) = K(t)f(t)$. Similarly, the operator $[L]$ is defined by $(Lf)(t) = L(t)f(t)$ with $L(t)$ given by $L(t) = -R^{-1}(t)B^T(t)V(t, t)$. Now, it's easy to show that

$$(V_h^* f)(t) = V(t, t + h(t))f(t + h(t))\dot{h}^*(t), \tag{40}$$

where h^* is the function defined in (4). Thus for $h = 0$, h^* is the identity, and $V_h^* = V$. On account that $V_0^* = V_0 = V$, whereas all the remaining h -depending operators vanish, the family $V(t, \tau)$ needs to be calculated just for $\tau = t$, thus by setting $V(t) = V(t, t)$, $V(t)$ is the matrix function satisfying

$$\dot{V} + A^T V + V A + Q - V B R^{-1} B^T V = 0, \tag{41}$$

which is, as expected, the classical backward Riccati equation for the Linear-Quadratic optimal control problem. •

Remark 3. From formula (30), the optimal control results in a (static) feedback from the state. Nevertheless, the optimal state, namely x^o , is the (unique) solution of the following *state-delayed* autonomous system:

$$\dot{x}^o(t) = A(t)x^o(t) + B(t)L^*(t)\Phi^T(t)x^o(t - h(t)), \tag{42}$$

with

$$\begin{aligned}
L^*(t) &= L(t + h^*(t) - h(t)), \\
\Phi^T(t) &= \Phi(t + h^*(t) - h(t), t - h(t)),
\end{aligned}$$

which immediately ensues substituting (30) in the state equation. •

Remark 4. In the paper [6], for a constant delay $h \equiv h(t)$, the optimal control was found in the following form

$$u^o(t) = K(t)x(t) + \int_{t-h}^t L(t, \tau)u(\tau). \tag{43}$$

What has been shown in the present paper is that a *single* instantaneous, state-feedback indeed exists for the optimal control, namely eq. (30). In other words, the past control-segment, which the control at current time depends of, can be taken over indeed by a different linear feedback of the state.

5. CONCLUSIONS AND FUTURE WORKS

The optimal control problem, as regards to the quadratic index (8), has been considered for the input-delayed system (2). The main result is summarized in eq. (30), showing that the optimal control – provided that a *gain* $L(t)$ can be found by solving eqs. (19)-(23) – has the form of an instantaneous feedback of the state, thereby generalizing some known former results, as argued in Remark 4. Moreover, such result holds even for a time-varying delay, thus it contributes to an issue that has not been enough explored up to now. Much work is needed in order to find features of the system of equations (19)-(23), such as conditions for existence of solutions and so on. There again, when solutions do exist, building up numerical methods for finding there should also been worked out. In conclusion, though further work is needed, the state-feedback Quadratic Optimal Control problem for time-varying delay systems has been moved to finding the solution of a system of integral-differential equations which are *Riccati-like* in that – as we have shown in Remark 2 – they are a generalization of the well known Riccati equation arising in Linear-Quadratic optimal control for undelayed systems.

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