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2-CLIQUE-BOND OF STABLE SET POLYHEDRA

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## Abstract

A *2-clique-bond* is a generalization 2-clique-join where the subsets of nodes that are connected on each shore of the partition are two (not necessarily disjoint) cliques.

We consider a graph  $G$  obtained as the 2-clique-bond of two graphs  $G_1$  and  $G_2$  and we study the polyhedral properties of the stable set polytopes associated with these graphs. In particular, we prove that a linear description of the stable set polytope of  $G$  is obtained by properly composing the linear inequalities describing the stable set polytopes of smaller graphs that are related to  $G_1$  and  $G_2$ .

*Key words:* Stable set polytope, graph compositions, 2-join, polyhedral combinatorics.



## 1. Introduction

Given a graph  $G = (V, E)$  and a vector  $w \in \mathbb{Q}_+^V$  of node weights, the *stable set problem* is to find a set of pairwise nonadjacent nodes (*stable set*) of maximum weight.

The *stable set polytope* of  $G$  is the convex hull of the incidence vectors of the stable sets of  $G$ . This polytope has full dimension and it is usually denoted as  $STAB(G)$ . A linear system  $Ax \leq b, x \geq 0$  is said to be *defining* for  $STAB(G)$  if  $STAB(G) = \{x \in \mathbb{R}^V : Ax \leq b, x \geq 0\}$ . Finding the defining linear system for  $STAB(G)$  is equivalent to transform the original optimization problem into the linear program  $\max\{w^T x : Ax \leq b, x \geq 0\}$ . Since the stable set problem is *NP*-hard, it is unlikely to find such a system for general graphs. Nevertheless the facial structure of the stable set polytope has been one of the major topics in polyhedral combinatorics and results concerning the facets of  $STAB(G)$  have been provided continuously since early 70's [29, 35, 24, 19, 6, 27, 33]. Besides the description of new classes of facets, there have been widely investigated composition procedures that are able to build new families of facets for the stable set polytope starting from facets of lower dimensional polytopes. These compositions are usually based on graph operations: for instance, sequential lifting is based on the extension of a graph with an additional node [29, 24], the Wolsey's lifting procedure [36] is based on edge subdivision, the composition of Barahona and Mahjoub is based on the 2-node cutset [1].

In addition to those listed above, a number of graph compositions were introduced in the attempt to solve the Strong Perfect Graph Conjecture (now Perfect Graph Theorem [7]): clique substitution [2], graph substitution [22], join [3, 12], amalgam [5], 2-amalgam [11]. All these operations were proved to preserve perfectness but many of them also have interesting polyhedral counterparts. In particular the knowledge of polyhedral descriptions of the smaller systems yields a description for the composed system in the case of: graph substitution [10], clique cutset composition [10], join [13], amalgam [4, 26].

The graph composition we consider is the following:

Let  $G_1$  and  $G_2$  be two graphs. Let  $a_0^i$  and  $b_0^i$  be two adjacent nodes of  $G_i$  such that  $A_i = N(a_0^i) \setminus \{b_0^i\}$  and  $B_i = N(b_0^i) \setminus \{a_0^i\}$  are two subsets of  $V(G_i)$ ,  $i = 1, 2$ . The *2-bond* of  $G_1$  and  $G_2$  along the edges  $a_0^1 b_0^1$  and  $a_0^2 b_0^2$  is the graph  $G$  obtained by deleting the nodes  $a_0^i$  and  $b_0^i$ , for each  $i = 1, 2$ , and joining every node in  $A_1$  with every node in  $A_2$  and every node of  $B_1$  with every node of  $B_2$ .

If the sets  $A_i$  and  $B_i$ ,  $i = 1, 2$ , are disjoint the 2-bond is actually a 2-join as defined in [32]. If  $A_i$  and  $B_i$ ,  $i = 1, 2$ , are cliques we speak of a *2-clique-bond*; if the neighborhood of each node of  $A_i \cap B_i$  coincides with  $A_i \cup B_i \cup \{a_0^i, b_0^i\}$ ,  $i = 1, 2$ , the 2-clique-bond is known as a *generalized 2-join* (see [9]).

In this paper, we study the polyhedral properties of the 2-clique-bond. In particular, we describe the structure of the stable set polytope of a graph  $G$  obtained as the 2-clique-bond of two smaller graphs  $G_1$  and  $G_2$  by showing how to obtain a linear description of  $STAB(G)$  once the linear descriptions of lower dimensional polytopes related to  $G_1$  and  $G_2$  are known.

Recently, the generalized 2-join received special attention because it reveals one of the fundamental operations to build claw-free graphs with stability number greater than 3 [9]. The stable set polytope of claw-free graphs is well studied in polyhedral combinatorics because it shows an anomaly with respect to the polynomial-time equivalence of the optimization problem and the separation problem over polyhedra stated by Grötschel, Lovász and Schrijver [20]. Presently, claw-free graphs are the only graphs for which optimizing over their stable set polytope

is polynomial time solvable [23, 28, 15] but no explicit linear description of the same polytope is known. This unreasonable asymmetry made the study of the structure of the stable set polytope of claw-free graphs a challenging problem in polyhedral combinatorics as many papers on the topic testify, see e.g. [14, 18, 19, 21, 27, 34, 31, 30, 8, 16]. Chudnovsky and Seymour made the first polyhedral considerations on the class of claw-free graphs obtained by composing fuzzy linear interval strips with the generalized 2-join (see [8] for definitions). They showed that the stable set polytope of these graphs can be obtained by basically composing the Edmonds inequalities associated with each strip (see [34] for a proof of this result).

In sections 3 and 4 we define an instrumental graph composition, named  $W_3$ -composition, that will allow us to identify which facet defining inequalities play a role in the 2-clique-tie of stable set polyhedra and how these inequalities have to be composed.

In section 5, we prove the main result of the paper. We show that the stable set polytope of a graph  $G$ , obtained as the 2-clique-bond of  $G_1$  and  $G_2$  along the edges  $e_1$  and  $e_2$ , has a linear description that depends only on the linear descriptions of  $STAB(G_i)$  and  $STAB(G_i/e_i)$ ,  $i = 1, 2$ , where  $G_i/e_i$  represents the graph  $G_i$  after contracting the edge  $e_i$ . It is worth noticing that this result can be applied iteratively. As a consequence, a defining linear system for the stable set polytope of a graph  $G$  is known provided that  $G$  is obtained as the 2-clique-bond of graphs  $G_1, \dots, G_k$  along edges  $e_1, \dots, e_k$  and the defining linear systems of  $STAB(G_i)$  and  $STAB(G_i/e_i)$ ,  $i = 1, \dots, k$ , are known. We also prove that the 2-clique-bond preserves the properties of inequalities of being facet defining and we present some nontrivial facet defining inequalities for the stable set polytope.

In a final section, we exploit some consequences of our result. The first non trivial consequence concerns the stable set polytope of claw-free graphs. In fact, as the generalized 2-join is a special type of 2-clique-bond, our result implies that finding a linear description for the stable set polytope of claw-free graphs with stability number at least 4 reduces to finding a linear description of the stable set polytopes of the building blocks of the decomposition provided by Chudnovsky and Seymour [9].

Finally we present some new interesting classes of (not claw-free) graphs whose stable set polyhedra can be easily provided via the 2-clique-bond of polyhedra.

## 2. Definitions

Let  $G = (V, E)$  be any finite, simple and connected graph with node set  $V$  and edge set  $E$ . An edge  $e \in E$  with endnodes  $u$  and  $v$  is denoted by  $uv$ . A graph  $G$  together with a node weighting  $w \in \mathbb{Q}_+^V$  is denoted as  $(G, w)$ . We denote by  $\alpha(G, w)$  the maximum weight of a stable set of  $G$  and we refer to  $\alpha(G) = \alpha(G, \mathbf{1})$  ( $\mathbf{1}$  being the vector of all ones) as the *stability number* of  $G$ .

