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THE STABLE SET POLYTOPE OF CLAW-FREE GRAPHS II: XX-GRAPHS ARE $\mathcal{G}$-PERFECT

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Abstract

A graph is said to be $\mathcal{G}$-perfect if its stable set polytope is described by: nonnegativity inequalities, rank inequalities, lifted 5-wheel inequalities and some special inequalities called *multiple geared inequalities*.

We prove that a large number of claw-free graphs with stability number greater than three are $\mathcal{G}$-perfect. This result moves a significant step towards the solution of the longstanding open problem of finding a linear description of the stable set polytope of claw-free graphs.

Key words: stable set polytope, claw-free graphs, polyhedral combinatorics.

1. Introduction

A well-known result of Grötschel, Lovász and Schrijver [12] states that the existence of a polynomial time algorithm to optimize over a polyhedron P is equivalent to the existence of a polynomial time algorithm for the separation problem over P. Therefore, a largely accepted conjecture in the Mixed-Integer Programming community is that if there exists a polynomial time algorithm to optimize over a polyhedron P then an explicit description of the defining linear system of P can also be found. Only for very few known problems [4] this conjecture is still open and one of them is the stable set problem for claw-free graphs.

Given a graph $G = (V, E)$ and a vector $w \in \mathbb{Q}_+^V$ of node weights, the *stable set problem* is the problem of finding a set of pairwise nonadjacent nodes *(stable set)* of G of maximum weight. We denote by $\alpha(G, w)$ the maximum weight of a stable set of G and we refer to $\alpha(G) = \alpha(G, \mathbb{1})$ ($\mathbb{1}$ being the vector of all ones) as the *stability number* of G. The *stable set polytope*, denoted by $STAB(G)$, is the convex hull of the incidence vectors of the stable sets of G. A linear system $Ax \leq b$ is said to be *defining* for $STAB(G)$ if $STAB(G) = \{x \in \mathbb{R}^V : Ax \leq b\}$. So, finding the defining linear system for $STAB(G)$ is equivalent to transform the original optimization problem into the linear program $\max\{w^T x : Ax \leq b\}$. Since the stable set problem is NP-hard, it is unlikely to find a defining linear system of $STAB(G)$ for general graphs. Nevertheless the study of the facial structure of the stable set polytope remains one of the main research topic in combinatorial optimization [3, 18, 25, 27].

For the claw-free graphs there exists a polynomial time algorithm to optimize over $STAB(G)$ [14, 15, 17] but finding an explicit linear description of $STAB(G)$ is still an open problem [22].

A linear inequality $\sum_{j\in V} \pi_j x_j \leq \pi_0$ is said to be a *rank inequality* for $STAB(G)$ if $\pi_i = 1$ for each $i \in S \subseteq V$, $\pi_i = 0$ for each $i \in V \setminus S$ and $\pi_0 = \alpha(G[S])$, where $G[S]$ is the subgraph of G induced by S. In 1965 Edmonds proved that the stable set polytope of a subclass of claw-free graphs, the *line graphs*, is described only by rank inequalities [5]. The line graph $L(G)$ of a graph G is obtained by considering the edges of G as nodes of $L(G)$ and two nodes of $L(G)$ are adjacent if and only if the corresponding edges of G have a common endnode.

After that pioneering result it seemed natural to look for the linear description of the stable set polytope of graphs that generalize line graphs: in particular, *claw-free graphs*, i.e., graphs such that the neighborhood of each node has no stable set of size three, and *quasi-line graphs*, i.e., graphs such that the neighborhood of each node can be partitioned into two cliques. Notice that the class of claw-free graphs properly contains the class of quasi-line graphs and both classes properly contain the line graphs.

Despite many research efforts [11, 21, 10, 16, 13], all the conjectures concerning the inequalities that are facet defining for $STAB(G)$ when G is claw-free or quasi-line were disproved. Here we mention four of them: the first three were disproved in [11] and the last one is more recent [24] and it was disproved in [8]:

Let $G = (V, E)$ be a claw-free graph and let $\sum_{i\in V} \pi_i x_i \leq \pi_0$ be an inequality that is facet defining for $STAB(G)$.

> *Conjecture 1. If G is quasi-line then $\sum_{i\in V} \pi_i x_i \leq \pi_0$ is a rank inequality,*
>
> *Conjecture 2. $\pi_i \in \{0, 1, 2\}$ for all $i \in V$,*
>
> *Conjecture 3. $\sum_{i\in V} \pi_i x_i \leq \pi_0$ can be obtained by a single application of the Chvatál's procedure to clique inequalities,*
>
> *Conjecture 4. If $\alpha(G) \geq 4$ and G not quasi-line, then $\sum_{i\in V} \pi_i x_i \leq \pi_0$ is either a rank inequality or a (lifted) 5-wheel inequality.*

The disproval of all these conjectures indicates that the extension of the results on the matching polytope to the stable set polytope is not so trivial. In particular, the disproval of Conjecture 1 implies that rank inequalities are not sufficient to describe $STAB(G)$ as long as G is not a line graph.

We had to wait for the results of Chudnovsky and Seymour on claw-free graphs [2] to gain a new perspective on all the work that has been done so far about the facial structure of $STAB(G)$. They proved that every claw-free graph that does not admit a 1-join either has stability number at most 3 or it is fuzzy circular interval or it can be obtained by composing three types of graphs, called *strips*: fuzzy linear interval strips, fuzzy XX-strips and fuzzy antihat strips. Moreover, they showed that every quasi-line graph belongs to either $\mathcal{Q}^\ell$ or $\mathcal{Q}^c$, where $\mathcal{Q}^\ell$ denote the set of quasi-line graphs that are composition of fuzzy linear interval strips and $\mathcal{Q}^c$ denote the set of quasi-line graphs that are fuzzy circular interval [1].

All these different classes of graphs behave quite differently from the polyhedral point of view: in fact, while the *Edmonds inequalities* suffice to provide a linear description of $STAB(G)$ when $G \in \mathcal{Q}^\ell$ [1], more complicated inequalities, i.e., the *clique-family inequalities*, come into play when $G \in \mathcal{Q}^c$ as shown by Eisenbrand et al. [6]. Moreover, while the case $\alpha(G) = 2$ has been solved by Cook years ago (see [23]), for $\alpha(G) = 3$ the roots of the facet defining inequalities of $STAB(G)$ have been studied [20]. It is worth noticing that up to now the most difficult facet defining inequalities of $STAB(G)$, i.e., those with arbitrarily many different coefficients [19, 13], appear only when G has stability number three. So, there is a chance that the stable set polytope of claw-free graphs with stability number greater than three that are obtained by composing fuzzy linear interval strips, XX-strips and antihat-strips has a simpler polyhedral structure.

In this paper we investigate this possibility and we provide a defining linear system for the stable set polytope of the XX*-graphs*, namely those graphs obtained by composing only two types of strips: fuzzy linear interval strips and XX-strips. By the Chudnovsky-Seymour decomposition theorem, the class of XX-graphs coincide with the class of all claw-free, not quasi-line graphs with stability number at least 4 that do not contain an antihat strip. Thus our result provides a significant contribution to the problem of finding a defining linear system for the stable set polytope of “genuine” claw-free graphs, namely claw-free graphs that are not quasi-line.

To prove our result we use the characterization of XX-graphs provided in [9] and the terminology introduced by Grötschel, Lovász and Schrijver in [12]. If $\mathcal{L}$ denotes a set of inequalities that are valid for $STAB(G)$ and $\mathcal{L}STAB(G) = \{x \in \mathbb{R}^V_+ |\ x \ \textit{satisfies}\ \ \mathcal{L}\}$ denotes the polytope of points satisfying all inequalities in $\mathcal{L}$, then a graph G is said to be $\mathcal{L}$*-perfect* if $\mathcal{L}STAB(G) = STAB(G)$. Different sets $\mathcal{L}$ of inequalities have been considered in the literature together with the corresponding classes of $\mathcal{L}$-perfect graphs, e.g. edge plus odd hole inequalities and t-perfect graphs [3]; clique plus odd hole inequalities and h-perfect graphs; rank inequalities and rank-perfect graphs [26].

In [9] we extended the notion of gear composition introduced in [8] in order to make it suitable to handle claw-free graphs. This *extended gear composition* builds a graph G called *geared graph* by suitably composing a given graph H with a fixed graph B (*extended gear*) along an edge e. We used this new graph operation to obtain a decomposition of the XX-graphs that is alternative to the one defined by Chudnosky and Seymour, i.e., we showed that an XX-graph can be obtained from a graph in $\mathcal{Q}^\ell$ by iteratively applying the extended gear composition.

In Section 3, we generalize the polyhedral properties of the simple gear composition studied in [7] to the extended gear composition. Then, in Section 4, we define the family $\mathcal{G}$ consisting of linear inequalities that are (sequential liftings) of: *rank inequalities*, 5*-wheel inequalities* and some special inequalities deriving from the gear composition called *multiple geared inequalities*. Finally, we prove that the inequalities in $\mathcal{G}$ are sufficient to describe $STAB(G)$ when G is an XX-graph. In other words, we prove that XX-graphs are $\mathcal{G}$-perfect. Apart from contributing to the problem of finding a linear description for the stable set polytope of claw-free, not quasi-line graphs (see [9] for an extensive introduction to the topic), our result supports the intuition that, from the polyhedral point of view, claw-free, not quasi-line graphs with large stability number are a *natural* generalization of the line graphs (see [7] for a more formal conjecture).

2. Preliminaries

We denote by G any connected, finite graph with node set V_G and edge set E_G. Given a vector $\beta \in \mathbb{R}^m$ and a subset $S \subseteq \{1, \ldots, m\}$, we define $\beta_S \in \mathbb{R}^{|S|}$ as the subvector of β restricted on the indices of S and we denote by $x^S \in \mathbb{R}^m$ the incidence vector of S. When no confusion arises we shall write β_G to indicate β_{V_G}. Moreover, we let $\beta(S) = \sum_{i \in S} \beta_i$.

A linear inequality $\sum_{j \in V_G} \pi_j x_j \leq \pi_0$ is *valid* for $STAB(G)$ if it holds for all $x \in STAB(G)$. A valid inequality for $STAB(G)$ *defines* a facet for $STAB(G)$ if and only if it is satisfied as an equality by $|V_G|$ affinely independent incidence vectors of stable sets of G (called *roots* or *tight solutions*). Since $STAB(G)$ is full dimensional, the *facet defining inequalities* for $STAB(G)$ are those inequalities that constitute the unique nonredundant defining linear system of $STAB(G)$. For short, we also denote a linear inequality $\pi^T x \leq \pi_0$ as (π, π_0). The subgraph G' of G induced by the nonzero components of a facet defining inequality (π, π_0) of $STAB(G)$ is called the *supporting graph* of (π, π_0). We also say that a stable set S is *tight* for (π, π_0) if $\pi(S) = \pi_0$ and that S *violates* (π, π_0) if $\pi(S) > \pi_0$.

We denote by $\delta(v)$ the set of edges of G having v as endnode and by $N(v)$ the set of nodes of V_G adjacent to v. We also denote by $G \setminus A$ the subgraph of G induced by $V_G \setminus A$ where $A \subseteq V_G$ and by $G \setminus e$ the subgraph of G obtained by removing the edge e.

A *k-hole* $C_k = (v_1, v_2, \ldots, v_k)$ is a chordless cycle of length k. A 5-*wheel* $W = (h : C_5)$ is a graph consisting of a 5-hole C_5 and a node h (*hub* of W) adjacent to every node of C_5. A *claw* is a bipartite graph $K_{1,3}$. A *gear* B is a graph of eight nodes $\{a, b_1, b_2, c, d_1, d_2, h_1, h_2\}$ such that $W_1 = (h_1 : a, d_1, b_1, c, h_2)$ and $W_2 = (h_2 : a, d_2, b_2, c, h_1)$ are 5-wheels; moreover, the edges of these 5-wheels are the only edges of B.

In [9] we extended the definitions of gear and gear composition given in [8] to make them suitable to treat claw-free graphs; here, we recall those definitions.

Definition 2.1. *Let $B = (V_B, E_B)$ be a gear and let u_{11} and u_{12} be two new nodes such that $N(u_{11}) = \{d_1, a, c, h_1, h_2, b_2\}$ and $N(u_{12}) = \{d_2, a, c, h_1, h_2, b_1\}$. Let $Y \subseteq \{u_{11}, u_{12}\}$ and $\delta(Y) = \bigcup_{u \in Y} \delta(u)$.*
An extended gear B_Y is a graph with node set $V_B \cup Y$ and edge set $E_B \cup \delta(Y)$ (see Fig. 1).

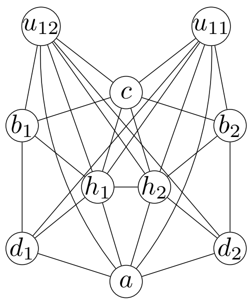


Figure 1: The extended gear B_Y with $Y = \{u_{11}, u_{12}\}$.

An edge $v_1 v_2$ of a graph H is said to be *simplicial* if $K_1 = N(v_1) \setminus \{v_2\}$ and $K_2 = N(v_2) \setminus \{v_1\}$ are cliques of H. Notice that K_1 and K_2 might have nonempty intersection.

Definition 2.2. *Let $H = (V_H, E_H)$ be a graph with a simplicial edge $e = v_1 v_2$ and let $B_Y = (V_{B_Y}, E_{B_Y})$ be an extended gear where $Y \subseteq \{u_{11}, u_{12}\}$. The* extended gear composition *of H and B_Y along e pro-*

duces a new graph G such that:

$V_G = V_H \setminus \{v_1, v_2\} \cup V_{B_Y}$,
$E_G = E_H \setminus (\delta(v_1) \cup \delta(v_2)) \cup E_{B_Y} \cup F_1 \cup F_2$, *where* $F_i = \{d_i u | u \in K_i\} \cup \{b_i u | u \in K_i\}$ *for* $i = 1, 2$.

The graph G is called geared graph *and to remark the fact that it is obtained by an extended gear composition of H and B_Y along e, it will be denoted by $G = (H, B_Y, e)$.*

When $Y = \emptyset$ an extended gear coincides with a gear and the extended gear composition coincides with the gear composition as defined in [8]. In [7] we proved the main polyhedral properties of the gear composition when $Y = \emptyset$. In particular, when $G = (H, B_\emptyset, e)$ is a geared graph, we built valid inequalities for $STAB(G)$ starting from valid inequalities for $STAB(H)$ or $STAB(H^e)$; then we showed that these inequalities are facet defining for $STAB(G)$ provided that the original inequalities are facet defining for $STAB(H)$ or $STAB(H^e)$. We recall here the definitions of geared inequalities and g-lifted inequalities given in [7].