We denote by  $\delta(v)$  the *star of  $v$* , i.e., the set of edges of  $E$  having  $v$  as an endnode and by  $N(v)$  the *neighborhood of  $v$* , i.e., the set of nodes of  $V$  adjacent to  $v$ . We also denote by  $G \setminus A$  the subgraph of  $G$  induced by  $V \setminus A$  where  $A \subseteq V$  and by  $G/e$  the graph obtained by contracting the edge  $e$  of  $G$ .

A node of  $V$  is *simplicial* if its neighborhood consists of a clique. An edge  $e = v_1v_2$  of  $E$  is *simplicial* if  $N(v_1) \setminus \{v_2\} = K_1$  and  $N(v_2) \setminus \{v_1\} = K_2$  are two nonempty cliques of  $G$ . A simplicial edge is said to be *super simplicial* if  $N(K_1) \cup N(K_2) \subseteq N(v_1) \cup N(v_2)$ . A *clique cutset* of  $G$  is a complete subgraph whose removal disconnects  $G$ .

Given a vector  $\beta \in \mathbb{R}^m$  and a subset  $S \subseteq \{1, \dots, m\}$ , let  $\beta(S) = \sum_{i \in S} \beta_i$ . Moreover, define  $x^S \in \mathbb{R}^m$  as the incidence vector of  $S$ . A linear inequality  $\sum_{j \in V_G} \beta_j x_j \leq \beta_0$  is *valid* for  $STAB(G)$  if it holds for all  $x \in STAB(G)$ . For short, we also denote a linear inequality  $\beta^T x \leq \beta_0$  as  $(\beta, \beta_0)$ .

A valid inequality for  $STAB(G)$  defines a facet of  $STAB(G)$  if and only if it is satisfied as an equality by  $|V|$  affinely independent incidence vectors of stable sets of  $G$  (called *roots* or *tight solutions*). The *facet defining inequalities* for  $STAB(G)$  constitute the unique nonredundant defining linear system of  $STAB(G)$ . We say that a stable set  $S$  is tight for  $(\beta, \beta_0)$  if  $\beta(S) = \beta_0$  and that  $S$  violates  $(\beta, \beta_0)$  if  $\beta(S) > \beta_0$ .

We denote by  $\mathcal{S}(G)$  the set of all the stable sets of  $G$  and by  $\mathcal{S}_{MAX}(G, w)$  the set  $\{S \in \mathcal{S}(G) \mid w(S) = \alpha(G, w)\}$ .

The proofs in this paper use basic concepts of (integer) linear programming that we summarize in the following.

**Lemma 2.1.** *Let  $(\gamma, \gamma_0)$  be a facet defining inequality for  $STAB(G)$ . Then, for every valid inequality  $(\beta, \beta_0)$  that, up to positive multiplications, is not  $(\gamma, \gamma_0)$ , there exists a stable set  $S \in \mathcal{S}_{MAX}(G, \gamma)$  that is not tight for  $(\beta, \beta_0)$  (i.e.  $\beta(S) < \beta_0$ ).*

*Proof.* Let  $(\beta, \beta_0)$  be one of such inequalities. If  $\beta^T x^S = \beta_0$  for all  $S \in \mathcal{S}_{MAX}(G, \gamma)$ , then  $(\beta, \beta_0)$  contains all the roots of  $(\gamma, \gamma_0)$ . As a consequence, either  $(\gamma, \gamma_0)$  is a positive multiple of  $(\beta, \beta_0)$  or it is not a facet defining inequality. In both cases we have a contradiction. ■

**Observation 1.** *Let  $G = (V, E)$  be a graph. For any node weighting function  $w \in \mathbb{Q}_+^{|V|}$ , there exists a facet defining inequality  $(\beta, \beta_0)$  of  $STAB(G)$  such that  $\beta(S) = \beta_0$  for any stable set  $S \in \mathcal{S}_{MAX}(G, w)$ , i.e., with weight  $w(S) = \alpha(G, w)$ .*

A basic result of linear programming [25] states that a point  $x^* \in P$  is an optimal solution of the optimization problem defined by the objective function  $c^T x$  over the polyhedron  $P$  if  $c$  can be expressed as a conic combination of the left hand side of the linear inequalities that define  $P$  and that are satisfied as equalities by  $x^*$ . As a consequence we have that the following holds:

**Observation 2.** *Let  $G = (V, E)$  be a graph,  $w \in \mathbb{Q}_+^{|V|}$  a node weighting function, and  $A'x \leq b'$  a system of  $p$  linear inequalities that are facet defining for  $STAB(G)$  and such that  $\beta(S) = \beta_0$  for any  $(\beta, \beta_0) \in (A'x \leq b')$  and any  $S \in \mathcal{S}_{MAX}(G, w)$ . If there does not exist  $\lambda \in \mathbb{R}_+^p$  such that  $\lambda^T A' = w$ , then there exists  $(\beta', \beta'_0)$  that is facet defining for  $STAB(G)$  and with the following properties:*

- there is no  $\tilde{\lambda} \in \mathbb{R}_+^p$  such that  $\tilde{\lambda}^T A' = \beta'$ ;
- $\beta'(S) = \beta'_0$  for any  $S \in \mathcal{S}_{MAX}(G, w)$ .

Chvátal [10] proved the following fundamental result concerning the stable set polytope of graphs containing clique cutsets:

**Theorem 2.2.** *Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two graphs. Let  $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$  and  $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ . If  $G_1 \cap G_2$  is a complete graph, then the defining linear system of  $STAB(G_1 \cup G_2)$  is given by the union of the defining linear systems of  $STAB(G_1)$  and  $STAB(G_2)$ .*

This implies that a graph supporting a facet defining inequality cannot contain a clique cutset.

### 3. $W_3$ -graphs

In this section we introduce a new graph structure that will be used as an intermediate step to prove our final result.

**Definition 3.1.** *Let  $H$  be a graph, let  $A$  and  $B$  be two cliques of  $V(H)$ , and let  $W_3$  be a clique of size four with  $V(W_3) = \{t_A, t_B, t_{AB}, t_\emptyset\}$ . The graph  $G$  obtained as follows:*

$$\begin{aligned} V(G) &= V(H) \cup V(W_3) \\ E(G) &= E(H) \cup E(W_3) \cup F_1 \cup F_2, \text{ where} \\ &F_1 = \{uv | u \in \{t_A, t_{AB}\}, v \in A\} \text{ and } F_2 = \{uv | u \in \{t_B, t_{AB}\}, v \in B\}. \end{aligned}$$

*is called a  $W_3$ -graph and denoted by  $G = (H, A, B, W_3)$ .*

We now study the stable set polytope of  $W_3$ -graphs. The first consideration is an easy consequence of Theorem 2.2:

**Corollary 3.2.** *If  $G$  is a  $W_3$ -graph and  $Dx \leq d$  defines  $STAB(G \setminus \{t_\emptyset\})$ , then  $\{Dx \leq d, x_{t_A} + x_{t_{AB}} + x_{t_B} + x_{t_\emptyset} \leq 1, x_{t_\emptyset} \geq 0\}$  defines  $STAB(G)$ .*

Thus, when  $G$  is a  $W_3$ -graph, the only graph supporting a non trivial facet defining inequality for  $STAB(G)$  that contains  $t_\emptyset$  is the clique  $W_3$  itself.

In [17] we proved the following:

**Proposition 3.3.** *Let  $G$  be a graph and let  $(\beta, \beta_0)$  be a facet defining inequality of  $STAB(G)$  which is not a clique inequality or a nonnegativity inequality. If  $uv$  is a simplicial edge then  $\beta_u = \beta_v$ .*

This result has some interesting consequences on the coefficients of the nodes of  $W_3$  in facet defining inequalities of the stable set polytope of  $W_3$ -graphs. Indeed:

**Proposition 3.4.** *Let  $G$  be a  $W_3$ -graph and let  $(\beta, \beta_0)$  be a facet defining inequality  $(\beta, \beta_0)$  of  $STAB(G)$  which is not a clique inequality or a nonnegativity inequality. If  $\beta_{t_A}$  or  $\beta_{t_B}$  is nonzero then  $\beta_{t_A} = \beta_{t_B} = \beta_{t_{AB}}$ .*

*Proof.* Since, by Corollary 3.2, the edge  $t_A t_B$  is simplicial in every graph supporting a facet defining inequality  $(\beta, \beta_0)$  of  $STAB(G)$  which is not a clique or a nonnegativity inequality, Proposition 3.3 implies that  $t_A$  and  $t_B$  have the same coefficient in every facet defining inequality that is not a clique or a nonnegativity inequality

By Lemma 2.1, there exists a stable set  $S \in \mathcal{S}_{MAX}(G, \beta)$  such that  $S \cap \{A \cup \{t_A, t_{AB}\}\} = \emptyset$ . This implies that  $t_B \in S$  otherwise  $S \cup \{t_A\}$  would contradict the fact that  $S \in \mathcal{S}_{MAX}(G, \beta)$ . Then  $\beta_{t_B} = \beta_{t_A} \geq \beta_{t_{AB}}$  because  $S \setminus \{t_B\} \cup \{t_{AB}\}$  is a stable set of  $G$ . As  $(\beta, \beta_0)$  is facet defining for  $STAB(G)$ , there exists  $S \in \mathcal{S}_{MAX}(G, \beta)$  such that  $t_{AB} \in S$ ; then  $\beta_{t_{AB}} \geq \beta_{t_B} = \beta_{t_A}$ . Hence  $\beta_{t_A} = \beta_{t_B} = \beta_{t_{AB}}$  in any facet defining inequality  $(\beta, \beta_0)$  of  $STAB(G)$  where  $t_A$  and  $t_B$  appear with nonzero coefficient. ■

Given a  $W_3$ -graph  $G$ , we say that an inequality  $(\beta, \beta_0)$  is a  $k$ -inequality,  $k \in \{0, 1, 2, 3, 4\}$ , if it is facet defining for  $STAB(G)$  and it has  $k$  nonzero coefficients among  $\{\beta_{t_\emptyset}, \beta_{t_A}, \beta_{t_B}, \beta_{t_{AB}}\}$ . Due to Proposition 3.4 and Corollary 3.2, we can classify the set of inequalities that define the stable set polytope of a  $W_3$ -graph  $G$  as follows:

**Observation 3.** Let  $G = (H, A, B, W_3)$  be a  $W_3$ -graph and let  $(\beta, \beta_0)$  be a facet defining inequality of  $STAB(G)$  that is not a nonnegativity inequality. Then  $(\beta, \beta_0)$  is one of the following:

- i) a 0-inequality, i.e.,  $\beta_{t_A} = \beta_{t_B} = \beta_{t_{AB}} = \beta_{t_0} = 0$ ,
- ii) a 1-inequality with  $\beta_{t_A} = \beta_{t_B} = \beta_{t_0} = 0$  and  $\beta_{t_{AB}} \neq 0$ ,
- iii) the 2-inequality  $\sum_{u \in A} x_u + x_{t_A} + x_{t_{AB}} \leq 1$ ,
- iv) the 2-inequality  $\sum_{u \in B} x_u + x_{t_B} + x_{t_{AB}} \leq 1$ ,
- v) a 3-inequality with  $\beta_{t_A} = \beta_{t_B} = \beta_{t_{AB}} \neq 0$ ,
- vi) the 4-inequality  $x_{t_A} + x_{t_B} + x_{t_{AB}} + x_{t_0} \leq 1$ .

#### 4. Compositions of $W_3$ -graphs

Here we introduce a composition operation between  $W_3$ -graphs.

**Definition 4.1.** Given two  $W_3$ -graphs  $G_i = (H_i, A_i, B_i, W_3^i)$  with  $i = 1, 2$ , the  $W_3$ -composition of  $G_1$  and  $G_2$  produces a new graph  $G$  by deleting the nodes  $W_3^i = \{t_A^i, t_B^i, t_{AB}^i, t_0^i\}$ , for each  $i = 1, 2$ , and joining every node of  $A_1$  with every node of  $A_2$  and every node of  $B_1$  with every node of  $B_2$  (see Figure 1).

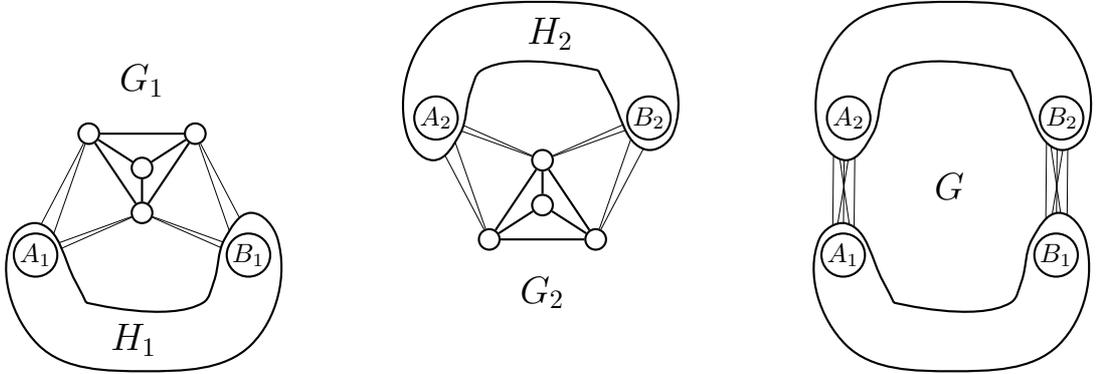


Figure 1:  $G$  is the  $W_3$ -composition of  $G_i = (H_i, A_i, B_i, W_3^i)$  for  $i = 1, 2$ .

We now introduce a new class of inequalities for graphs obtained by  $W_3$ -composition. Let  $\beta^1 x \leq \beta_0^1$  be a 1-inequality that is valid for  $STAB(G_1)$  and let  $\beta^2 x \leq \beta_0^2$  be a 3-inequality that is valid for  $STAB(G_2)$ . By Observation 3, it is not difficult to see that the coefficients of  $\beta^1 x \leq \beta_0^1$  and  $\beta^2 x \leq \beta_0^2$  can be normalized so that  $\beta^1(t_{AB}^1) = \beta^2(t_A^2) = \beta^2(t_B^2) = \beta^2(t_{AB}^2) = 1$ . Then the composition of  $(\beta^1, \beta_0^1)$  and  $(\beta^2, \beta_0^2)$  is the following inequality:

$$\sum_{v \in V(H_1)} \beta_v^1 x_v + \sum_{v \in V(H_2)} \beta_v^2 x_v \leq \beta_0^1 + \beta_0^2 - 1 \quad (1)$$

In the rest of the paper when composing inequalities  $(\beta^1, \beta_0^1)$  and  $(\beta^2, \beta_0^2)$  we will always assume that the nonzero coefficients on  $W_3^1$  and  $W_3^2$  equal 1.

It holds the following:

**Lemma 4.2.** *Let  $G$  be the  $W_3$ -composition of two  $W_3$ -graphs  $G_1 = (H_1, A_1, B_1, W_3^1)$  and  $G_2 = (H_2, A_2, B_2, W_3^2)$ , let  $(\beta^1, \beta_0^1)$  be a 1-inequality that is valid for  $STAB(G_1)$ , and  $(\beta^2, \beta_0^2)$  a 3-inequality that is valid for  $STAB(G_2)$ . If  $(\beta, \beta_0)$  is the composition of  $(\beta^1, \beta_0^1)$  and  $(\beta^2, \beta_0^2)$  then it is valid for  $STAB(G)$ .*

*Proof.* By contradiction, let  $S$  be a maximal stable set of  $G$  that violates  $(\beta, \beta_0)$ , i.e.,  $\beta(S) = \beta^1(S \cap V(H_1)) + \beta^2(S \cap V(H_2)) > \beta_0^1 + \beta_0^2 - 1$ . Clearly  $|S \cap (A_1 \cup A_2 \cup B_1 \cup B_2)| \leq 2$ .