Definition 2.3. *Let H be a graph with a simplicial edge $e = v_1 v_2$. Let H^e be the graph obtained from H by subdividing the edge e with a new node t.*

An inequality (π, π_0) which is valid for $STAB(H)$ is said to be g-extendable *(with respect to e) if $\pi_{v_1} = \pi_{v_2} = \lambda > 0$ and it is not the inequality $x_{v_1} + x_{v_2} \leq 1$.*

An inequality (π, π_0) which is valid for $STAB(H^e)$ is said to be g-liftable *(with respect to e) if $\pi_{v_1} = \pi_{v_2} = \pi_t = \lambda > 0$.*

Definition 2.4. *Let $H = (V_H, E_H)$ be a graph containing the simplicial edge $v_1 v_2$, let $B = (V_B, E_B)$ be a gear and let (π, π_0) be a valid inequality for $STAB(H)$ that is g-extendable with respect to e. Then the inequalities*

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B \setminus \{h_1, h_2\}} x_i + 2\lambda (x_{h_1} + x_{h_2}) \leq \pi_0 + 2\lambda \tag{1}$$

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B \setminus A} x_i \leq \pi_0 + \lambda \tag{2}$$
$$where \quad A \in \{\{b_1, c\}, \{b_2, c\}, \{d_1, a\}, \{d_2, a\}, \{a, c\}\}$$

are called geared inequalities *associated with (π, π_0). The unique geared inequality that has full support on V_B is (1) and it will be called* proper geared inequality.

Definition 2.5. *Let $H = (V_H, E_H)$ be a graph containing the simplicial edge $e = v_1 v_2$, let $B = (V_B, E_B)$ be a gear and let (π, π_0) be a valid inequality for $STAB(H^e)$ that is g-liftable with respect to e. Then the inequalities*

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B} x_i \leq \pi_0 + \lambda, \tag{3}$$

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B \setminus A} x_i \leq \pi_0 \tag{4}$$
$$where \quad A \in \{\{b_1, c, b_2, h_1, h_2\}, \{d_1, a, d_2, h_1, h_2\}\}$$

are called g-lifted inequalities *associated with (π, π_0). The unique g-lifted inequality that has full support on V_B is (3) and it will be called* proper g-lifted inequality.

Examples of the above defined inequalities can be found in [9]. In [7] we showed that the inequalities (1), (2), (3) and (4) are sufficient to give a linear description of a geared graph G (when $Y = \emptyset$) provided that a linear description of $STAB(H)$ and $STAB(H^e)$ is known.

Theorem 2.6. *[7] Let $G = (H, B_Y, e)$ be a geared graph with $Y = \emptyset$. Then the stable set polytope $STAB(G)$ is described by the following linear inequalities:*

- *nonnegativity inequalities,*
- *clique inequalities,*
- *(lifted) 5-wheel inequalities,*
- *geared inequalities of type (1) and (2) associated with g-extendable facet defining inequalities of $STAB(H)$,*
- *g-lifted inequalities of type (3) and (4) associated with g-liftable facet defining inequalities of $STAB(H^e)$,*
- *facet defining inequalities of $STAB(H)$ with zero coefficients on the endnodes of e.*

In [9] we supplied the lifting coefficients of the nodes in Y for the geared inequalities and the g-lifted inequalities. Then we proved that the sequential lifting of inequalities (1), (2), (3) and (4) are not anymore sufficient to provide a linear description for $STAB(G)$ if $Y \neq \emptyset$. In fact, new facet defining inequalities arise when the nodes u_{11} and u_{12} are added to a gear B (see Theorem 10 in [9]). The new inequalities are similar to those listed in (4). More precisely they are the following:

$$\sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_{B_Y} \setminus A} x_i \leq \pi_0 \tag{5}$$
$$where \quad A \in \{\{b_1, a, c, d_2, h_1, h_2, u_{12}\}, \{d_1, a, c, b_2, h_1, h_2, u_{11}\}\},$$

where (π, π_0) is a g-liftable facet defining inequality of $STAB(H^e)$; from now on we include them in the list of g-lifted inequalities.

In this paper we prove that the sequential lifting of inequalities (1)-(4) plus the inequalities (5) suffice to describe $STAB(G)$ when G is a geared graph $G = (H, B_Y, e)$ and $Y \neq \emptyset$. We denote as $G_\emptyset, G_{11}, G_{12}$, and G_Ω the geared graphs $G = (H, B_Y, e)$ where Y equals the sets $\emptyset$, $\{u_{11}\}$, $\{u_{12}\}$, and $\Omega = \{u_{11}, u_{12}\}$, respectively. Similarly, we denote as $B_\emptyset$, B_{11}, B_{12}, B_Ω the extended gears B_Y.

In all our proofs we take advantage of the symmetric structure of the extended gear and we use three different kinds of symmetries.

Given a graph $G = (V, E)$ we say that a permutation $\sigma : V \to V$ is a *symmetry* of G if and only if $\sigma(N(v)) = N(\sigma(v))$ for each $v \in V$ (where $\sigma(N(v)) = \{\sigma(u) | u \in N(v)\}$). To simplify the notation we write $\sigma(v_1, v_2, \ldots, v_k) = (u_1, u_2, ..., u_k)$ instead of $\sigma(v_1) = u_1$, $\sigma(v_2) = u_2$, …, $\sigma(v_k) = u_k$.

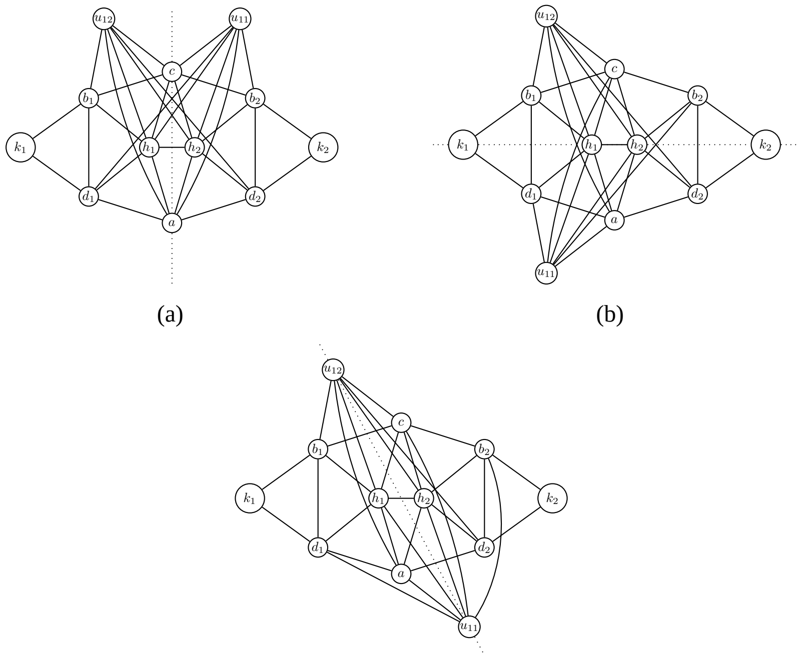
Consider an extended gear B_Y with $Y = \{u_{11}, u_{12}\}$ and let B_Y^* be the graph obtained from B_Y by adding two new nodes, say k_1 and k_2, such that $N(k_i) = \{b_i, d_i\}$, $i = 1, 2$.

It is easy to verify that the following observation holds.

Observation 1. *The permutation functions*

$$\sigma_v(a, c, b_1, d_1, b_2, d_2, h_1, h_2, u_{11}, u_{12}, k_1, k_2) = (a, c, b_2, d_2, b_1, d_1, h_2, h_1, u_{12}, u_{11}, k_2, k_1)$$
$$\sigma_h(a, c, b_1, d_1, b_2, d_2, h_1, h_2, u_{11}, u_{12}, k_1, k_2) = (c, a, d_1, b_1, d_2, b_2, h_1, h_2, u_{12}, u_{11}, k_1, k_2)$$
$$\sigma_d(a, c, b_1, d_1, b_2, d_2, h_1, h_2, u_{11}, u_{12}, k_1, k_2) = (c, a, d_2, b_2, d_1, b_1, h_2, h_1, u_{11}, u_{12}, k_2, k_1)$$

are symmetries for the graph B_Y^* and will be referred to as vertical symmetry, horizonthal symmetry, *and* diagonal symmetry, *respectively.*



(a) (b) (c)

Figure 2: (a) vertical symmetry; (b) horizonthal symmetry; (c) diagonal symmetry.

Each symmetry maps each node on one side of the dotted line onto the corresponding node on the other side (see Fig.2). Notice that $\sigma_v(u_{11}) = \sigma_h(u_{11}) = u_{12}$ and $\sigma_v(u_{12}) = \sigma_h(u_{12}) = u_{11}$, while $\sigma_d(u_{11}) = u_{11}$ and $\sigma_d(u_{12}) = u_{12}$. Therefore, when we delete both nodes u_{11} and u_{12}, the permutations σ_v, σ_h, and σ_d restricted to $V_{B^*_\emptyset}$ are still symmetries, while if we delete only the node u_{12} (u_{11}), the permutation σ_d restricted to $V_{B^*_Y} \setminus \{u_{12}\}$ ($V_{B^*_Y} \setminus \{u_{11}\}$) is still a symmetry, while σ_v and σ_h are not defined and so they are not symmetries anymore. Hence, the only symmetry for G_{11} and G_{12} is the diagonal symmetry.

Furthermore, for each induced subgraph of B^*_Y with $Y = \{u_{11}, u_{12}\}$ there exist three symmetric induced subgraphs. Indeed, for $A \subseteq V_{B^*_Y}$, let $B^*_Y \setminus A$ be the subgraph induced by $V_{B^*_Y} \setminus A$: the subgraphs obtained by applying the vertical, horizonthal and diagonal symmetry to $B^*_Y \setminus A$ are denoted as $\sigma_v(B^*_Y \setminus A)$, $\sigma_h(B^*_Y \setminus A)$, and $\sigma_d(B^*_Y \setminus A)$, respectively. In Fig. 3 we depict the induced subgraph $B^*_Y \setminus A$ for $A = \{b_1, c\}$ together with its three symmetric subgraphs. Notice that if $A = \{u_{12}\}$ then $\sigma_v(G \setminus A) = \sigma_v(G_{11}) = G_{12}$. Similarly $\sigma_h(G_{11}) = G_{12}$ and $\sigma_d(G_{11}) = G_{11}$.

3. Extended gear composition of polyhedra

In this section we prove that a linear description for $STAB(G)$ can be obtained from facet defining inequalities of $STAB(G_\emptyset)$ plus inequalities of type (5).

First observe that the two 5-wheels contained in the extended gear B_Y, i.e., $W_1 = (h_1 : a, d_1, b_1, c, h_2) = (h_1 : C_1)$ and $W_2 = (h_2 : a, d_2, b_2, c, h_1) = (h_2 : C_2)$ (see Fig. 1), produce two inequalities that do not have full support on V_{B_Y}. Trivially these inequalities are facet defining for $STAB(G)$.

Observation 2. *Let $G = (H, B_Y, e)$ be a geared graph with $Y \subseteq \{u_{11}, u_{12}\}$. Then*

$$\sum_{u \in C_i \cup Y} x_u + 2x_{h_i} \leq 2 \quad i = 1, 2 \qquad (6)$$

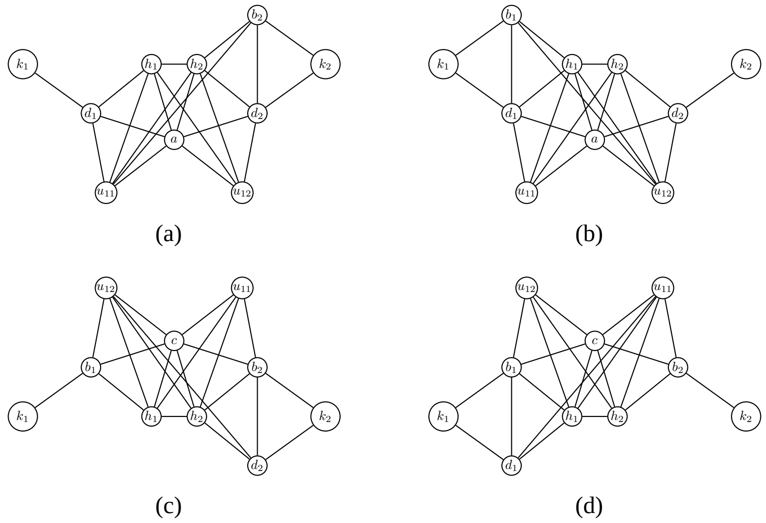


Figure 3: (a) The induced subgraph $B_Y^* \setminus A$ with $A = \{b_1, c\}$; (b) the vertical symmetric subgraph $\sigma_v(B_Y^* \setminus A) = B_Y^* \setminus \{b_2, c\}$; (c) the horizonthal symmetric subgraph $\sigma_h(B_Y^* \setminus A) = B_Y^* \setminus \{d_1, a\}$; (d) the diagonal symmetric subgraph $\sigma_d(B_Y^* \setminus A) = B_Y^* \setminus \{d_2, a\}$

are two lifted 5*-wheel inequalities that are facet defining for* $STAB(G)$*.*

The second consideration follows from the Chvátal's result on composition of polyhedra. A graph G has a clique-cutset if there exists a complete subgraph whose removal disconnects G.

Theorem 3.1. *[3] The supporting graph of a facet defining inequality for* $STAB(G)$ *does not have a clique-cutset.*

Now, if e is a simplicial edge with $K_1 = K_2$ then the geared graph G generated by H and B_Y along e has a clique-cutset. When this happens the results of Chvátal on the composition of polyhedra [3] explain how to find a defining linear system for $STAB(G)$ from the defining linear systems of $STAB(H)$ and $STAB(L)$, where $L = (K, B_Y, e)$ and $K = K_1 \cup \{v_1, v_2\}$. So, in the rest of the paper we will focus on the composition of polyhedra resulting from applying the extended gear composition along a simplicial edge that has $K_1 \neq K_2$.