Assume now that  $S \cap A_2 = \emptyset$ . As  $(\beta^1, \beta_0^1)$  is valid for  $STAB(G_1)$  and  $S \cap V(H_1)$  is a stable set of  $G_1$ ,  $\beta^1(S \cap V(H_1)) \leq \beta_0^1$  and, consequently,  $\beta^2(S \cap V(H_2)) > \beta_0^2 - 1$ . But then  $(S \cap V(H_2)) \cup t_A^2$  is a stable set of  $G_2$  violating  $(\beta^2, \beta_0^2)$ , a contradiction. So  $S \cap A_2 \neq \emptyset$  and symmetric arguments prove also that  $S \cap B_2 \neq \emptyset$ . As a consequence,  $S \cap A_1 = S \cap B_1 = \emptyset$ . Since  $(\beta^2, \beta_0^2)$  is valid for  $STAB(G_2)$  and  $S \cap V(H_2)$  is a stable set of  $G_2$ , we have that  $\beta^2(S \cap V(H_2)) \leq \beta_0^2$  and then  $\beta^1(S \cap V(H_1)) > \beta_0^1 - 1$ . But then  $(S \cap V(H_1)) \cup \{t_{AB}^1\}$  would be a stable set of  $G_1$  violating  $(\beta^1, \beta_0^1)$ , a contradiction. ■

As the  $W_3$ -composition is commutative, a result analogous to Lemma 4.2 holds when  $(\beta, \beta_0)$  is the composition of a 3-inequality for  $STAB(G_1)$  and a 1-inequality for  $STAB(G_2)$ .

In the remaining of the paper we show how to obtain a linear description of  $STAB(G)$  starting from the linear description of the stable set polytopes of the two  $W_3$ -graphs  $G_1$  and  $G_2$ .

**Definition 4.3.** *Let  $G$  be the  $W_3$ -composition of two  $W_3$ -graphs  $G_i = (H_i, A_i, B_i, W_3^i)$ ,  $i = 1, 2$ . For any node weighting function  $w$  for  $G$ , we define the generating weighting functions  $w^i$  on  $V(G_i)$  for  $i = 1, 2$  as follows:*

$$w^i(u) = \begin{cases} w(u), & \text{if } u \in V(H_i) \\ \alpha(H_j \setminus B_j, w), & \text{if } u = t_A^i \\ \alpha(H_j \setminus A_j, w), & \text{if } u = t_B^i \\ \alpha(H_j \setminus (A_j \cup B_j), w), & \text{if } u = t_\emptyset^i \\ \alpha(H_j, w), & \text{if } u = t_{AB}^i \end{cases}$$

with  $i, j = 1, 2$  and  $j \neq i$ . We also say that the weighted graph  $(G, w)$  is generated by the weighted graphs  $(G_1, w^1)$  and  $(G_2, w^2)$ .

It is not difficult to observe that  $w^i(t_\emptyset^i) \leq w^i(t_A^i), w^i(t_B^i) \leq w^i(t_{AB}^i)$  for  $i = 1, 2$  and that  $\alpha(G, w) = \alpha(G_1, w^1) = \alpha(G_2, w^2)$ .

**Definition 4.4.** *Two stable sets  $S_1 \in \mathcal{S}_{MAX}(G_1, w^1)$  and  $S_2 \in \mathcal{S}_{MAX}(G_2, w^2)$  are said to generate  $S \in \mathcal{S}_{MAX}(G, w)$  if  $S = S_1 \cup S_2 \setminus (W_3^1 \cup W_3^2)$ .*

Notice that for each  $S \in \mathcal{S}_{MAX}(G, w)$  we can produce two stable sets  $S_1 \in \mathcal{S}_{MAX}(G_1, w^1)$  and  $S_2 \in \mathcal{S}_{MAX}(G_2, w^2)$  that generate  $S$  in the following way:  $S_i \cap V(H_i) = S \cap V(H_i)$  and

$$S_i \cap \{t_A^i, t_B^i, t_{AB}^i, t_\emptyset^i\} = \begin{cases} t_A^i, & \text{if } S \cap A_j \neq \emptyset, S \cap B_j = \emptyset \\ t_B^i, & \text{if } S \cap A_j = \emptyset, S \cap B_j \neq \emptyset \\ t_{AB}^i, & \text{if } S \cap A_j \neq \emptyset, S \cap B_j \neq \emptyset \\ t_\emptyset^i, & \text{if } S \cap A_j = \emptyset, S \cap B_j = \emptyset \end{cases}$$

with  $i, j = 1, 2$  and  $j \neq i$ .

If  $S_1 \in \mathcal{S}_{MAX}(G_1, w^1)$  and  $S_2 \in \mathcal{S}_{MAX}(G_2, w^2)$  generate  $S \in \mathcal{S}_{MAX}(G, w)$  then the following conditions hold:

$$\begin{aligned} w(S) &= \alpha(G, w) = w(S_1 \cap V(H_1)) + w(S_2 \cap V(H_2)) = \\ &= w^1(S_1) = \alpha(G_1, w^1) = w(S_1 \cap V(H_1)) + w^1(S_1 \cap V(W_3^1)) = \\ &= w^2(S_2) = \alpha(G_2, w^2) = w(S_2 \cap V(H_2)) + w^2(S_2 \cap V(W_3^2)) = \\ &= w^1(S_1 \cap V(W_3^1)) + w^2(S_2 \cap V(W_3^2)). \end{aligned}$$

It is easy to prove the following:

**Observation 4.** *If  $S_i \in \mathcal{S}_{MAX}(G_i, w^i)$ , then there always exists  $S_j \in \mathcal{S}_{MAX}(G_j, w^j)$  such that  $S_i$  and  $S_j$  generate a stable set  $S \in \mathcal{S}_{MAX}(G, w)$ , for  $i, j \in \{1, 2\}$  and  $j \neq i$ .*

Moreover, it holds the following property:

**Lemma 4.5.** *Let  $(G, w)$  be generated by  $(G_1, w^1)$  and  $(G_2, w^2)$ , and let  $S_i \in \mathcal{S}_{MAX}(G_i, w^i)$  and  $S_j \in \mathcal{S}_{MAX}(G_j, w^j)$  generate  $S \in \mathcal{S}_{MAX}(G, w)$ . If  $t_{AB}^i \in S_i$  and  $u \in S_j$ , with  $u \in \{t_A^j, t_B^j, t_{AB}^j\}$ ,  $i, j \in \{1, 2\}$  and  $j \neq i$ , then  $w^j(u) = w^j(t_{AB}^j)$ .*

*Proof.* We first consider  $u = t_A^j$ . Let  $S' = S_i \setminus \{t_{AB}^i\}$  and then assume, by contradiction, that  $w^j(t_A^j) > w^j(t_{AB}^j)$ . It follows that there exists a stable set  $S''$  in  $H_i \setminus B_i$  such that  $w^i(S'') > w^j(t_{AB}^j) = \alpha(H_i \setminus (A_i \cup B_i), w) = w^i(S')$ . But then  $S \setminus S' \cup S''$  would be a stable set of  $G_i$  whose weight is greater than  $\alpha(G, w)$ , contradicting the fact that  $S \in \mathcal{S}_{MAX}(G, w)$ . Similar arguments prove the cases  $u = t_B^j$  and  $u = t_{AB}^j$ . ■

**Theorem 4.6.** *Let  $G$  be the  $W_3$ -composition of two  $W_3$ -graphs  $G_1 = (H_1, A_1, B_1, W_3^1)$  and  $G_2 = (H_2, A_2, B_2, W_3^2)$ . Then  $STAB(G)$  is described by the following inequalities:*

- *nonnegativity constraints;*
- *clique inequalities induced by  $A_1 \cup A_2$  and  $B_1 \cup B_2$ ;*
- *0-inequalities of  $STAB(G_1)$  or  $STAB(G_2)$ ;*
- *inequalities that are composition of: either a 3-inequality that is facet defining for  $STAB(G_1)$  and a 1-inequality that is facet defining for  $STAB(G_2)$ , or a 1-inequality that is facet defining for  $STAB(G_1)$  and a 3-inequality that is facet defining for  $STAB(G_2)$ .*

*Proof.* Let  $(\gamma, \gamma_0)$  be a facet defining inequality for  $STAB(G)$  that (up to positive multiplications) is different from the inequalities listed in the thesis. Without loss of generality, assume that  $(\gamma, \gamma_0)$  is fully supported by  $G$ .