We indicate with (β, β_0) a generic facet defining inequality for $STAB(G)$ where $G = (H, B_Y, e)$; then we split the vector of coefficients β into two subvectors $(\beta_{G \setminus B_Y}, \beta_{B_Y})$ where $\beta_{G \setminus B_Y}$ is the vector of coefficients associated with the nodes $V_G \setminus V_{B_Y}$ and β_{B_Y} is the vector of coefficients associated with the nodes V_{B_Y}. Finally, notice that:

Proposition 3.2. *[7] Let* G *be a graph and* G' *be a subgraph of* G *that supports a facet defining inequality* (β, β_0) *of* $STAB(G)$*. If* G' *contains a simplicial edge* $v_1 v_2$*, then* $\beta_{v_1} = \beta_{v_2}$*. If* G' *contains a simplicial edge* $v_1 v_2$ *subdivided with a node* t*, then* $\beta_{v_1} = \beta_{v_2} = \beta_t$*.*

Let $\mathscr{S}(G)$ denote the family of stable sets of G. If $\sum_{j \in V_G \setminus \{v\}} \pi_j x_j \leq \pi_0$ is a facet defining inequality of $STAB(G \setminus \{v\})$, then the inequality

$$\sum_{j \in V_G \setminus \{v\}} \pi_j x_j + \pi_v x_v \leq \pi_0 \;\; with \;\; \pi_v = \pi_0 - \max_{S \in \mathscr{S}(G \setminus (N(v) \cup \{v\}))} \pi(S)$$

is facet defining for $STAB(G)$ [18]. The resulting inequality is called *sequential lifting of* (π, π_0) and π_v is the *lifting coefficient of* v. This procedure can be iterated to generate facet defining inequalities, simply called *lifted inequalities*, of the stable set polytope of a larger graph.

We now state the main theorem of this paper.

Theorem 3.3. *Let $G = (H, B_Y, e)$ be a geared graph generated by H and B_Y along the simplicial edge $e = v_1v_2$. Then the stable set polytope $STAB(G)$ is described by the following linear inequalities:*

- *nonnegativity inequalities,*
- *clique inequalities,*
- *lifted* 5*-wheel inequalities,*
- *lifted geared inequalities of type (1) and (2) associated with a g-extendable facet defining inequality of $STAB(H)$,*
- *lifted g-lifted inequalities of type (3), (4) , and (5) associated with a g-liftable facet defining inequality of $STAB(H^e)$,*
- *facet defining inequalities of $STAB(H)$ with zero coefficients on the endnodes of e.*

Proof. Since the proof of this result is quite technical and up to some extent repetitive, we split it into three main steps that are illustrated below (each step is proved in a separate subsection). We consider a facet defining inequality (β, β_0) for $STAB(G)$ that is neither a clique inequality nor a lifted 5-wheel inequality. We assume that the components of β_{B_Y} are not all zero since otherwise (β, β_0) would be a facet defining inequality of $STAB(H)$.

If $Y = \emptyset$ the result follows by Theorem 2.6. So, we assume that $Y \neq \emptyset$ and $\beta_u > 0$ for each $u \in Y$ since otherwise (β, β_0) would be a facet defining inequality for $STAB(G \setminus \{u\})$.

We first consider the case when (β, β_0) does not have full support on V_{B_Y}, meaning that the set $A = \{u \in V_{B_Y} : \beta_u = 0\}$ is nonempty. In this case we show that:

a.1) If $Y = \{u_{11}\}$ then (β, β_0) is

either a geared inequality of type (2) with $A \in \{\{b_1, c\}, \{d_2, a\}\}$ lifted with node u_{11}
or a g-lifted inequality of type (5) with $A = \{b_1, a, c, d_2, h_1, h_2\}$.

This result follows by Theorem 3.7 in Subsection 3.1.

a.2) if $Y = \{u_{12}\}$ then (β, β_0) is

either a geared inequality of type (2) with $A \in \{\{b_2, c\}, \{d_1, a\}\}$ lifted with node u_{12}
or a g-lifted inequality of type (5) with $A = \{d_1, a, c, b_2, h_1, h_2\}$.

This result follows by Theorem 3.8 in Subsection 3.1.

a.3) If $Y = \{u_{11}, u_{12}\}$ then (β, β_0) is a lifted 5-wheel inequality (see Theorem 3.9 in Subsection 3.1).

Then we consider the case when (β, β_0) has full support on V_{B_Y}. In this case we prove that:

b.1) If $Y = \{u_{11}\}$ then (β, β_0) is obtained from a facet defining inequality of $STAB(G_\emptyset)$ by lifting the node u_{11} (see Theorem 3.19 in Subsection 3.2).

b.2) If $Y = \{u_{12}\}$ then (β, β_0) is obtained from a facet defining inequality of $STAB(G_\emptyset)$ by lifting the node u_{12} (see Theorem 3.20 in Subsection 3.2).

b.3) If $Y = \{u_{11}, u_{12}\}$ then (β, β_0) is obtained from a facet defining inequality of $STAB(G_\emptyset)$ by lifting u_{11} and u_{12} (see Theorem 3.21 in Subsection 3.2).

As a consequence of the above results, we have that each facet defining inequality for $STAB(G)$ which is different from clique inequalities and lifted 5-wheel inequalities, and has $\beta_{B_Y} \neq \mathbb{0}$ is

either the sequential lifting of an inequality of type (1) or (2) associated with a g-extendable facet defining inequality of $STAB(H)$,

or the sequential lifting of an inequality of type (3) or (4) associated with a g-liftable facet defining inequality of $STAB(H^e)$,

or an inequality of type (5) associated with a g-liftable facet defining inequality of $STAB(H^e)$.

Now, let (π, π_0) be a facet defining inequality (π, π_0) of $STAB(H)$ that is not the clique inequality $x_{v_1} + x_{v_2} \leq 1$. If π_{v_1} and π_{v_2} are both nonzero then, by Proposition 3.2, (π, π_0) is either a g-extendable or a g-liftable inequality. If exactly one between π_{v_1} and π_{v_2} is zero, say $\pi_{v_1} = 0$, then the node v_2 is simplicial in the supporting graph G' of (π, π_0). Since v_2 is simplicial and G' cannot contain a clique-cutset by Theorem 3.1, it follows that G' is a clique and so (π, π_0) is the clique inequality $\sum_{u \in K_2} x_u + x_{v_2} \leq 1$. Such inequality is extended to a clique inequality for $STAB(G)$ by simply replacing the node v_2 with $\{b_2, d_2\}$. Finally, if $\pi_{v_1} = \pi_{v_2} = 0$ then it is easy to see that (π, π_0) can be lifted to a facet defining inequality of $STAB(G)$ with zero coefficients on the nodes of B_Y. Thus the thesis follows. ∎

3.1. Inequalities not having full support on B_Y

In this section we deal with inequalities that do not have full support on V_{B_Y}. We denote by A the set $\{u \in V_{B_Y} : \beta_u = 0\}$. If a facet defining inequality (β, β_0) does not have full support on V_{B_Y} then $A \neq \emptyset$. We can suppose without loss of generality that $G \setminus A$ is the supporting graph of the inequality (β, β_0).

In the following we illustrate the arguments that will be used in the proofs of this subsection. The next three observations concern the lifting coefficients of the nodes in A.

Observation 3. *If $A \neq \emptyset$, then every node $u \in A$ has lifting coefficient $\beta_u = 0$.*

Observation 4. *If a node $u \in A$, then there exists a stable set S_u in $G \setminus (A \cup N(u))$ that is tight for (β, β_0).*

Observation 5. *If u and v are two adjacent nodes of G such that $N(u) \setminus (A \cup \{v\}) \supseteq N(v) \setminus (A \cup \{u\})$, then $\beta_u \geq \beta_v$. In particular this implies that: if $\beta_v > 0$ then the node $u \notin A$.*

We also use the following arguments:

Observation 6. *Let G be a graph and let (π, π_0) and (β, β_0) be two facet defining inequalities for $STAB(G)$. If (β, β_0) is not a positive scalar multiple of (π, π_0) then there exists a stable set S that is tight for (β, β_0), i.e., $\beta(S) = \beta_0$, but not for (π, π_0), i.e., $\pi(S) < \pi_0$.*

In the next proofs clique inequalities or lifted 5-wheel inequalities will play the role of (π, π_0) in the previous observation. In these cases, we will say that there exists a tight stable set S for (β, β_0) that *misses* a certain clique in $V_{B_Y} \cup K_1 \cup K_2$ or one of the two lifted 5-wheels contained in B_Y.

Observation 7. *Let G be a graph and let (β, β_0) be a facet defining inequality for $STAB(G)$. Then for each $u \in V_G$ there exists at least one stable set containing u that is tight for (β, β_0).*

Observation 8. *Let $B_Y \setminus A$ contain an induced 4-hole $C = (u_1, u_2, u_3, u_4)$ and a clique K such that $u_3, u_4 \in K$, $N(u_3) = K \setminus \{u_3\} \cup \{u_2\}$, and $N(u_4) = K \setminus \{u_4\} \cup \{u_1\}$. If (β, β_0) is facet defining for $STAB(G)$, then there is no tight stable set for (β, β_0) missing the clique K and, by Observation 6, (β, β_0) is equivalent to the clique inequality defined by K.*

Proof. Suppose that S is a tight stable set for (β, β_0) such that $S \cap K = \emptyset$. If $u_1 \in S$, then $S \cup \{u_3\}$ is feasible. If $u_2 \in S$, then $S \cup \{u_4\}$ is feasible. Finally, if $u_1, u_2 \notin S$, then $S \cup \{u_3\}$ is feasible. All the cases contradict the hypothesis that S is tight. ∎

To find all the facets for $STAB(G)$ when $G = (H, B_Y, e)$, we need to consider three different cases: $Y = \{u_{11}\}$, $Y = \{u_{12}\}$ and $Y = \{u_{11}, u_{12}\}$. The proofs of the first two cases are the same (up to vertical symmetry), so we consider only the case $Y = \{u_{11}\}$ and $Y = \{u_{11}, u_{12}\}$.

In particular, in the next two lemmas we consider a facet defining inequality for $STAB(G_{11})$ that is not a clique or a lifted 5-wheel inequality: in the first lemma we identify the supporting graph of such an inequality and, in the second lemma, we prove that all the coefficients on this graph have the same value.

Lemma 3.4. *Let $G = (H, B_Y, e)$ be a geared graph with $Y = \{u_{11}\}$ and let (β, β_0) be a facet defining inequality for $STAB(G)$ that does not have full support on V_{B_Y}. Let $A = \{u \in V_{B_Y} : \beta_u = 0\}$. If (β, β_0) is neither a clique inequality nor a lifted 5-wheel inequality and moreover $\beta_{u_{11}} > 0$, then one of the following cases occurs:*

1) $A = \{b_1, a, c, d_2, h_1, h_2\}$,
2) $A = \{d_2, a\}$,
3) $A = \{b_1, c\}$.

Proof. Let $G \setminus A$ be the supporting graph of the inequality (β, β_0). First observe that if S is a tight stable set of G, then $S \setminus A$ is also tight. So, in the following, also if not explicitely remarked, we will always refer to tight stable sets in $G \setminus A$.

As (β, β_0) is facet defining for $STAB(G)$, clearly, $G \setminus A$ has to be connected and, by Theorem 3.1, it has no clique-cutset. If $G \setminus A$ satisfies these two properties we say that $G \setminus A$ is admissible. The proof of the lemma is done by enumerating all possible subsets A such that $G \setminus A$ is admissible and by showing that only the three cases listed in the thesis do not lead to a contradiction. The use of the diagonal symmetry allows us to considerably reduce the number of possible cases (see Appendix A.1). Since the case (3) is diagonally symmetric to case (2), we shall provide an explicit proof of the lemma only for the cases (1) and (2).

It is not difficult to check that if $G \setminus A$ is admissible then $|A| \leq 6$. If $|A| = 6$ and $G \setminus A$ is admissible, then case (1) occurs. If $|A| = 5$ and $G \setminus A$ is admissible, then the unique nonsymmetric case is $A = \{d_1, d_2, a, c, h_2\}$ (see Appendix A.1). Since $N(h_1) \setminus (A \cup \{d_1\}) \subseteq N(d_1) \setminus (A \cup \{h_1\})$, by Observation 5, we have that $\beta_{d_1} \geq \beta_{h_1} > 0$, a contradiction. In the following we prove three easy claims.

Claim 1. If $h_1 \in A$, then $a \in A$. If $h_2 \in A$, then $c \in A$.

Let $h_1 \in A$. Then, by Observation 4, there exists a tight stable set S_{h_1} in $G \setminus (A \cup N(h_1))$. We have that $b_2 \in S_{h_1}$ since otherwise $S_{h_1} \cup \{u_{11}\}$ would violate (β, β_0). Thus if $a \notin A$, then $S_{h_1} \cup \{a\}$ violates (β, β_0), a contradiction. The second half of the statement follows by diagonal symmetry, i.e., by replacing each node u in the above proof with its diagonally symmetric node $\sigma_d(u)$. In particular, let $\sigma_d(h_1)(= h_2) \in A$. Then there exists a tight stable set S_{h_2} in $G \setminus (A \cup N(h_2))$. We have that $\sigma_d(b_2)(= d_1) \in S_{h_1}$ since otherwise $S_{h_2} \cup \{\sigma_d(u_{11})(= u_{11})\}$ would violate (β, β_0). Again, if $\sigma_d(a)(= c) \notin A$, then $S_{h_2} \cup \{c\}$ violates (β, β_0), a contradiction. *(End of Claim 1)*

Claim 2. $\{d_1, a\} \not\subseteq A$, $\{b_2, c\} \not\subseteq A$.

Since the two cases are diagonally symmetric, we prove the Claim explicitly only for $\{d_1, a\} \not\subseteq A$. Suppose that $\beta_a = \beta_{d_1} = 0$ and consider a tight stable set S in $G \setminus A$ missing the clique $\{c, b_2, h_2, u_{11}\}$. Clearly $h_1 \in S$, since otherwise $S \cup \{u_{11}\}$ would violate (β, β_0), and so $\beta_{h_1} \geq \beta_c$, $\beta_{h_1} \geq \beta_{u_{11}}$. Let S' be a tight stable set in $G \setminus A$ missing the clique $\{c, h_1, h_2, u_{11}\}$; then $b_1 \in S'$ (otherwise $S' \cup \{h_1\}$ would violate (β, β_0)) and thus $\beta_{b_1} \geq \beta_{h_1} \geq \beta_c$. By Observation 4, since $d_1 \in A$, there exists a tight stable set S_{d_1} in $G \setminus (A \cup N(d_1))$. Clearly $c \in S_{d_1}$, otherwise $S_{d_1} \cup \{b_1\}$ would violate (β, β_0). This implies that $S_{d_1} \setminus \{c\} \cup \{b_1, u_{11}\}$ violates (β, β_0), a contradiction. *(End of Claim 2)*

Claim 3. If $|A| \leq 4$, *then* $\{a, c\} \not\subseteq A$.

Suppose that $\beta_a = \beta_c = 0$. By Observation 4, there exist two tight stable sets S_a and S_c in $G \setminus (A \cup N(a))$ and $G \setminus (A \cup N(c))$, respectively. Because of Claim 2, we have $b_2 \notin A$. Consider now the case that $b_1 \in A$. Since $N(h_1) \setminus (A \cup \{a\}) \subseteq N(a) \setminus (A \cup \{h_1\})$, by Observation 5, $\beta_{h_1} \leq \beta_a = 0$, that is $h_1 \in A$. Since, by hypothesis $|A| \leq 4$, it follows that $A = \{a, c, b_1, h_1\}$, $b_2 \in S_a$ (otherwise $S_a \cup \{h_2\}$ would violate (β, β_0)), and $S_a \cap K_1 \neq \emptyset$; consequently $\beta_{b_2} \geq \beta_{u_{11}} + \beta_{d_2}$ and so $\beta_{b_2} > \beta_{d_2}$. Now, if $d_2 \in S_c$, then $S_c \setminus \{d_2\} \cup \{b_2\}$ violates (β, β_0) and if $d_2 \notin S_c$, then $S_c \cup \{h_2\}$ violates (β, β_0). Therefore, $b_1 \notin A$, $b_2 \notin A$, and, consequently, $\{b_1, b_2\} \subseteq S_a$; by diagonal symmetry $d_1 \notin A$, $d_2 \notin A$, and $\{d_1, d_2\} \subseteq S_c$. It follows that $\beta_{b_i} = \beta_{d_i}$, $i = 1, 2$, and the stable set $S_a \setminus \{b_2\} \cup \{d_2, u_{11}\}$ violates (β, β_0), a contradiction. *(End of Claim 3)*

If $|A| = 4$ and $G \setminus A$ is admissible, then arguments analogously to those in Appendix A.1 show that A is one of the following nonsymmetric sets:

i) $A = \{d_1, h_1, h_2, b_2\}$,
ii) $A = \{b_1, c, h_1, d_2\}$,
iii) $A = \{a, h_1, h_2, d_2\}$,
iv) $A = \{d_1, a, h_2, d_2\}$,
v) $A = \{a, c, h_2, d_2\}$,
vi) $A = \{d_1, a, c, d_2\}$,
vii) $A = \{b_1, a, c, d_2\}$.

The first four cases cannot occur because of Claim 1. The remaining cases cannot occur by Claim 3.

If $|A| = 3$ and $G \setminus A$ contains no clique-cutset, then A is one of the following nonsymmetric sets:

i) $A = \{d_1, h_1, h_2\}$,
ii) $A = \{d_1, c, h_2\}$,
iii) $A = \{d_1, h_2, b_2\}$,
iv) $A = \{a, h_2, d_2\}$,
v) $A = \{b_1, a, d_2\}$,
vi) $A = \{a, c, d_2\}$,
vii) $A = \{d_1, a, c\}$,
viii) $A = \{d_1, a, d_2\}$.

In the first three cases h_2 has a nonzero lifting coefficient, since Observation 5 applies to h_2, a. Case (iv) cannot occur because of Claim 1. In case (v), we have that $u_{11} \in S_{b_1}$; this implies that $\beta_{u_{11}} \geq \beta_{d_1} + \beta_c$. Similarly we have that $c \in S_a$ and so $\beta_{u_{11}} \leq \beta_c$, a contradiction. The last three cases cannot occur by Claim 2 and Claim 3.

If $|A| = 2$ and $G \setminus A$ contains no clique-cutset, then A is one of the following nonsymmetric sets:

i) $A = \{d_2, a\}$,
ii) $A = \{h_1, a\}$,
iii) $A = \{h_1, h_2\}$,
iv) $A = \{h_1, d_2\}$,
v) $A = \{d_1, h_1\}$,
vi) $A = \{d_1, h_2\}$,
vii) $A = \{d_1, c\}$,
viii) $A = \{c, d_2\}$,
ix) $A = \{d_1, b_2\}$,
x) $A = \{b_1, d_2\}$,
xi) $A = \{a, c\}$,
xii) $A = \{d_1, a\}$.

Case (i) matches case (2) in the thesis. Case (ii) cannot occur since Observation 8 applies to the clique $\{c, h_2, b_2, u_{11}\}$ and the 4-hole (c, u_{11}, d_1, b_1). Cases (iii)÷(vi) do not occur because of Claim 1. Case (vii) does not occur since Observation 5 applies to nodes b_1 and d_1.

Consider case (viii): by Observation 4, there exist two tight stable sets S_c and S_{d_2} in $G \setminus (A \cup N(c))$ and $G \setminus (A \cup N(d_2))$, respectively. It is not difficult to see that $a \in S_c$, and as a consequence $\beta_a \geq \beta_{u_{11}}$. If $h_1 \in S_{d_2}$ then $S_{d_2} \cup \{b_2\}$ violates (β, β_0), a contradiction. Thus $u_{11} \in S_{d_2}$ and $S_{d_2} \setminus \{u_{11}\} \cup \{a, b_2\}$

violates (β, β_0), a contradiction.

Consider case (ix): by Observation 4, there exist two tight stable sets S_{d_1} in $G \setminus (A \cup N(d_1))$ and S_{b_2} in $G \setminus (A \cup N(b_2))$. It is easy to see that $\{a, b_1\} \subseteq S_{b_2}$ and $\{c, d_2\} \subseteq S_{d_1}$. So, $\beta_c \leq \beta_{b_1}$, and then, $S_{d_1} \setminus \{c\} \cup \{b_1, u_{11}\}$ violates (β, β_0), a contradiction.

Finally, consider case (x) and let S be a tight stable set missing the clique $\{c, b_2, h_2, u_{11}\}$. Clearly, $h_1 \in S$ and so $\beta_{h_1} \geq \beta_a + \beta_c$. By Observation 4, there exists S_{b_1} in $G \setminus (A \cup N(b_1))$. If $\{a, b_2\} \subseteq S_{b_1}$ then $\beta_a \geq \beta_{h_1}$, a contradiction. Thus, $u_{11} \in S_{b_1}$ and $\beta_{u_{11}} \geq \beta_{d_1} + \beta_{h_2}$. Let S' be a tight stable set missing the clique $\{a, d_1, h_1, u_{11}\}$. Hence, $h_2 \in S'$ and $\beta_{h_2} \geq \beta_{u_{11}}$, a contradiction.

The last two cases cannot occur by Claim 3 and Claim 2, respectively.

If $|A| = 1$, then there are four nonsymmetric cases to be considered: $A = \{b_1\}$, $A = \{c\}$, $A = \{b_2\}$, and $A = \{h_1\}$. Since the last case contradicts Claim 1, we are left with only three cases.

Case 1. $A = \{b_1\}$.

Let T be a tight stable set missing the clique $\{c, b_2, h_2, u_{11}\}$. Clearly $h_1 \in T$ (otherwise $T \cup \{c\}$ would violate (β,β_0)), and so $\beta_{h_1} \geq \beta_{u_{11}}$. By Observation 4, there exists a tight stable set S_{b_1} in $G \setminus (A \cup N(b_1))$. It is not difficult to see that S_{b_1} contains $\{a, b_2\}$ or u_{11}.

Subcase 1a. $S_{b_1} \supseteq \{a, b_2\}$. Then $\beta_a \geq \beta_{h_1}$ and $\beta_a \geq \beta_{d_1}$. Since $S_{b_1} \setminus \{a, b_2\} \cup \{d_1, c, d_2\}$ is a stable set, it follows that $\beta_a + \beta_{b_2} \geq \beta_{d_1} + \beta_c + \beta_{d_2}$.

If $\beta_a = \beta_{d_1}$, then $\beta_{b_2} \geq \beta_c + \beta_{d_2}$. Since all coefficients of β_{B_Y} apart from β_{b_1} are positive, we have that $\beta_{b_2} > \beta_{d_2}$. This implies that $d_2 \notin T$ (otherwise $T \setminus \{d_2\} \cup \{b_2\}$ would violate (β, β_0)) and so $T \setminus \{h_1\} \cup \{a, c\}$ violates (β, β_0), a contradiction.

Hence $\beta_a > \beta_{d_1}$. Thus every tight stable set S containing b_2 contains either a or h_1 and every tight stable set S containing c contains either a or d_2. Indeed, in all other cases, $d_1 \in S$ and $S \setminus \{d_1\} \cup \{a\}$ violates (β, β_0), a contradiction. Moreover, every tight stable set S containing a also contains either b_2 or c and every tight stable set containing d_2 also contains either h_1 or c or u_{11} (otherwise $S \cup \{c\}$ would violate (β, β_0)). Furthermore every tight stable set S containing h_1 also contains either b_2 or d_2 (otherwise $S \setminus \{h_1\} \cup \{a, c\}$ would violate (β, β_0)) and every tight stable set S containing u_{11} also contains d_2. Indeed, if not, then $\beta_{u_{11}} \geq \beta_a + \beta_c$ and so $\beta_{h_1} \geq \beta_{u_{11}} \geq \beta_a + \beta_c \geq \beta_{h_1} + \beta_c$ yields a contradiction.

Hence we proved that every tight solution of (β, β_0) either contains h_2 or two nodes in $N(h_2)$, and so it is also tight for the 5-wheel inequality supported by $W_2 = (h_2 : a, d_2, b_2, c, h_1)$ lifted with the node u_{11}. It is not difficult to see that all nodes in $N(W_2) \setminus \{u_{11}\}$ have a zero lifting coefficient, and so (β, β_0) is a lifted 5-wheel inequality, contradicting the hypothesis.

Subcase 1b. $u_{11} \in S_{b_1}$. It follows that $\beta_{u_{11}} \geq \beta_c + \beta_{d_1}$ and, as $\beta_{h_1} \geq \beta_{u_{11}}$, $\beta_{h_1} > \beta_{d_1}$. Assume that $\{a, b_2\}$ is not contained in any tight stable set. Let S' be a tight stable set missing the clique $\{c, h_1, h_2, u_{11}\}$, then $d_1 \in S'$ and so $\beta_{d_1} \geq \beta_{h_1}$, a contradiction. Hence there exists a tight stable set S containing $\{a, b_2\}$. If $S \subseteq V_G \setminus (N(b_1) \cup \{b_1\})$, then subcase 1a) applies to S. So let us assume that $S \cap K_1 \neq \emptyset$; as a consequence, $\beta_a + \beta_{b_2} \geq \beta_{u_{11}} + \beta_{d_2}$ and $\beta_a \geq \beta_{h_1}$. If $d_2 \in S_{b_1}$, then $S_{b_1} \setminus \{u_{11}, d_2\} \cup \{a, b_2\}$ leads to the previous case. Thus consider the case $d_2 \notin S_{b_1}$ and consequently $S_{b_1} \cap K_2 \neq \emptyset$; then $\beta_{h_1} \geq \beta_{u_{11}} \geq \beta_a + \beta_c \geq \beta_{h_1} + \beta_c$, a contradiction. *(End of Case 1)*

Case 2. $A = \{c\}$.

By Observation 4, there exists a tight stable set S_c in $G \setminus (A \cup N(c))$. It is not difficult to see that S_c contains $\{d_1, d_2\}$ or $\{a\}$.

Suppose first that $\{d_1, d_2\} \subseteq S_c$. Hence $\beta_{d_i} \geq \beta_{h_i}$ and $\beta_{d_i} \geq \beta_{b_i}$, $i = 1, 2$. Consider a tight stable set S_i missing the clique $\{b_i, d_i\} \cup K_i$, $i = 1, 2$: then $h_i \in S_i$, $i = 1, 2$. Hence $\beta_{h_i} \geq \beta_{d_i}$ and so $\beta_{h_i} = \beta_{d_i}$ for $i = 1, 2$. Since $\beta_{d_1} + \beta_{d_2} \geq \beta_{b_1} + \beta_{u_{11}} + \beta_{d_2}$, it follows that $\beta_{d_1} > \beta_{b_1}$ and $\beta_{d_1} > \beta_{u_{11}}$.

As $\beta_{h_1} = \beta_{d_1} > \beta_{b_1}$ and $\beta_{h_1} = \beta_{d_1} > \beta_{u_{11}}$, every tight solution T containing u_{11} contains also b_1 (otherwise $T \setminus \{u_{11}\} \cup \{h_1\}$ would violate (β, β_0)) and every tight stable set S containing b_1 contains also a or u_{11} (otherwise $S \setminus \{b_1\} \cup \{d_1\}$ would violate (β, β_0)).

Consider now a tight stable set S missing the clique $\{a, d_2, h_2\}$. Then one among d_1, h_1, u_{11} belongs to S. It follows that $\beta_{h_1} = \beta_{d_1} \geq \beta_a$ since otherwise the stable set obtained from S by replacing d_1 or h_1 or u_{11} with a would violate (β, β_0), a contradiction.

Now, suppose that $\beta_{h_1} > \beta_a$. Then, every tight stable set containing a also contains b_1 (otherwise $S \setminus \{a\} \cup \{h_1\}$ would violate (β, β_0)). As every tight solution containing u_{11} also contains b_1 and every tight solution containing b_1 also contains either a or u_{11}, the tight solutions of (β, β_0) are not linearly independent. In fact, consider the matrix $M \in \{0,1\}^{|V_G| \times |V_G|}$ whose columns are the incidence vectors of the roots of (β, β_0): the row indexed by the node b_1 is obtained as the sum of the rows indexed by a and u_{11}, contradicting the nonsingularity of M.

Hence, $\beta_{h_1} = \beta_a$. Let S' be a tight stable set missing the clique $\{a, d_1, h_1, u_{11}\}$. Since $\beta_{d_1} > \beta_{b_1}$, we have that $b_1 \notin S'$ (otherwise $S' \setminus \{b_1\} \cup \{d_1\}$ would violate (β, β_0)) and $h_2 \in S'$. So $\beta_{h_2} \geq \beta_a$. Thus, every tight stable set T that contains b_1 and a also contains b_2; indeed, if not, $T \setminus \{b_1, a\} \cup \{d_1, h_2\}$ would violate (β, β_0).

Since $S_c \setminus \{d_1, d_2\} \cup \{a, b_1, b_2\}$ is a stable set, it follows that $\beta_{d_1} + \beta_{d_2} \geq \beta_a + \beta_{b_1} + \beta_{b_2}$. As $\beta_a = \beta_{h_1} = \beta_{d_1}$, we have that $\beta_{d_2} > \beta_{b_2}$. Thus, every tight stable set T' containing b_2 contains a (otherwise $T' \setminus \{b_2\} \cup \{d_2\}$ would violate (β, β_0)) and contains b_1 (otherwise $T' \setminus \{a, b_2\} \cup \{h_1, d_2\}$ would violate (β, β_0)).