Now consider the vector  $\gamma$  as a node weighting of  $G$  and let  $\gamma^i$  be the generating weighting functions  $\gamma^i$  of  $\gamma$  on  $G_i$ ,  $i = 1, 2$ , as in Definition 4.3.

Because of Observation 1, there exist two inequalities  $(\beta^1, \beta_0^1)$  and  $(\beta^2, \beta_0^2)$  such that  $(\beta^i, \beta_0^i)$  is facet defining for  $STAB(G_i)$  and such that all  $S_i \in \mathcal{S}_{MAX}(G_i, \gamma^i)$  are tight for  $(\beta^i, \beta_0^i)$ ,  $i = 1, 2$ . Since  $G_i$  are  $W_3$ -graphs, each non trivial inequality that is facet defining for  $STAB(G_i)$ ,  $i = 1, 2$ , is of type i), . . . , vi) by Observation 3. Furthermore, as  $(\gamma, \gamma_0)$  is fully supported by  $G$ , there is at least one node  $u_i \in V(H_i)$  with  $\gamma^i(u_i) > 0$  for  $i = 1, 2$ . By Observation 2, this implies that, if  $S_i \in \mathcal{S}_{MAX}(G_i, \gamma^i)$ , then there exists at least one facet defining inequality for  $STAB(G_i)$  that is

satisfied as an equality by  $S_i$  and that has nonzero coefficient on the node  $u_i$ . As a consequence, there exists a facet defining inequality for  $STAB(G_i)$  that has nonzero coefficient on the node  $u_i$  and that is satisfied to equality by all  $S_i \in \mathcal{S}_{MAX}(G_i, \gamma^i)$ . Hence, we can assume that both  $(\beta^1, \beta_0^1)$  and  $(\beta^2, \beta_0^2)$  are restricted to be in the set i),  $\dots$ , v).

We first consider the case when  $(\beta^i, \beta_0^i)$  is a 0-inequality of type i), i.e. for a fixed  $i \in \{1, 2\}$ ,  $\beta^i(W_3^i) = 0$ . Without loss of generality let  $i = 1$ . Then, let  $S \in \mathcal{S}_{MAX}(G, \gamma)$ , and let  $S_1 \in \mathcal{S}_{MAX}(G_1, \gamma^1)$  and  $S_2 \in \mathcal{S}_{MAX}(G_2, \gamma^2)$  generate  $S$ : as  $S \cap V(H_1) = S_1 \cap V(H_1)$  and, by assumption,  $S_1$  is tight for  $(\beta^1, \beta_0^1)$ , then  $S$  also satisfies  $(\beta^1, \beta_0^1)$  as an equality. As this holds for any  $S \in \mathcal{S}_{MAX}(G, \gamma)$  and  $(\gamma, \gamma_0)$  is not a positive multiple of  $(\beta^1, \beta_0^1)$ , we got a contradiction of Lemma 2.1.

We now assume that  $(\beta^i, \beta_0^i)$  is the 2-inequality of type iii) and without loss of generality let  $i = 1$ . Let  $S \in \mathcal{S}_{MAX}(G, \gamma)$  with  $S \cap A = \emptyset$  (such a stable set exists because of Lemma 2.1 applied to the clique inequality induced by the nodes of  $A$ ) and let  $S_1 \in \mathcal{S}_{MAX}(G_1, \gamma^1)$  and  $S_2 \in \mathcal{S}_{MAX}(G_2, \gamma^2)$  generate  $S$ . As  $S_1 \in \mathcal{S}_{MAX}(G_1, \gamma^1)$  and  $(\beta^1, \beta_0^1)$  is of type ii),  $S_1 \cap (A_1 \cup \{t_A^1, t_{AB}^1\}) \neq \emptyset$ , i.e. there exists  $u \in \{t_A^1, t_{AB}^1\}$  contained in  $S_1$ . Furthermore, as  $S \cap A = \emptyset$ , we have  $S_2 \cap A_2 = \emptyset$  and, as a consequence,  $\gamma(S_2 \cap V(H_2)) = \gamma^1(t_B^1)$ . If  $u = t_{AB}^1$ , then  $S_1 \cap (A_1 \cup B_1) = \emptyset$ . Moreover, the stable set  $\tilde{S} = S_1 \setminus \{t_{AB}^1\} \cup \{t_B^1\}$  is not tight for  $(\beta^1, \beta_0^1)$  and, as by definition every stable set in  $\mathcal{S}_{MAX}(G_1, \gamma^1)$  is tight for  $(\beta^1, \beta_0)$ , we have that  $\tilde{S}$  cannot belong to  $\mathcal{S}_{MAX}(G_1, \gamma^1)$ . As a consequence,  $\gamma^1(t_{AB}^1) > \gamma^1(t_B^1)$ , and so there exists a stable set  $S'$  of  $H_2$  such that  $\gamma(S') > \alpha(H_2 \setminus A_2, \gamma) = \gamma^1(t_B^1) = \gamma(S_2 \cap V(H_2))$ . But then  $\tilde{S} = (S \cap V(H_1)) \cup S'$  would be a stable set (because  $S \cap (A_1 \cup B_1) = \emptyset$ ) with  $\gamma(\tilde{S}) > \gamma(S)$ , so violating the fact that  $S \in \mathcal{S}_{MAX}(G, \gamma)$ . If  $u = t_A^1$  we can use arguments similar to the above ones to prove that  $\gamma^1(t_A^1) > \gamma^1(t_\emptyset^1)$ . This implies that there exists a stable set  $S''$  of  $H_2 \setminus B_2$  such that  $\gamma(S'') > \alpha(H_2 \setminus (A_2 \cup B_2), \gamma) = \gamma^1(t_\emptyset^1)$ . As  $\tilde{S} = (S_1 \cap V(H_1)) \cup S''$  is a stable set (because  $S_1 \cap A_1 = \emptyset$ ) with  $\gamma(\tilde{S}) > \gamma(S)$ , it would violate again the fact that  $S \in \mathcal{S}_{MAX}(G, \gamma)$ . Symmetrically, we can prove that  $(\beta^i, \beta_0^i)$  cannot be the 2-inequality of type iv).

It remains to consider the following non symmetric cases:

a)  $(\beta^1, \beta_0^1)$  and  $(\beta^2, \beta_0^2)$  are both 1-inequalities

Let  $S \in \mathcal{S}_{MAX}(G, \gamma)$  be such that  $S \cap (A_1 \cup A_2) = \emptyset$  (such a stable set exists by Lemma 2.1). Observe that  $|S \cap (B_1 \cup B_2)| \leq 1$  and assume, without loss of generality,  $S \cap B_1 = \emptyset$ . Let  $S_1 \in \mathcal{S}_{MAX}(G_1, \gamma^1)$  and  $S_2 \in \mathcal{S}_{MAX}(G_2, \gamma^2)$  be two stable sets generating  $S$ . In particular, as  $S_1 \in \mathcal{S}_{MAX}(G_1, \gamma^1)$ ,  $S_1$  is tight for  $(\beta^1, \beta_0^1)$  and it is not difficult to see that  $t_{AB}^1 \in S_1$ . Indeed, if  $t_{AB}^1 \notin S_1$ , then  $S_1 \setminus \{t_A^1, t_B^1, t_\emptyset^1\} \cup \{t_{AB}^1\}$  violates  $(\beta^1, \beta_0^1)$ , a contradiction.