Summing up, we proved that every tight solution containing u_{11} contains b_1, every tight solution containing b_2 contains b_1, and every tight solution containing b_1 contains either b_2 or u_{11}. This implies that the tight solutions of (β, β_0) are not linearly independent. Indeed, consider the matrix $M \in \{0,1\}^{|V_G| \times |V_G|}$ whose columns are the incidence vectors of the roots of (β, β_0): the row indexed by the node b_1 is obtained as the sum of the rows indexed by b_2 and u_{11}, contradicting the nonsingularity of M.

Suppose now that no tight stable set contains both d_1 and d_2. Hence, $a \in S_c$ and $\beta_a \geq \beta_{u_{11}}$. Let S be a tight stable set missing $\{a, h_1, h_2, u_{11}\}$. Since d_1 and d_2 cannot be both contained in a tight stable set, we have that S contains $\{d_1, b_2\}$ and moreover $\beta_{b_2} > \beta_{d_2}$. Let T be any tight stable set in $G \setminus A$ containing d_2. Then T contains u_{11}, otherwise $T \setminus \{d_2\} \cup \{b_2\}$ would violate (β, β_0); thus, $\beta_{u_{11}} > \beta_a$ (otherwise $T \setminus \{d_2, u_{11}\} \cup \{b_2, a\}$ would violate (β, β_0)), a contradiction. *(End of Case 2)*

Case 3. $A = \{b_2\}$.

First observe that every tight stable set S containing a also contains b_1 or c and every tight stable set S containing d_1 also contains h_2 or c (otherwise $S \cup \{c\}$ would violate (β, β_0)). Moreover, every tight stable set S containing b_1 contains also a or h_2 or u_{11} (otherwise $S \cup \{u_{11}\}$ would violate (β, β_0)). By Observation 4, there exists a tight stable set S_{b_2} in $G \setminus (A \cup N(b_2))$. It is not difficult to see that S_{b_2} contains $\{a, b_1\}$ and so $\beta_a \geq \beta_{h_2}$ and $\beta_a \geq \beta_{u_{11}} + \beta_{d_2}$. The former relation implies that every tight stable set S containing h_2 also contains b_1 or d_1 (otherwise $S \setminus \{h_2\} \cup \{a, c\}$ would violate (β, β_0)). The latter implies $\beta_a > \beta_{d_2}$ and $\beta_a > \beta_{u_{11}}$. Consider now a tight stable set T missing the clique $\{a, d_1, h_1, u_{11}\}$; T does not contain d_2, otherwise $T \setminus \{d_2\} \cup \{a\}$ would violate (β, β_0), and so it contains h_2. Hence, $\beta_{h_2} = \beta_a > \beta_{u_{11}}$. Thus every tight stable set S containing u_{11} also contains d_2 (otherwise $S \setminus \{u_{11}\} \cup \{h_2\}$ would violate (β, β_0)). We also claim that S must contain b_1. Suppose $b_1 \notin S$: we have that $\beta_{u_{11}} \geq \beta_{h_1}$, $\beta_{u_{11}} \geq \beta_c$ and, consequently, $\beta_{h_2} > \beta_c$. Let T be a tight stable set missing the clique $\{a, d_2, h_2\}$; clearly, $u_{11} \notin T$ and $h_1 \notin T$ (because $T \setminus \{u_{11}, h_1\} \cup \{h_2\}$ violates (β, β_0)). Thus $\{c, d_1\} \subseteq T$ and $T \setminus \{c\} \cup \{h_2\}$ violates (β, β_0), a contradiction. Thus every tight stable set S containing u_{11} also contains b_1 and d_2.

Finally, a tight stable set S containing c also contains either a or d_1, otherwise S contains d_2, and so $S \setminus \{d_2\} \cup \{a\}$ violates (β, β_0), a contradiction. Observe now that every tight stable set of (β, β_0) either contains h_1 or two nodes in $N(h_1)$. Summing up all the previous considerations, we have that such a stable set is also a tight solution of the 5-wheel inequality associated with $W_1 = (h_1 : a, h_2, c, b_1, d_1)$ lifted with the node u_{11}. It is not difficult to see that all nodes in $N(W_1) \setminus \{u_{11}\}$ have zero lifting coefficients, and so (β, β_0) is equivalent to the lifted 5-wheel inequality contradicting the hypothesis.

(End of Case 3) ∎

Lemma 3.5. *Let $G = (H, B_Y, e)$ be a geared graph with $Y = \{u_{11}\}$ and let (β, β_0) be a facet defining inequality for $STAB(G)$ with $\beta_{u_{11}} = \lambda > 0$. If (β, β_0) does not have full support on V_{B_Y} and it is neither a clique inequality nor a lifted 5-wheel inequality, then (β, β_0) has the following form:*

$$\beta^T_{G \setminus B_Y} x_{G \setminus B_Y} + \lambda x_{B_Y \setminus A} \leq \beta_0$$

where $A \in \{\{b_1, a, c, h_1, h_2, d_2\}, \{b_1, c\}, \{d_2, a\}\}$.

Proof. By Lemma 3.4, β_{B_Y} is supported by $B_Y \setminus A$ for A as in the thesis; hence, we only need to prove that, for every set A, the coefficients $\beta_{B_Y \setminus A}$ have the same value. If $A = \{b_1, a, c, h_1, h_2, d_2\}$ the supporting graph of (β, β_0) is isomorphic to H^e and, by Proposition 3.2, we are done.

The cases $A = \{b_1, c\}$ and $A = \{d_2, a\}$ are diagonally symmetric, so we prove in detail the former and symmetric arguments will prove the latter.

Let S be a tight stable set for (β, β_0) missing the clique $\{a, d_1, h_1, u_{11}\}$, then $h_2 \in S$ and thus $\beta_{h_2} \geq \beta_a$, $\beta_{h_2} \geq \beta_{h_1}$, and $\beta_{h_2} \geq \beta_{u_{11}}$. Moreover, by Observation 5, $\beta_{u_{11}} \geq \beta_{h_1}$ and $\beta_a \geq \beta_{h_1}$.

As $b_1 \in A$ there exists a stable set S_{b_1} in $G \setminus (A \cup N(b_1))$ which is tight for (β, β_0) by Observation 4. Clearly there are two cases: either $a \in S_{b_1}$ or $u_{11} \in S_{b_1}$ (otherwise $S_{b_1} \cup \{d_1\}$ would violate (β, β_0)).

Case 1. $a \in S_{b_1}$.

It follows that $\beta_a \geq \beta_{d_1}$. If $b_2 \notin S_{b_1}$, then $\beta_a \geq \beta_{h_2} + \beta_{d_1}$. Thus, $\beta_a > \beta_{h_2}$, a contradiction. Hence, $b_2 \in S_{b_1}$.

Since the node c has a zero lifting coefficient with respect to (β, β_0), there exists a stable set S_c in $G \setminus (A \cup N(c))$ which is tight for (β, β_0). Two possibilities may occur: either $\{d_1, d_2\} \subseteq S_c$ or $a \in S_c$.

Suppose first that $\{d_1, d_2\} \subseteq S_c$. Clearly, $\beta_{d_i} \geq \beta_{h_i}$ for $i = 1, 2$ and $\beta_{d_1} \geq \beta_{u_{11}}$. Moreover, since $S_{b_1} \setminus \{a, b_2\} \cup \{d_1, d_2\}$ and $S_c \setminus \{d_1, d_2\} \cup \{a, b_2\}$ are both feasible, we have that $\beta_a + \beta_{b_2} = \beta_{d_1} + \beta_{d_2}$. Now, if $\beta_a > \beta_{d_1}$ then $\beta_{d_2} > \beta_{b_2}$. Consider a stable set S which is tight for (β, β_0) and misses the clique $\{a, d_2, h_2\}$. Then, $b_2 \notin S$ (since otherwise $S \setminus \{b_2\} \cup \{d_2\}$ would violate (β, β_0)) and then either $h_1 \in S$ or $u_{11} \in S$ (otherwise $S \cup \{h_2\}$ would violate (β, β_0)). It follows that the stable set obtained from S by replacing h_1 or u_{11} with a violates (β, β_0) (as $\beta_{h_1} \leq \beta_{d_1} < \beta_a$ and $\beta_{u_{11}} \leq \beta_{d_1} < \beta_a$), a contradiction. Hence, $\beta_{d_1} = \beta_a$ and then $\beta_{b_2} = \beta_{d_2}$. Now, if $\beta_a > \beta_{h_1}$ then consider a stable set S which is tight for (β, β_0) and contains h_1. We have that S contains d_2 (otherwise $S \setminus \{h_1\} \cup \{a\}$ would violate (β, β_0)) and so $S \setminus \{h_1, d_2\} \cup \{a, b_2\}$ violates (β, β_0), a contradiction. Thus, $\beta_{h_1} = \beta_a = \beta_{d_1}$ and consequently $\beta_{h_1} = \beta_{u_{11}}$.

If $\beta_{d_2} > \beta_{h_2}$ then consider a stable set S which is tight for (β, β_0) and misses the clique $K_2 \cup \{d_2, b_2\}$. Clearly S has to contain h_2 (since otherwise $S \cup \{b_2\}$ or $S \cup \{d_2\}$ would violate (β, β_0)), but then $S \setminus \{h_2\} \cup \{d_2\}$ would violate (β, β_0). Thus, $\beta_{d_2} = \beta_{h_2}$.

If $\beta_{h_2} > \beta_{h_1}$ then consider a tight stable set S for (β, β_0) missing the clique $\{h_2, b_2, d_2\}$. Hence S contains either a, or h_1, or u_{11}. As $\beta_{h_1} = \beta_a = \beta_{u_{11}}$, the stable set obtained from S by replacing a, or h_1, or u_{11} with h_2 violates (β, β_0), a contradiction. Hence $\beta_{h_2} = \beta_{h_1}$. This implies that all nonzero entries of β_{B_Y} have the same value and we are done.

Suppose now that there does not exist S_c such that $\{d_1, d_2\} \subseteq S_c$. Then S_c contains a and $\beta_a = \beta_{h_2}$. Let S be a stable set which is tight for (β, β_0) and misses $\{a, h_1, h_2, u_{11}\}$; then $d_1 \in S$, $d_2 \notin S$ (otherwise S would be a tight stable set containing $\{d_1, d_2\}$), and $b_2 \in S$ (otherwise $S \cup \{h_2\}$ would violate (β, β_0)). Hence, $\beta_{b_2} > \beta_{d_2}$. Let S' be a stable set which is tight for (β, β_0) and contains d_2. It follows that $u_{11} \in S'$ (otherwise $S' \setminus \{d_2\} \cup \{b_2\}$ would violate (β, β_0)), and thus $\beta_{d_2} + \beta_{u_{11}} \geq \beta_a + \beta_{b_2}$. Since $\beta_{b_2} > \beta_{d_2}$ it follows that $\beta_{u_{11}} > \beta_a = \beta_{h_2} \geq \beta_{u_{11}}$, a contradiction.

Case 2. $u_{11} \in S_{b_1}$.

We may assume that every tight solution T containing a is such that $T \cap (N(b_1) \cup \{b_1\}) \neq \emptyset$, otherwise Case 1 applies.

If $d_2 \notin S_{b_1}$ then $\beta_{u_{11}} > \beta_a$ else $S_{b_1} \setminus \{u_{11}\} \cup \{a\}$ contradicts the above assumption. Let T' be a tight stable set containing h_1, then $b_2 \in T'$, otherwise $T' \setminus \{h_1\} \cup \{u_{11}\}$ violates (β, β_0) as $\beta_{u_{11}} > \beta_a \geq \beta_{h_1}$, and thus $\beta_{b_2} > \beta_{d_2}$ (otherwise $T' \setminus \{h_1, b_2\} \cup \{u_{11}, d_2\}$ would violate (β, β_0)). There also must exist a tight stable set S_c in $G \setminus (N(c) \cup A)$. Two cases may then occur: either $S_c \supseteq \{d_1, d_2\}$ or $a \in S_c$. In both cases there exists a stable set ($S_c \setminus \{d_2\} \cup \{b_2\}$ or $S_c \setminus \{a\} \cup \{u_{11}\}$, respectively) that violates (β, β_0), a contradiction.

Hence $d_2 \in S_{b_1}$ and we can assume that $\beta_{u_{11}} + \beta_{d_2} > \beta_a + \beta_{b_2}$, since otherwise Case 1 applies. The last inequality implies that any tight stable set T containing a does not contain b_2. Then $\beta_a \geq \beta_{u_{11}}$ and thus $\beta_{d_2} > \beta_{b_2}$. Now let T' be a tight stable set such that $b_2 \in T'$. Since a cannot belong to T', the stable set $T' \setminus \{b_2\} \cup \{d_2\}$ violates (β, β_0), a contradiction. ∎

We recall here a result in [9] that will be used later in the proofs. Notice that $G \setminus B_Y = H \setminus \{v_1, v_2\}$.

Lemma 3.6. *[9] Let $G = (H, B_Y, v_1v_2)$ be a geared graph with $Y = \{u_{11}\}$ and let (π_H, π_0) be a g-extendable facet defining inequality of $STAB(H)$ with $\pi_{v_1} = \pi_{v_2} = \lambda > 0$. Then the lifted geared inequalities*

$$\sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_{B_Y} \setminus A} x_i \leq \pi_0 + \lambda$$

where $A \in \{\{b_1, c\}, \{d_2, a\}\}$ are facet defining for $STAB(G)$.

Theorem 3.7. *Let $G = (H, B_Y, v_1v_2)$ be a geared graph with $Y = \{u_{11}\}$ and let (β, β_0) be a valid inequality for $STAB(G)$ such that: it is not a clique or a lifted 5-wheel inequality, it does not have full support on B_Y and it has $\beta_{u_{11}} = \lambda > 0$. Then (β, β_0) is facet defining for $STAB(G)$ if and only if it has the form $\beta^T_{G \setminus B_Y} x_{G \setminus B_Y} + \lambda x_{B_Y \setminus A} \leq \beta_0$ and one of the following cases occurs:*

i) $A \in \{\{b_1, c\}, \{d_2, a\}\}$ and $\beta^T_{H \setminus \{v_1, v_2\}} x_{H \setminus \{v_1, v_2\}} + \lambda(x_{v_1} + x_{v_2}) \leq \beta_0 - \lambda$ is a g-extendable facet defining inequality of $STAB(H)$.

ii) $A = \{b_1, a, c, d_2, h_1, h_2\}$ and $\beta^T_{H \setminus \{v_1, v_2\}} x_{H \setminus \{v_1, v_2\}} + \lambda(x_{v_1} + x_{v_2} + x_t) \leq \beta_0$ is g-liftable facet defining inequality for $STAB(H^e)$.