Moreover, we have that  $\gamma^1(u) < \gamma^1(t_{AB}^1)$ , for  $u \in \{t_A^1, t_B^1, t_\emptyset^1\}$ . Indeed, if  $\gamma(u) = \gamma(t_{AB}^1)$  for some  $u \in \{t_A^1, t_B^1, t_\emptyset^1\}$ , then  $S' = S_1 \setminus \{t_{AB}^1\} \cup \{u\}$  would be a maximum stable set in  $\mathcal{S}_{MAX}(G_1, \gamma^1)$  such that  $\beta^1(S') = \beta_0^1 - \beta^1(t_{AB}^1) < \beta_0^1$ , violating the hypothesis that all the stable sets in  $\mathcal{S}_{MAX}(G_1, \gamma^1)$  are tight for  $(\beta^1, \beta_0^1)$ .

Hence  $\gamma^1(u) < \gamma^1(t_{AB}^1)$  for  $u \in \{t_A^1, t_B^1, t_\emptyset^1\}$ . This implies that there exists a stable set  $S'' \subseteq V(H_2)$  such that  $\gamma(S'') = \alpha(H_2, \gamma) = \gamma^1(t_{AB}^1) > \gamma^1(t_B^1) = \alpha(H_2 \setminus A_2, \gamma) = \gamma(S \cap V(H_2))$ . But then, as  $S \cap A_1 = \emptyset$  and  $S \cap B_1 = \emptyset$ ,  $\tilde{S} = S \setminus S_2 \cup S''$  would be a stable set of  $G$  with  $\gamma(\tilde{S}) > \gamma(S)$ , violating the fact that  $S \in \mathcal{S}_{MAX}(G, \gamma)$ . A contradiction.

b)  $(\beta^1, \beta_0^1)$  and  $(\beta^2, \beta_0^2)$  are both 3-inequalities

As for Case a), let  $S \in \mathcal{S}_{MAX}(G, \gamma)$  be such that  $S \cap (A_1 \cup A_2) = \emptyset$ . Since  $|S \cap (B_1 \cup B_2)| \leq 1$  we can assume without loss of generality that  $S \cap B_1 = \emptyset$ . Let  $S_1 \in \mathcal{S}_{MAX}(G_1, \gamma^1)$  and

$S_2 \in \mathcal{S}_{MAX}(G_2, \gamma^2)$  be two stable sets generating  $S$ . As  $\gamma^1(t_{AB}^1) \geq \gamma^1(u)$  for any  $u \in \{t_A^1, t_B^1, t_\emptyset^1\}$ , there exists a stable set  $\tilde{S}_1 \in \mathcal{S}_{MAX}(G_1, \gamma^1)$  with  $t_{AB}^1 \in \tilde{S}_1$  and  $\tilde{S}_1 \cap V(H_1) = S_1 \cap V(H_1)$ . Moreover, as  $\gamma^2(t_{AB}^2) \geq \gamma^2(t_A^2), \gamma^2(t_B^2) \geq \gamma^2(t_\emptyset^2)$ , there exists a stable set  $\tilde{S}_2 \in \mathcal{S}_{MAX}(G_2, \gamma^2)$  containing  $v \in \{t_A^2, t_{AB}^2\}$  and such that  $\tilde{S}_2 \cap V(H_2) = S_2 \cap V(H_2)$ . Now observe that  $\tilde{S}_1$  and  $\tilde{S}_2$  do generate  $S$ . Furthermore, as  $\tilde{S}_2$  is tight for  $(\beta^2, \beta_0^2)$  and  $\tilde{S}_2 \setminus \{v\} \cup \{t_\emptyset^1\}$  is not tight for  $(\beta^2, \beta_0^2)$ , then also  $\gamma^2(v) > \gamma^2(t_\emptyset^2)$ , contradicting Lemma 4.5.

c)  $(\beta^1, \beta_0^1)$  is a 1-inequality and  $(\beta^2, \beta_0^2)$  is a 3-inequality

Let  $(\beta, \beta_0)$  be the inequality (1) obtained by composing  $(\beta^1, \beta_0^1)$  and  $(\beta^2, \beta_0^2)$ . Here, we show that  $(\beta, \beta_0)$  is satisfied as an equality by each  $S \in \mathcal{S}_{MAX}(G, \gamma)$ , so contradicting Lemma 2.1. Suppose on the contrary that there exists a stable set  $S \in \mathcal{S}_{MAX}(G, \gamma)$  such that  $\beta(S \cap V(H_1)) + \beta(S \cap V(H_2)) < \beta_0 = \beta_0^1 + \beta_0^2 - 1$ . As the coefficients of  $(\beta^1, \beta_0^1)$  and  $(\beta^2, \beta_0^2)$  are normalized so that  $\beta^1(t_{AB}^1) = 1$  and  $\beta^2(t_{AB}^2) = \beta^2(t_A^2) = \beta^2(t_B^2) = 1$ , this implies that  $S$  is not tight for  $(\beta, \beta_0)$  if and only if  $t_{AB}^1 \in S_1$  and  $u \in S_2$ , where  $u \in \{t_A^2, t_B^2, t_{AB}^2\}$  and, as usual,  $S_1 \in \mathcal{S}_{MAX}(G_1, \gamma^1)$  and  $S_2 \in \mathcal{S}_{MAX}(G_2, \gamma^2)$  generate  $S$ . As by hypothesis every stable set in  $\mathcal{S}_{MAX}(G_2, \gamma^2)$  is tight for  $(\beta^2, \beta_0^2)$ , it follows that  $\gamma^2(t_u^2) > \gamma^2(t_\emptyset^2)$ , so contradicting Lemma 4.5.

■

Here we show that the description of  $STAB(G)$  given by Theorem 4.6 is also minimal.

**Theorem 4.7.** *Let  $(\beta, \beta_0)$  be a composition of two  $(\beta^1, \beta_0^1)$ , that is a facet defining 1-inequality of  $STAB(G_1)$  and  $(\beta^2, \beta_0^2)$ , that is a facet defining 3-inequality of  $STAB(G_2)$ . Then  $(\beta, \beta_0)$  is facet defining for  $STAB(G)$ .*

*Proof.* Assume that  $(\beta, \beta_0)$  is not facet defining for  $STAB(G)$ ; then there exists an inequality  $(\gamma, \gamma_0)$  that defines a facet of  $STAB(G)$  containing all the roots of  $(\beta, \beta_0)$ . By Theorem 4.6,  $(\gamma, \gamma_0)$  is one of the following:

- nonnegativity constraints;
- clique inequalities induced by  $A_1 \cup A_2$  or  $B_1 \cup B_2$ ;
- 0-inequalities of  $STAB(G_1)$  or  $STAB(G_2)$ ;
- inequalities that are composition of: either a 3-inequality that is facet defining for  $STAB(G_1)$  and a 1-inequality that is facet defining for  $STAB(G_2)$ , or a 1-inequality that is facet defining for  $STAB(G_1)$  and a 3-inequality that is facet defining for  $STAB(G_2)$ .

First consider the case where  $(\gamma, \gamma_0)$  is a nonnegativity constraint or a 0-inequality and, without loss of generality, assume that it is defined over the nodes of  $G_1$ . As a consequence, its restriction  $(\gamma^1, \gamma_0^1)$  over  $V(G_1)$  is facet defining for  $STAB(G_1)$  and then, by Lemma 2.1, there exists  $S^1 \in \mathcal{S}_{MAX}(G_1, \beta^1)$  such that  $S^1$  is not tight for  $(\gamma^1, \gamma_0^1)$ . By Observation 4 there exists also  $S^2 \in \mathcal{S}_{MAX}(G_2, \beta^2)$  such that  $S^1$  and  $S^2$  generate  $S \in \mathcal{S}_{MAX}(G, \beta)$ . By construction, such  $S$  is not tight for  $(\gamma, \gamma_0)$ .