Proof. (If). If (ii) holds, then $G \setminus A$ is isomorphic to H^e. It is not difficult to check that each node in A has a zero lifting coefficient and the claim follows. Consider now the case $A \in \{\{b_1, c\}, \{d_2, a\}\}$ and let $(\beta_H, \beta_0 - \lambda)$ be a g-extendable facet defining inequality for $STAB(H)$. Then (β, β_0) is facet defining for $STAB(G)$ by Lemma 3.6.

(Only if). Suppose now that (β, β_0) is facet defining for $STAB(G)$. By Lemma 3.5, (β, β_0) has the form $\beta^T_{G \setminus B_Y} x_{G \setminus B_Y} + \lambda x_{B_Y \setminus A} \leq \beta_0$ with $A \in \{\{b_1, a, c, d_2, h_1, h_2\}, \{b_1, c\}, \{d_2, a\}\}$. If $A = \{b_1, a, c, d_2, h_1, h_2\}$, then again the result can be easily proven since $G \setminus A$ is a graph isomorphic to H^e.

So, let us consider the case with $A \in \{b_1, c\}$ (the case $\{d_2, a\}$ will be proved using the diagonal symmetry) and suppose that $(\beta_H, \beta_0 - \lambda)$ as in point (i) is not facet defining for $STAB(H)$. Then there exists an inequality (π, π_0) that is facet defining for $STAB(H)$ and such that all the roots of $(\beta_H, \beta_0 - \lambda)$ are roots of (π, π_0). By Proposition 3.2, $\pi_{v_1} = \pi_{v_2}$. If $\pi_{v_1} = 0$ then (π, π_0) can be lifted to a facet defining inequality for $STAB(G)$ that has $\pi_v = 0$ for each $v \in V_{B_Y}$ and contains all the roots of (β, β_0); as $STAB(G)$ has full support and (β, β_0) is facet defining, we have a contradiction with $\beta_{B_Y} \neq \mathbb{0}$, i.e. $\lambda > 0$. Hence $\pi_{v_1} > 0$ and we assume without loss of generality that $\pi_{v_1} = \lambda$ and consider the following non proper geared inequality lifted with the node u_{11}:

$$\sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_{B_Y} \setminus \{b_1, c\}} x_i \leq \pi_0 + \lambda. \tag{7}$$

Since (π, π_0) is g-extendable and facet defining for $STAB(H)$, it follows, by Lemma 3.6 that (7) is facet defining for $STAB(G)$. Consider now a tight stable set S of (β, β_0). Notice that $\beta(S \cap B_Y)$ equals either λ or 2λ; hence, every tight solution x^S of (β, β_0) can be reduced to a tight solution x^{S_H} of $(\beta_H, \beta_0 - \lambda)$ by removing from S an appropriate node u with coefficient λ contained in B_Y and, for $i = 1, 2$, $v_i \in S_H$ if and only if $(S \setminus \{u\}) \cap \{b_i, d_i\} \neq \emptyset$. By assumption x^{S_H} is also a tight solution for (π, π_0). It is not difficult to see that x^S is also a root of (7) once we reintroduce the previously removed node. Therefore, (β, β_0) and (7) are equivalent. As $STAB(G)$ is full dimensional, the two inequalities only differ by a positive scalar factor. Hence, (π, π_0) is equivalent to $(\beta_H, \beta_0 - \lambda)$ and both are g-extendable facet defining of $STAB(H)$. ∎

By using the vertical symmetry we can also prove the following:

Theorem 3.8. *Let $G = (H, B_Y, v_1v_2)$ be a geared graph with $Y = \{u_{12}\}$ and let (β, β_0) be a valid inequality for $STAB(G)$ such that: it is not a clique or a lifted 5-wheel inequality, it does not have full support on $STAB(G)$, and $\beta_{u_{12}} = \lambda > 0$. Then (β, β_0) is facet defining for $STAB(G)$ if and only if it has the form $\beta^T_{G\setminus B_Y} x_{G\setminus B_Y} + \lambda x_{B_Y \setminus A} \leq \beta_0$ and one of the following cases occurs:*

i) $A \in \{\{b_2, c\}, \{d_1, a\}\}$ and $\beta^T_{H\setminus\{v_1,v_2\}} x_{H\setminus\{v_1,v_2\}} + \lambda(x_{v_1} + x_{v_2}) \leq \beta_0 - \lambda$ is a g-extendable facet defining inequality of $STAB(H)$.

ii) $A = \{d_1, a, c, b_2, h_1, h_2\}$ and $\beta^T_{H\setminus\{v_1,v_2\}} x_{H\setminus\{v_1,v_2\}} + \lambda(x_{v_1} + x_{v_2} + x_t) \leq \beta_0$ is a g-liftable facet defining inequality of $STAB(H^e)$.

The previous two theorems complete the proof of steps a.1) and a.2) of Theorem 3.3. It remains to consider the case when $Y = \{u_{11}, u_{12}\}$ and (β, β_0) does not have full support on the extended gear B_Y. .8

Theorem 3.9. *Let $G = (H, B_Y, e)$ be a geared graph with $Y = \{u_{11}, u_{12}\}$ and let (β, β_0) be a facet defining inequality for $STAB(G)$ that does not have full support on V_{B_Y}. If $\beta_{u_{11}}$ and $\beta_{u_{12}}$ are both positive then (β, β_0) is a lifted 5-wheel inequality.*

The proof follows the same pattern of the proof given for $Y = \{u_{11}\}$. All details can be found in Appendix A.2.

3.2. Inequalities having full support on B_Y

Throughout this subsection we consider an inequality (β, β_0) that is facet defining inequality for $STAB(G)$ and has full support on V_{B_Y}, i.e., $\beta_v > 0$ for each $v \in V_{B_Y}$. In particular, (β, β_0) is neither a clique inequality nor a (lifted) 5-wheel inequality (see Observation 2); this implies that for each clique and for each 5-wheel of B_Y there exists a tight solution of (β, β_0) that misses the prescribed clique or 5-wheel by Observation 6. Let $\mathscr{S}(G)$ denote the set of stable sets of G. Since (β, β_0) has full support on V_{B_Y}, it follows that $S \cap V_{B_Y}$ is maximal in B_Y for any stable set $S \in \mathscr{S}(G)$ that is tight for (β, β_0).

Let $\mathcal{R}$ denote the set of the roots of (β, β_0) and let $M_{(\beta,\beta_0)}$ be the matrix whose rows are indexed by the nodes of V_G and whose columns are the vectors in $\mathcal{R}$. Since (β, β_0) is facet defining, the matrix $M_{(\beta,\beta_0)}$ has full rank. Consider now the matrix $M'_{(\beta,\beta_0)}$ obtained by summing up all the rows indexed by the nodes $u \in K_i$ into a single row indexed by k_i, $i = 1, 2$. This matrix may be interpreted in terms of graphs as follows: let B^*_Y be the graph obtained from B_Y by adding two new nodes to V_{B_Y}, say k_1 and k_2, such that $N(k_i) = \{b_i, d_i\}$, $i = 1, 2$. Then $\tilde{S} \in \mathscr{S}(B^*_Y)$ if and only if there exists a stable set $S \in \mathscr{S}(G)$ such that: $\tilde{S} \setminus \{k_1, k_2\} = S \cap V_{B_Y}$ and $K_i \cap S \neq \emptyset$ if and only if $k_i \in \tilde{S}$; we say that S is *associated* with the *configuration* $\tilde{S}$.

It is not difficult to verify that if $rank(M'_{(\beta,\beta_0)}) < |V_G| - \sum_{i=1,2} |K_i| + 2$ then $rank(M_{(\beta,\beta_0)}) < |V_G|$. In other words, $M_{(\beta,\beta_0)}$ cannot have full rank if $M'_{(\beta,\beta_0)}$ has not full rank.

Now, let $M''_{(\beta,\beta_0)}$ be the submatrix of $M'_{(\beta,\beta_0)}$ whose rows are indexed by the nodes of B^*_Y. The columns of $M''_{(\beta,\beta_0)}$ may have repetitions, since all the stable sets of G that differ only on the nodes not in $V_{B^*_Y}$ produce the same $(0,1)$-column.

We say that a configuration $\tilde{S} \in \mathscr{S}(B^*_Y)$ is tight for (β, β_0) if and only if there exists a stable set S associated with $\tilde{S}$ such that $x^S \in \mathcal{R}$. So, denote by $\mathcal{R}'$ the set of the incidence vectors of the tight configurations of (β, β_0) and by $\tilde{M}_{(\beta,\beta_0)}$ the matrix of dimension $|V_{B^*_Y}| \times |\mathcal{R}'|$ obtained by canceling multiple columns from $M''_{(\beta,\beta_0)}$. Clearly, we have that if $M_{(\beta,\beta_0)}$ has full rank then $\tilde{M}_{(\beta,\beta_0)}$ has full rank. In particular, we can state the following:

Proposition 3.10. *Let $G = (H, B_Y, e)$ be a geared graph. If (β, β_0) is facet defining for $STAB(G)$, then the matrix $\tilde{M}_{(\beta,\beta_0)}$ has rank $|V_{B^*_Y}| + 2 = 10 + |Y|$.*

It is not difficult to verify that there are 32 possible configurations (i.e., maximal stable sets) in $\mathscr{S}(B^*_\Omega)$, $\Omega = \{u_{11}, u_{12}\}$: the first 22 configurations, not containing nodes in Ω, are depicted in Fig. 4 and denoted by S_i, $i = 1, \ldots, 19$, and by T_j, $j = 1, 2, 3$; the remaining 10 configurations, containing at least one node in Ω, are depicted in Fig. 5 and denoted by S_i, $i = 20, \ldots, 28$, and by T_4.

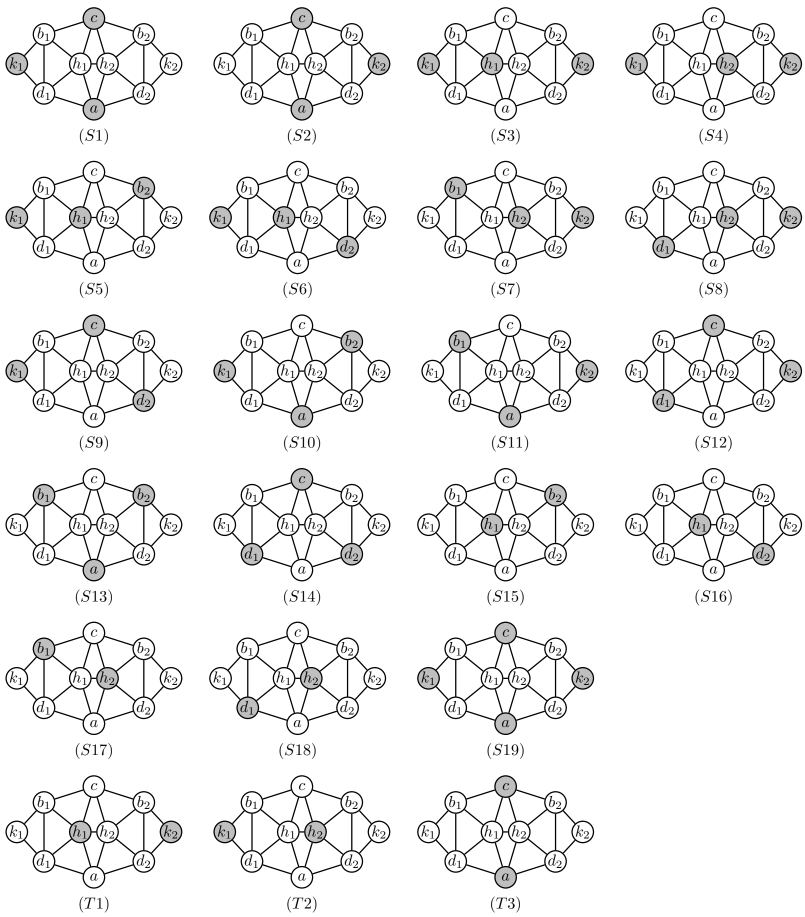


Figure 4: Maximal solutions in B^*_Ω containing neither u_{11} nor u_{12}

Note that the configurations in $\mathscr{S}(B^*_{11})$ are those in Fig. 4, plus S_{20}, S_{21}, S_{22}, $S_{23} \setminus \{u_{12}\}$, and $S_{28} \setminus \{u_{12}\}$.

In the remaining of this section we say that a configuration S_i (or T_j) is tight for (β, β_0) if there exists a solution $S \in \mathscr{S}(G)$ associated with S_i (or T_j) that is tight for (β, β_0), and we say that S_i (or T_j) is not tight if no such solution exists. Moreover, we say that S_i (or T_j) violates (β, β_0) if there exists a solution $S \in \mathscr{S}(G)$ associated with S_i that violates (β, β_0).

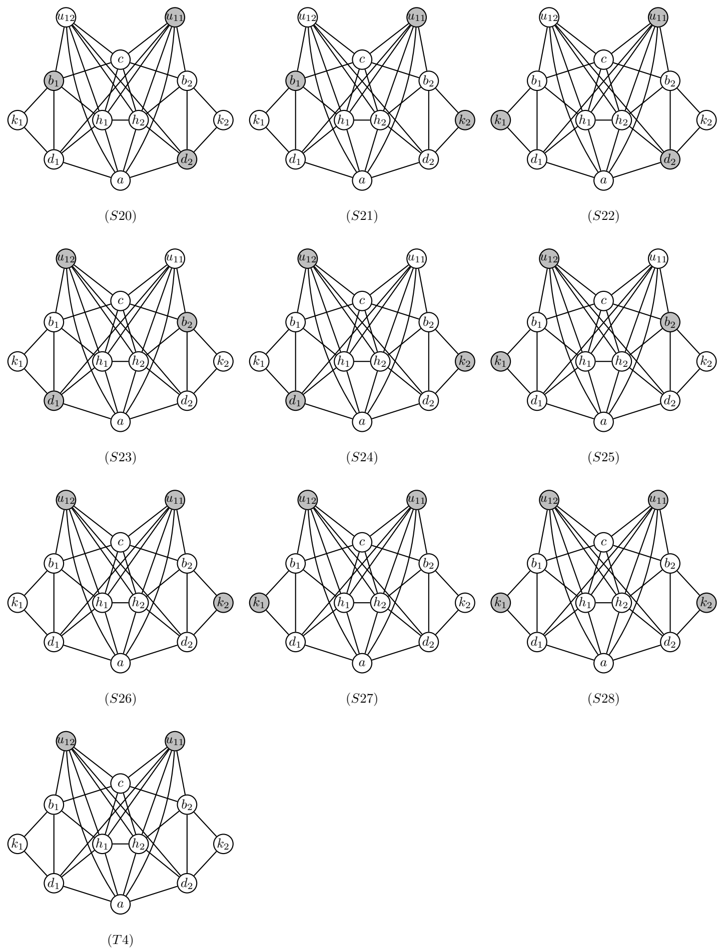


Figure 5: Maximal solutions in B^*_Ω containing u_{11} and/or u_{12}

In this subsection we prove that all the facet defining inequalities with full support on B_Y are sequential lifting of the facet defining inequalities for $STAB(G_\emptyset)$ that have full support on $B_\emptyset$, i.e. either proper geared inequalities (1) or proper g-lifted inequalities (3).

We start showing that some configurations cannot be tight for a facet defining inequality (β, β_0) that has full support on B_Y with $Y \neq \emptyset$.

Lemma 3.11. *Let $G = (H, B_Y, e)$ be a geared graph with $Y \neq \emptyset$ and let (β, β_0) be a facet defining inequality for $STAB(G)$ that has full support on B_Y. If $Y = \{u_{11}\}$ then $S_{28} \setminus \{u_{12}\}$ is not tight for (β, β_0). If $Y = \{u_{12}\}$ then $S_{28} \setminus \{u_{11}\}$ is not tight for (β, β_0).*

Proof. We prove the case $Y = \{u_{11}\}$; the other case follows by vertical symmetry. Suppose conversely that the configuration $S_{28} \setminus \{u_{12}\}$ is tight for (β, β_0). Therefore $\beta_{u_{11}} \geq \beta_{h_i}$, $i = 1, 2$, $\beta_{u_{11}} \geq \beta_a + \beta_c$, and so $\beta_{u_{11}} > \beta_a$, $\beta_{u_{11}} > \beta_c$. Hence, S_1, S_2, T_1, T_2, and T_3 are not tight.

Let S be a tight stable set missing the clique $\{c, b_2, h_2, u_{11}\}$. If $h_1 \notin S$ then b_1 and a belong to S and so $S \setminus \{a\} \cup \{u_{11}\}$ violates (β, β_0), a contradiction. Hence $h_1 \in S$ and $\beta_{h_1} \geq \beta_{u_{11}}$. By diagonal symmetry it can be proved that h_2 belongs to a tight stable set missing the clique $\{a, d_1, h_1, u_{11}\}$ and so $\beta_{h_2} \geq \beta_{u_{11}}$. Thus $\beta_{u_{11}} = \beta_{h_i}$, $i = 1, 2$ and $\beta_{h_i} > \beta_a$, $\beta_{h_i} > \beta_c$, $i = 1, 2$; hence S_9, S_{10}, S_{11}, S_{12} cannot be tight. Now, consider a tight stable set S' missing the clique $\{a, h_1, h_2, u_{11}\}$. It is not difficult to see that S' contains either $\{d_1, c, d_2\}$ or $\{d_1, b_2\}$. In the former case, we have that $\beta_{d_1} > \beta_{b_1}$ (since otherwise $S' \setminus \{d_1, c\} \cup \{b_1, u_{11}\}$ would violate (β, β_0)) and the same holds in the latter case (since otherwise $S' \setminus \{d_1\} \cup \{b_1, a\}$ would violate (β, β_0)). By diagonal symmetry, we can prove that $\beta_{b_2} > \beta_{d_2}$. It follows that S_6, S_7, S_{16}, and S_{17} are not tight. Moreover, neither S_{21} nor S_{22} are tight since otherwise $S_{21} \setminus \{b_1, u_{11}\} \cup \{d_1, h_2\}$ or $S_{22} \setminus \{d_2, u_{11}\} \cup \{b_2, h_1\}$ would violate (β, β_0).

Summing up, the tight solutions of (β, β_0) can be chosen only among the following configurations: $S_3, S_4, S_5, S_8, S_{13}, S_{14}, S_{15}, S_{18}, S_{19}, S_{20}, S_{23} \setminus \{u_{12}\}$, and $S_{28} \setminus \{u_{12}\}$. Now let $\tilde{M}$ be the matrix whose columns are the incidence vectors of the above configurations and whose rows are indexed on the nodes of $V^*_{B_Y}$; then the matrix $\tilde{M}_{(\beta,\beta_0)}$ whose columns are the incidence vectors of the configurations corresponding to the stable sets that are tight for (β, β_0) is a submatrix of $\tilde{M}$. Since (β, β_0) is facet defining for $STAB(G_{11})$, then $rank(\tilde{M}_{(\beta,\beta_0)}) \leq rank(\tilde{M})$ should be equal to $|V_{B^*_Y}| + 2 = 11$ (see Proposition 3.10). However, it is straighforward to compute that $rank(\tilde{M}) = 10$, a contradiction. ∎

Lemma 3.12. *Let $G = (H, B_Y, e)$ be a geared graph with $Y \neq \emptyset$ and let (β, β_0) be a facet defining inequality for $STAB(G)$. If (β, β_0) has full support on B_Y then the configurations T_1, T_2, and T_3 are not tight for (β, β_0). Moreover, if $Y = \{u_{11}, u_{12}\}$ then also T_4 is not tight.*

Proof. We begin with T_3 and suppose on the contrary that T_3 is tight for (β, β_0). Then

a) $\beta_a \geq \beta_{d_1} + \beta_{d_2}$ and $\beta_c \geq \beta_{b_1} + \beta_{b_2}$;

b) $\beta_a + \beta_c \geq \beta_{h_1} + \beta_{d_2}$ and $\beta_a + \beta_c \geq \beta_{h_2} + \beta_{d_1}$;

c) $\beta_a + \beta_c \geq \beta_{b_1} + \beta_{u_{11}} + \beta_{d_2}$ and $\beta_a + \beta_c \geq \beta_{d_1} + \beta_{u_{12}} + \beta_{b_2}$.

From a) it follows that $\beta_a > \beta_{d_i}$, $\beta_c > \beta_{b_i}$, for $i = 1, 2$, thus S_9, S_{10}, S_{11}, and S_{12} are not tight. Moreover, if $Y = \{u_{11}\}$ then $S_{23} \setminus \{u_{12}\}$ is not tight since otherwise $S_{23} \setminus \{u_{12}, d_1\} \cup \{a\}$ violates (β, β_0). Analogously, if $Y = \{u_{12}\}$ then $S_{20} \setminus \{u_{11}\}$ is not tight (otherwise $S_{20} \setminus \{u_{11}, d_2\} \cup \{a\}$ violates (β, β_0)).

By condition b) it follows that $\beta_a + \beta_c > \beta_{h_i}$ for $i = 1, 2$, thus T_1, T_2, S_3, and S_4 are not tight. By condition c) it follows that $\beta_a + \beta_c > \beta_{b_1} + \beta_{u_{11}}$, $\beta_a + \beta_c > \beta_{d_2} + \beta_{u_{11}}$, $\beta_a + \beta_c > \beta_{d_1} + \beta_{u_{12}}$, and $\beta_a + \beta_c > \beta_{b_2} + \beta_{u_{12}}$, thus S_{21}, S_{22}, S_{24}, and S_{25} are not tight.

By Lemma 3.11, $S_{28} \setminus \{u_{12}\}$ is not tight if $Y = \{u_{11}\}$ and $S_{28} \setminus \{u_{11}\}$ is not tight if $Y = \{u_{12}\}$. Hence, in all cases the solutions that might be tight for (β, β_0) are also tight for the two lifted 5-wheel inequalities (6). Therefore the face defined by (β, β_0) is contained in the intersection of two facets, contradicting the hypothesis that (β, β_0) is facet defining.

Suppose now that T_1 is tight for (β, β_0), then

a) $\beta_{h_1} \geq \beta_{h_2} + \beta_{b_1}$ and $\beta_{h_1} \geq \beta_{h_2} + \beta_{d_1}$;

b) $\beta_{h_1} \geq \beta_a + \beta_c$;

c) $\beta_{h_1} \geq \beta_{u_{11}} + \beta_{b_1}$ and $\beta_{h_1} \geq \beta_{u_{12}} + \beta_{d_1}$.

By condition a) it follows that $\beta_{h_1} > \beta_{h_2}$, thus T_2 and S_4 are not tight. Moreover $\beta_{h_1} > \beta_{d_1}$, thus $S_{23} \setminus \{u_{12}\}$ is not tight when $Y = \{u_{11}\}$ ($\beta_{h_1} > \beta_{b_1}$, thus $S_{20} \setminus \{u_{11}\}$ is not tight when $Y = \{u_{12}\}$). By condition b) it follows that $\beta_{h_1} > \beta_a$ and $\beta_{h_1} > \beta_c$, thus S_9 and S_{10} are not tight. By condition c) it follows that $\beta_{h_1} > \beta_{u_{11}}$ and $\beta_{h_1} > \beta_{u_{12}}$, thus S_{22} and S_{25} are not tight.

By Lemma 3.11, $S_{28}\setminus\{u_{12}\}$ is not tight if $Y=\{u_{11}\}$ and $S_{28}\setminus\{u_{11}\}$ is not tight if $Y=\{u_{12}\}$. Hence, in all cases the solutions that might be tight for (β,β_0) are also tight for the lifted 5-wheel inequality with hub h_1. Therefore (β,β_0) is equivalent to or contained in it, contradicting the hypothesis. The proof that T_2 is never tight for (β,β_0) is derived by diagonal symmetry.

Finally, if $Y=\{u_{11},u_{12}\}$, suppose by contradiction that T_4 is tight for (β,β_0). Since $\beta_{u_{11}}+\beta_{u_{12}}\geq\beta_{h_1}+\beta_{b_2}$ and $\beta_{u_{11}}+\beta_{u_{12}}\geq\beta_{h_2}+\beta_{d_1}$, it follows that $\beta_{u_{11}}+\beta_{u_{12}}>\beta_{h_i}$, $i=1,2$. As a consequence the configurations S_3 and S_4 are not tight for (β,β_0). Moreover

i) $\beta_{u_{11}}+\beta_{u_{12}}\geq\beta_{u_{11}}+\beta_{b_1}+\beta_{d_2}$ implies that $\beta_{u_{12}}>\beta_{b_1}$ and $\beta_{u_{12}}>\beta_{d_2}$;

ii) $\beta_{u_{11}}+\beta_{u_{12}}\geq\beta_{u_{12}}+\beta_{d_1}+\beta_{b_2}$ implies that $\beta_{u_{11}}>\beta_{d_1}$ and $\beta_{u_{11}}>\beta_{b_2}$;

iii) $\beta_{u_{11}}+\beta_{u_{12}}\geq\beta_{d_1}+\beta_c+\beta_{d_2}$ implies that $\beta_{u_{11}}+\beta_{u_{12}}>\beta_{d_1}+\beta_c$ and $\beta_{u_{11}}+\beta_{u_{12}}>\beta_c+\beta_{d_2}$;

iv) $\beta_{u_{11}}+\beta_{u_{12}}\geq\beta_{b_1}+\beta_a+\beta_{b_2}$ implies that $\beta_{u_{11}}+\beta_{u_{12}}>\beta_{b_1}+\beta_a$ and $\beta_{u_{11}}+\beta_{u_{12}}>\beta_a+\beta_{b_2}$.

From (i) and (ii) it follows that solutions of type S_{21}, S_{22}, S_{24} and S_{25} are not tight for (β,β_0). From (iii) and (iv) it follows that solutions of type S_9, S_{10}, S_{11}, and S_{12} are not tight for (β,β_0). Hence every tight solution for (β,β_0) is also tight for both the lifted 5-wheel inequalities (6). But this contradicts the hypothesis that (β,β_0) is facet defining. ∎

The next two results show that some configurations are incompatible as tight configurations of the same inequality.

Lemma 3.13. *Let $G=(H,B_Y,e)$ be a geared graph with $Y\neq\emptyset$ and let (β,β_0) be a facet defining inequality for $STAB(G)$ that has full support on B_Y. If S_1 (S_2) is tight then S_4 (S_3, respectively) is not tight.*

Proof. Suppose that S_4 is tight, then $\beta_{h_2}\geq\beta_a+\beta_c$. Then if S_1 is tight, also T_2 must be tight contradicting Lemma 3.12. A symmetric argument shows that S_3 cannot be tight if S_2 is tight. ∎

Lemma 3.14. *Let $G=(H,B_Y,e)$ be a geared graph with $Y\neq\emptyset$ and let (β,β_0) be a facet defining inequality for $STAB(G)$ that has full support on B_Y. If at least one configuration in $\{S_{21},S_{22},S_{24},S_{25}\}$ is tight, then the solutions S_{15}, S_{16}, S_{17}, and S_{18} are not tight.*

Proof. We first prove the claim when S_{21} is tight and $u_{11}\in Y$. Then $\beta_{u_{11}}\geq\beta_{h_2}$ and thus S_{17} is not tight (since otherwise $S_{17}\setminus\{h_2\}\cup\{u_{11},d_2\}$ would violate (β,β_0)). Moreover, since S_{21} is tight and T_1 is not tight (see Lemma 3.12) then $\beta_{b_1}+\beta_{u_{11}}>\beta_{h_1}$ and, as a consequence, S_{16} is not tight.

Now, suppose on the contrary that S_{18} is tight; then $\beta_{h_2}\geq\beta_c+\beta_{d_2}$, $\beta_{d_1}>\beta_{b_1}$ (as S_{17} is not tight), and $\beta_{u_{11}}>\beta_{h_2}$ (otherwise $S_{21}\setminus\{u_{11},b_1\}\cup\{h_2,d_1\}$ would violate (β,β_0)). With these relations S_4, S_7, and S_9 ($\beta_c+\beta_{d_2}\leq\beta_{h_2}<\beta_{u_{11}}$) are not tight. If $Y=\{u_{11}\}$, then, as T_2 is not tight by Lemma 3.12, the clique $\{a,d_1,h_1,u_{11}\}$ is not missed by any tight solution of (β,β_0), a contradiction. Thus S_{18} is not tight. If $Y=\Omega$, S_{25} is tight because it is the unique configuration missing the clique $\{a,d_1,h_1,u_{11}\}$. Then $\beta_{u_{12}}+\beta_{b_2}>\beta_{h_2}$ (T_2 is not tight) and thus S_{18} is not tight.

Finally, suppose on the contrary that S_{15} is tight; then $\beta_{b_2}>\beta_{d_2}$ (since S_{16} is not tight) and so S_6 is not tight. Moreover, $\beta_{h_1}\geq\beta_{b_1}+\beta_a$ implies $\beta_{h_1}>\beta_a$ and $\beta_{h_1}>\beta_{b_1}$; so S_{10} is not tight. As S_{21} is tight and T_1 is not, it follows that $\beta_{b_1}+\beta_{u_{11}}>\beta_{h_1}\geq\beta_{b_1}+\beta_a$, i.e. $\beta_{u_{11}}>\beta_a$ and so S_{11} is not tight.