Assume now that  $(\gamma, \gamma_0)$  is a clique inequality induced by  $A = A_1 \cup A_2$ . Then again, as  $(\beta^1, \beta_0^1)$  is facet defining for  $STAB(G_1)$ , there exists  $S^1 \in \mathcal{S}_{MAX}(G_1, \beta^1)$  such that  $S^1$  is not tight for the clique inequality defined by  $A_1 \cup \{t_A^1, t_{AB}^1\}$ ; as a consequence  $S^1 \cap \{t_B^1, t_\emptyset^1\} \neq \emptyset$ . By Observation 4 there exists  $S^2 \in \mathcal{S}_{MAX}(G_2, \beta^2)$  such that  $S^1$  and  $S^2$  generate  $S \in \mathcal{S}_{MAX}(G, \beta)$ . As  $t_B^1 \in S^1$

or  $t_0^1 \in S^1$  then we can choose  $S^2$  such that  $S^2 \cap A_2 = \emptyset$ . As a consequence,  $S$  is not tight for  $(\gamma, \gamma_0)$ .

Consider now the case with  $(\gamma, \gamma_0)$  being a composition of a 1-inequality  $(\gamma^1, \gamma_0^1)$  that is facet defining for  $STAB(G_1)$  and a 3-inequality  $(\gamma^2, \gamma_0^2)$  that is facet defining for  $STAB(G_2)$ . If  $(\gamma^1, \gamma_0^1)$  is equivalent to  $(\beta^1, \beta_0^1)$  and  $(\gamma^2, \gamma_0^2)$  is equivalent to  $(\beta^2, \beta_0^2)$ , then  $(\gamma, \gamma_0)$  is equivalent to  $(\beta, \beta_0)$  and we are done. Then, without loss of generality, assume that  $(\gamma^1, \gamma_0^1)$  cannot be obtained as a positive multiple of  $(\beta^1, \beta_0^1)$ . Hence, there exists  $S^1 \in \mathcal{S}_{MAX}(G_1, \beta^1)$  such that  $\gamma^1(S^1) < \gamma_0^1$ . Again, by Observation 4 there exists  $S^2 \in \mathcal{S}_{MAX}(G_2, \beta^2)$  such that  $S^1$  and  $S^2$  generate  $S \in \mathcal{S}_{MAX}(G, \beta)$ . As  $\gamma^2(S^2) \leq \gamma_0^2$ , then it is not difficult to see that, in order to have  $S$  tight for  $(\gamma, \gamma_0)$ , it should be  $\gamma^1(S^1 \cap W_3^1) + \gamma^2(S^2 \cap W_3^2) < 1$ . However, the validity of the composition of  $(\gamma^1, \gamma_0^1)$  and  $(\gamma^2, \gamma_0^2)$  established by Lemma 4.2 is equivalent to the condition  $\gamma^1(\tilde{S} \cap W_3^1) + \gamma^2(\tilde{S} \cap W_3^2) \geq 1$  for any maximal stable set  $\tilde{S}$  of  $G$ . So we get a contradiction.

Analogous arguments apply to prove the case where  $(\gamma, \gamma_0)$  is a composition of a 3-inequality  $(\gamma^1, \gamma_0^1)$  that is facet defining for  $STAB(G_1)$  and a 1-inequality  $(\gamma^2, \gamma_0^2)$  that is facet defining for  $STAB(G_2)$ . ■

## 5. The main result

In this final section we prove the main result of the paper. We first recall the definition of 2-clique-bond.

**Definition 5.1.** Let  $G_1$  and  $G_2$  be two graphs. Let  $a_0^i$  and  $b_0^i$  be two adjacent nodes of  $G_i$  such that  $A_i = N(a_0^i) \setminus \{b_0^i\}$  and  $B_i = N(b_0^i) \setminus \{a_0^i\}$  are two cliques of  $V(G_i)$ ,  $i = 1, 2$ .

The 2-clique-bond of  $G_1$  and  $G_2$  along the edges  $a_0^1 b_0^1$  and  $a_0^2 b_0^2$  is the graph  $G$  obtained by deleting the nodes  $a_0^i$  and  $b_0^i$ , for  $i = 1, 2$ , and joining every node in  $A_1$  with every node in  $A_2$  and every node of  $B_1$  with every node of  $B_2$ .

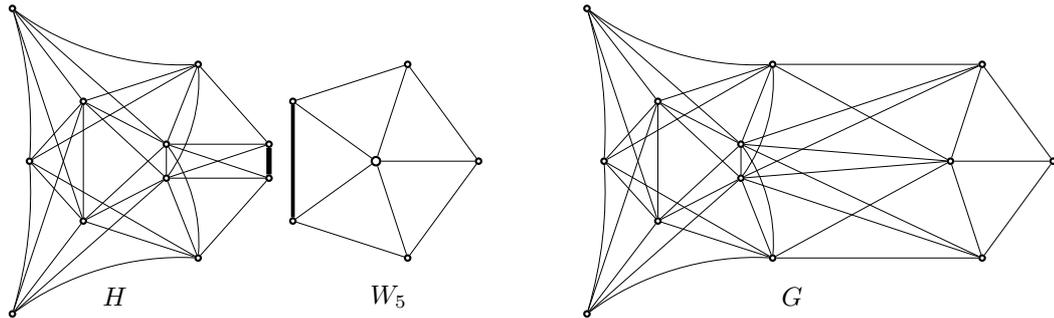


Figure 2: Graph  $G$  is the 2-clique-bond composition of  $H$  and  $W_5$ .

In Fig. 5 it is represented the 2-clique-bond composition of a graph  $H$  and a 5-wheel  $W_5$  along two simplicial edges depicted in bold. In the following we see how to compose inequalities that are valid for the stable set polytopes of some graphs associated with  $G_1$  and  $G_2$  in order to obtain valid inequalities for the stable set polytope of a graph  $G$  that is the 2-clique-bond of  $G_1$  and  $G_2$ .

**Definition 5.2.** Let  $G_i$  be a graph with a simplicial edge  $e_i = a_0^i b_0^i$ ,  $i = 1, 2$  and let  $G$  be the 2-clique-bond of  $G_1$  and  $G_2$  along  $e_1$  and  $e_2$ . Let  $G_i/e_i$  be the graph obtained by contracting the edge  $a_0^i b_0^i$  into a single node  $z_0^i$ .

An inequality  $(\beta, \beta_0)$  of  $STAB(G)$  is said to be an even-odd combination of inequalities of  $STAB(G_i)$  and  $STAB(G_j/e_j)$  for  $i, j = 1, 2$  and  $i \neq j$  if it has the following form:

$$\sum_{v \in V(G_1 \setminus \{a_0^1, b_0^1\})} \beta_v^1 x_v + \sum_{v \in V(G_2 \setminus \{a_0^2, b_0^2\})} \beta_v^2 x_v \leq \beta_0^1 + \beta_0^2 - 1, \quad (2)$$

where

either  $\beta^1 x \leq \beta_0^1$  is a valid inequality for  $STAB(G_1)$  different from  $x_{a_0^1} + x_{b_0^1} \leq 1$ ,  $\beta^2 x \leq \beta_0^2$  is a valid inequality for  $STAB(G_2/e_2)$  different from the clique inequalities supported by  $A_2 \cup \{z_0^2\}$  or  $B_2 \cup \{z_0^2\}$ , and  $\beta_{a_0^1}^1 = \beta_{b_0^1}^1 = \beta_{z_0^2}^2 = 1$ ,

or  $\beta^1 x \leq \beta_0^1$  is a valid inequality for  $STAB(G_1/e_1)$  different from the clique inequalities supported by  $A_1 \cup \{z_0^1\}$  or  $B_1 \cup \{z_0^1\}$ ,  $\beta^2 x \leq \beta_0^2$  is a valid inequality for  $STAB(G_2)$  different from  $x_{a_0^2} + x_{b_0^2} \leq 1$ , and  $\beta_{z_0^1}^1 = \beta_{a_0^2}^2 = \beta_{b_0^2}^2 = 1$ .

Note that the conditions  $\beta_{a_0^1}^1 = \beta_{b_0^1}^1$  and  $\beta_{a_0^2}^2 = \beta_{b_0^2}^2$  are not restrictive by Proposition 3.3. Therefore it is always possible to normalize  $(\beta^1, \beta_0^1)$  and  $(\beta^2, \beta_0^2)$  so that  $\beta_{a_0^1}^1 = \beta_{b_0^1}^1 = \beta_{z_0^2}^2 = 1$  or  $\beta_{z_0^1}^1 = \beta_{a_0^2}^2 = \beta_{b_0^2}^2 = 1$ .

We can now prove the main result of the paper.

**Theorem 5.3.** Let  $G_i$  be a graph with a simplicial edge  $e_i = a_0^i b_0^i$ ,  $i = 1, 2$ , and let  $G$  be the 2-clique-bond of  $G_1$  and  $G_2$ .

Then  $STAB(G)$  is described by the following inequalities:

- nonnegativity inequalities;
- clique inequalities induced by  $A_1 \cup A_2$  and  $B_1 \cup B_2$ ;
- facet defining inequalities of  $STAB(G_i)$  with zero coefficients on the endnodes of  $e_i$  for each  $i = 1, 2$ ;
- even-odd combinations of facet defining inequalities of  $STAB(G_i)$  and  $STAB(G_j/e_j)$  for each  $i, j = 1, 2$  and  $i \neq j$ .

*Proof.* Build two  $W_3$ -graphs  $\Gamma_i = (G_i \setminus \{a_0^i, b_0^i\}, A_i, B_i, W_3^i)$ ,  $i = 1, 2$  as described in Definition 3.1. It is not difficult to see that the  $W_3$ -composition of  $\Gamma_1$  and  $\Gamma_2$  is isomorphic to  $G$ , thus  $STAB(G)$  can be derived from Theorem 4.6 by observing that:

- The 0-inequalities of  $STAB(\Gamma_i)$  have the same supporting graph of facet defining inequalities of  $STAB(G_i)$  with zero coefficients on the endnodes of  $e_i$ , for  $i = 1, 2$ , and therefore they are equivalent.
- The 1-inequalities of  $STAB(\Gamma_i)$  have the same supporting graph of facet defining inequalities of  $STAB(G_i/e_i)$  that have nonzero coefficient on  $z_0^i$  and are different from the clique inequalities supported by  $A_1 \cup \{z_0^1\}$  or  $B_1 \cup \{z_0^1\}$ , for  $i = 1, 2$ .

- The graphs  $\Gamma_i \setminus \{t_0^i, t_{AB}^i\}$  are isomorphic to  $G_i$  and every 3-inequality of  $STAB(\Gamma_i)$  corresponds to the sequential lifting of a facet defining inequality of  $STAB(G_i)$  having nonzero coefficients on  $a_0^i$  and  $b_0^i$  with the additional node  $t_{AB}^i$ , for  $i = 1, 2$ .
- The composition of 1-inequalities and 3-inequalities as described in (1) produces the same inequalities obtained by the even-odd combination (2) as described in Definition 5.2.

Thus the theorem follows and, by Theorem 4.7, the provided linear description is also minimal. ■

In Fig. 5 we show how to obtain a nontrivial facet defining inequality for the stable set of a graph  $G$  resulting from the 2-clique-bond of a graph  $H$  and a 5-wheel  $W_5$ . By applying Theorem 5.3 it is not difficult to verify that the graph  $G$  supports a nontrivial facet defining inequality obtained as the even-odd combination of the 5-wheel inequality of  $STAB(W_5)$  and a facet defining inequality of  $STAB(H/e)$  (where  $e$  was the simplicial edge involved in the 2-clique-bond composition) defined by Giles and Trotter in [19].

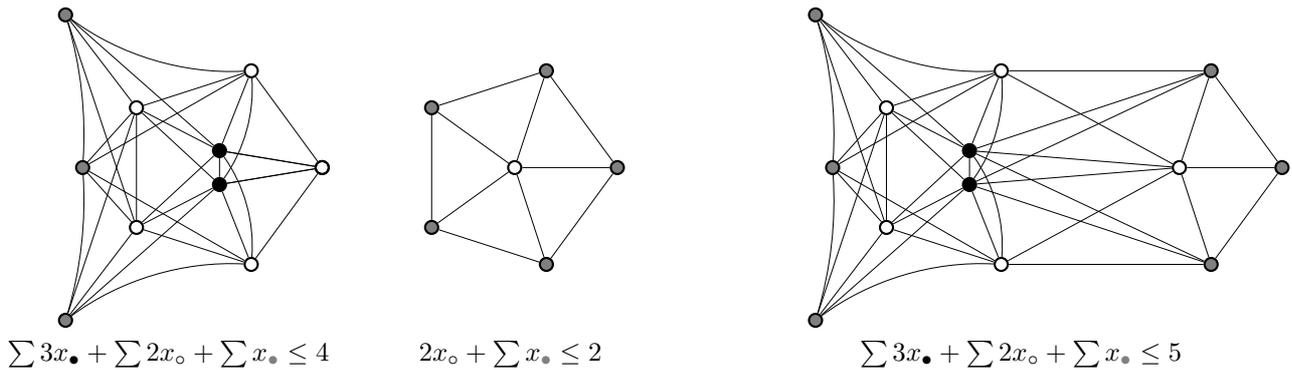


Figure 3: Even-odd composition of facet defining inequalities

## 6. Final remark

We end the paper with some final considerations on the consequences of the above results. If we drop the requirement that  $A_i$  and  $B_i$  are cliques, we speak of 2-bond. Since the sets  $A_i$  and  $B_i$  are not required to be disjoint, the 2-bond properly generalizes the 2-join as defined in [32]. It is also interesting to observe that the 2-bond also generalizes the 2-amalgam defined by Cornuéjols and Cunningham [11]. Indeed the 2-amalgam was conceived to preserve perfectness and so, the set  $A_i \cap B_i = K_i$  was required to be a clique for  $i = 1, 2$  such  $|K_1| = |K_2|$ . Moreover,  $N(K_i) \subseteq N(a_0^i) \cup N(b_0^i)$  and the 2-amalgam joined every node in  $A_1$  with every node in  $A_2$  and every node of  $B_1$  with every node of  $B_2$  while the two cliques  $K_1$  and  $K_2$  are identified. It is not difficult to see that the identification of the two cliques  $K_1$  and  $K_2$  was needed to preserve perfectness but graphs obtained as 2-amalgam can also be obtained as 2-bond and in this sense the 2-bond generalizes the 2-amalgam even though it does not preserve perfectness.

The 2-clique-bond composition reveals crucial to understand the facial structure of the stable set polytope for important classes of graphs. One of these is the class of claw-free graphs whose stable set polytope does not have an explicit linear description yet. In fact, under the

restriction that  $N(A_i \cap B_i) \subseteq N(a_0^i) \cup N(b_0^i)$ , the 2-clique-bond becomes the generalized 2-join introduced by Chudnovsky and Seymour in the decomposition theorem of claw-free graphs [9]. This decomposition theorem basically states that claw-free graphs that are not *quasi-line* (i.e., whose neighborhood cannot be partitioned into two cliques) and do not contain homogeneous pairs of clique (see [9] for definitions) are obtained by performing 1-joins or generalized 2-joins of graphs belonging to one of the following classes: linear interval graphs,  $XX$ -graphs, antihat graphs and claw-free graphs with stability number at most three. The polyhedral features of the 1-join are well understood since the 1-join is a special case of the composition of polyhedra described by Chvátal in [10]. It follows from Theorem 5.3 that an explicit linear description of  $STAB(G)$  when  $G$  is claw-free is obtained by composing the facet defining inequalities of the stable set polytopes of the building blocks  $G_1, \dots, G_k$  of the decomposition and their contractions along simplicial edges.

Notice that 2-clique-bond composition allows the construction of classes of graphs other than claw-free graphs and whose stable set polytope is easy to describe. For example, graphs obtained as the 2-clique-bond composition of linear interval graphs with any graph  $H$  such that  $STAB(H)$  and  $STAB(H/e)$ ,  $e$  simplicial, are known.

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