Consider first the case $Y=\{u_{11}\}$. The clique $\{c,b_2,h_2,u_{11}\}$ is missed only by the configuration S_3, which must be tight for (β,β_0). Then $\beta_{h_1}\geq\beta_a+\beta_c$ implies that $\beta_{h_1}>\beta_c$ and so S_9 is not tight. Since $S_{28}\setminus\{u_{12}\}$ is not tight by Lemma 3.11 and S_3 is tight, $\beta_{h_1}>\beta_{u_{11}}$. As a consequence, S_{22} is not tight and, as $\beta_{u_{11}}\geq\beta_{h_2}$, then $\beta_{h_1}>\beta_{h_2}$. Therefore S_4 is not tight for (β,β_0) and thus $S_{23}\setminus\{u_{12}\}$ is the only configuration missing the lifted 5-wheel inequality supported by $W_1\cup\{u_{11}\}$ and so it is tight. It follows that $\beta_{d_1} \geq \beta_{h_1}$ and, as $\beta_{h_1} > \beta_{b_1}$, S_7 is not tight. But then no tight solution for (β, β_0) misses the clique $\{a, d_1, h_1, u_{11}\}$, a contradiction. Hence S_{15} is not tight.

Consider now the case $Y = \Omega$. As S_{15} is tight, $\beta_{h_1} \geq \beta_{d_1} + \beta_{u_{12}}$; but then, since T_1 is not tight, S_{24} is not tight as well. Thus the clique $\{c, b_2, h_2, u_{11}\}$ is missed only by the configuration S_3, which must be tight for (β, β_0). It follows that $\beta_{h_1} \geq \beta_a + \beta_c$ and so S_9 is not tight; moreover, $\beta_{h_1} \geq \beta_{u_{11}} + \beta_{u_{12}}$ implies that $\beta_{h_1} > \beta_{u_{11}} \geq \beta_{h_2}$ and that $\beta_{h_1} > \beta_{u_{12}}$. Then S_{22}, S_4, and S_{25} are not tight. As a consequence, no tight configuration misses the lifted 5-wheel inequality supported by $W_1 \cup \{u_{11}, u_{12}\}$, a contradiction. Hence S_{15} is not tight and the proof of case S_{21} tight follows.

The case when S_{22} (or S_{24}, or S_{25}) is tight follows by diagonally (or horizonthally, or vertically) symmetric arguments. ∎

The following lemma states that at least two tight configurations for each node in Y are needed to have inequalities which are not sequential lifting of a lower dimensional polytope.

Lemma 3.15. *Let $G = (H, B_Y, e)$ be a geared graph with $Y \neq \emptyset$ and let (β, β_0) be a facet defining inequality for $STAB(G)$ that has full support on B_Y. If $u_{11} \in Y$ (or $u_{12} \in Y$) and (β, β_0) is not a sequential lifting of a facet defining inequality for $STAB(G \setminus \{u_{11}\})$ ($STAB(G \setminus \{u_{12}\})$), then there exist at least two tight stable sets containing u_{11} (u_{12}) associated with distinct tight configurations in B^*_Y.*

Proof. We prove the lemma only for $u_{11} \in Y$, as the proof for u_{12} follows by vertical symmetry. Let M be the matrix whose columns are $|V_G|$ linearly independent roots of (β, β_0).

Claim 1. M contains at least two tight solutions containing the node node u_{11}.

If the claim is false, the row of M corresponding to u_{11} contains only one nonzero entry, thus proving that the matrix $\overline{M}$ obtained by deleting this row and the column associated with the unique solution containing u_{11} is defined by $|V_G| - 1$ linearly independent vectors satisfying $(\beta_{G \setminus \{u_{11}\}}, \beta_0)$ as an equality. This implies that $(\beta_{G \setminus \{u_{11}\}}, \beta_0)$ is facet defining for $STAB(G \setminus \{u_{11}\})$ and that (β, β_0) is its sequential lifting, a contradiction. *(End of Claim 1)*

Suppose by contradiction that all the tight stable sets of (β, β_0) that contain u_{11} are associated with a unique tight configuration in B^*_Y, say S^*. Clearly, $S^* \in \{S_{20}, S_{21}, S_{22}, S_{26}, S_{27}, S_{28}$.

Claim 2. For any choice of S^*, there is no tight stable set T such that $u_{11} \notin T$ and $T \cap K_i \neq \emptyset$ if and only if $k_i \in S^*$ for $i = 1, 2$.

Assume on the contrary that such a T exists. By Claim 1, in M there are two different tight solutions $x^{S'}$ and $x^{S''}$ such that S' and S'' contain u_{11}. By hypothesis, S' and S'' are associated with the same configuration S^*; then $\beta(T \cap V_{B_Y}) = \beta(S' \cap V_{B_Y}) = \beta(S'' \cap V_{B_Y}) = \beta(S^* \cap V_{B_Y})$, otherwise either $(T \cap V_{B_Y}) \cup (S' \setminus V_{B_Y})$ or $(S' \cap V_{B_Y}) \cup (T \setminus V_{B_Y})$ would violate (β, β_0) and similarly for S''. Thus, the stable sets $T' = (T \cap V_{B_Y}) \cup (S' \setminus V_{B_Y})$ and $T'' = (T \cap V_{B_Y}) \cup (S'' \setminus V_{B_Y})$ are also tight for (β, β_0). As $x^{S'} - x^{S''} = x^{T'} - x^{T''}$, these four vectors are linearly dependent and they are not all present in M; so if we delete from M the column $x^{S'}$ and we add the columns $x^{T'}$, $x^{T''}$, we get a matrix with the same rank as M and with fewer solutions containing u_{11}. Repeating the above arguments for all the columns of M associated with stable sets containing u_{11}, we finally get a matrix $\overline{M}$ with only one column corresponding to a solution containing u_{11} and with the same rank as M, contradicting Claim 1. *(End of Claim 2)*

Thus, one of the following cases may occur:

1. $S^* = S_{20}$ and there are no tight solutions associated with $S_{21}, S_{22}, S_{26}, S_{27}, S_{28}$ (by uniqueness of S^*), $S_{13}, S_{14}, S_{15}, S_{16}, S_{17}, S_{18}, S_{23}$ or, if $Y = \{u_{11}\}$, $S_{23} \setminus \{u_{12}\}$ (by Claim 2);

2. either $S^* = S_{21}$ or $S^* = S_{26}$, and there are no tight solutions associated with S_{20}, S_{22}, S_{26} (or S_{21}, respectively), S_{27}, S_{28} (by uniqueness of S^*), $S_2, S_7, S_8, S_{11}, S_{12}, S_{24}$ (by Claim 2);

3. either $S^* = S_{22}$ or $S^* = S_{27}$, and there are no tight solutions associated with S_{20}, S_{21}, S_{26}, S_{27} (or S_{22}, respectively), S_{28} (by uniqueness of S^*), $S_1, S_5, S_6, S_9, S_{10}, S_{25}$ (by Claim 2);

4. $S^* = S_{28}$ and there are no tight solutions associated with $S_{20}, S_{21}, S_{22}, S_{26}, S_{27}$ (by uniqueness of S^*), S_3, S_4, S_{19} (by Claim 2).

If case 1 occurs, then the list of all possible tight configurations is S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9, S_{10}, S_{11}, S_{12}, S_{19}, S_{20} if $G = G_{11}$ (add S_{24} and S_{25} if $G = G_\Omega$). As S_{20} is tight while S_{13} and S_{14} are not tight, then $\beta_{u_{11}} + \beta_{d_2} > \beta_a + \beta_{b_2}$ and $\beta_{b_1} + \beta_{u_{11}} > \beta_{d_1} + \beta_c$. As a consequence, S_{10} and S_{12} cannot be tight. By Lemma 3.13 if S_1 (S_2) is tight, then S_4 (S_3) is not tight. This implies that, if $G = G_{11}$, we are left with only 10 possible tight configurations, thus, by Proposition 3.10, (β, β_0) cannot be facet defining.

So consider the case $G = G_\Omega$. We still have 12 configurations that are candidate to be tight for (β, β_0). Observe now that, if S_5 and S_6 are both tight, we have that $\beta_{d_2} = \beta_{b_2}$ and, if S_{24} is also tight (since S_{12} is not) then $\beta_{u_{12}} > \beta_c$. Thus S_9 cannot be tight, otherwise, $S_9 \setminus \{c, d_2\} \cup \{u_{12}, b_2\}$ would violate (β, β_0). Thus at least one among S_5, S_6, S_9, S_{24} cannot be tight; we are left with 11 possible tight configurations, so (β, β_0) cannot be facet defining.

Let now case 2 occur. If $S^* = S_{21}$ then, since S_7 is not tight, we have that $\beta_{u_{11}} > \beta_{h_2}$. Thus S_{17} and S_4 cannot be tight. This implies that S_{18} is tight because it is the only remaining configuration containing the node h_2. Since S_{21} is tight and S_8 not, we also have that $\beta_{u_{11}} + \beta_{b_1} > \beta_{d_1} + \beta_{h_2}$. But then $S_{18} \setminus \{d_1, h_2\} \cup \{u_{11}, b_1\}$ would violate (β, β_0), a contradiction. If $S^* = S_{26}$, then $Y = \Omega$ and $\beta_{u_{11}} + \beta_{u_{12}} > \beta_{b_1} + \beta_{h_2}$ (as S_7 is not tight), $\beta_{u_{11}} + \beta_{u_{12}} > \beta_a + \beta_c$ (as S_2 is not tight), and $\beta_{u_{11}} + \beta_{u_{12}} > \beta_{d_1} + \beta_{h_2}$ (as S_8 is not tight). Thus, S_{17}, S_1, and S_{18} cannot be tight, so implying that no tight configuration misses the clique $K_2 \cup \{b_2, d_2\}$, a contradiction. Case 3 can be proved using diagonal symmetric arguments with respect to case 2.

Finally, we prove case 4; notice that $Y = \Omega$. Since S_{28} is tight and S_{19} is not, we have that $\beta_{u_{11}} + \beta_{u_{12}} > \beta_a + \beta_c$. As a consequence, S_1 and S_2 cannot be tight. Consider now the subcase with S_{16} tight. Then $\beta_{h_1} \geq \beta_c + \beta_{d_1}$ (consequently, $\beta_{h_1} > \beta_c$). This implies that S_9 is not tight. Moreover, as T_1 is not tight, S_{12} cannot be tight as well. Now, assume that also S_{15} is tight: we have that $\beta_{h_1} \geq \beta_a + \beta_{b_1}$ and $\beta_{h_1} \geq \beta_{u_{12}} + \beta_{d_1}$; consequently, S_{10} and S_{25} are not tight. Thus there is no tight configuration missing the lifted 5-wheel inequality supported by $W_1 \cup \{u_{11}.u_{12}\}$, a contradiction. On the contrary, if S_{15} is not tight then S_5 must be tight since it is the only configuration missing the clique $\{a, d_2, h_2, u_{12}\}$. But, since S_{16} is tight and S_{15} not, we have that $\beta_{b_2} < \beta_{d_2}$. Consequently, $S_5 \setminus \{b_2\} \cup \{d_2\}$ violates (β, β_0), a contradiction.

We now consider the subcase with S_{16} not tight. Then S_{15} must be tight because it is the only remaining configuration missing the clique $K_1 \cup \{b_1, d_1\}$. As a consequence, $\beta_{h_1} \geq \beta_a + \beta_{b_1}$, $\beta_{h_1} \geq \beta_{u_{12}} + \beta_{d_1}$, and (since S_{16} is not tight) $\beta_{b_2} > \beta_{d_2}$. Then S_{11}, S_{24} (as T_1 is not tight), and S_6 cannot be tight. But now no remaining configuration misses the clique $\{c, b_2, h_2, u_{11}\}$, a contradiction. ∎

We now focus our attention on the graph $G_{11} = (H, B_Y, e)$ obtained as the extended gear composition of H and B_Y with $Y = \{u_{11}\}$. In other words we start by proving case b.1) of Theorem 3.3. The case with $Y = \{u_{12}\}$ will follow by applying vertical symmetry. The proof of the case $Y = \{u_{11}, u_{12}\}$ follows the same pattern but it is longer, so we confine it in Appendix A.3.

In the following we show that a given set of configurations can be tight only for a proper g-lifted inequality of $STAB(G_\emptyset)$ lifted with the node u_{11}. To prove this we need two preliminary results:

Proposition 3.16. *Let (γ, γ_0) be a facet defining inequality for $STAB(G_\emptyset)$ with $\gamma_{B_\emptyset} \neq \mathbb{0}$. If S_1, S_2, S_5, S_6, S_{10}, S_{12} and at least one between S_7 and S_9 are tight for (γ, γ_0), then (γ, γ_0) is a proper g-lifted inequality.*

Proof. By Theorem 2.6 we know that any facet defining inequality of $STAB(G_\emptyset)$ with $\gamma_B \neq \mathbb{0}$ is either a geared or a g-lifted or a clique or a 5-wheel inequality.

Since S_1 is a tight configuration for (γ, γ_0) then it is not difficult to see that (γ, γ_0) is neither a proper nor a non-proper geared inequality. As S_5 and S_6 are tight, then (γ, γ_0) cannot be a non-proper g-lifted inequality. As S_{10} and S_{12} are tight, then (γ, γ_0) is not any of the two 5-wheel inequalities (6).

Moreover, S_1 misses the cliques $K_2 \cup \{d_2, b_2\}$, $\{b_1, d_1, h_1\}$, and $\{b_2, d_2, h_2\}$, S_2 misses the clique $K_1 \cup \{b_1, d_1\}$, S_{10} misses the cliques $\{c, b_1, h_1\}$ and $\{c, h_1, h_2\}$, S_{12} misses the cliques $\{a, d_2, h_2\}$ and $\{a, h_1, h_2\}$, and S_6 misses the clique $\{c, b_2, h_2\}$. Finally, the clique $\{a, d_1, h_1\}$ is missed by S_7 and S_9, Thus,(γ, γ_0) cannot be the clique inequality supported by any of the above cliques.

It follows that the proper g-lifted inequality is the only facet defining inequality for $STAB(G_\emptyset)$ that admits the given set of configurations as tight configurations. ∎

Lemma 3.17. *[9] Let $G = (H, B_Y, e)$ with $Y = \{u_{11}\}$ and $e = v_1 v_2$. Let (γ, γ_0) be a proper g-lifted inequality (3) associated with a g-liftable facet defining inequality (π, π_0) of $STAB(H^e)$. Then the inequality $\gamma_{G_\emptyset} x_{G_\emptyset} + \gamma_{u_{11}} x_{u_{11}} \leq \gamma_0$, obtained by lifting the node u_{11}, is facet defining for $STAB(G)$. The lifting coefficient of u_{11} is $\gamma_{u_{11}} = \pi_{v_1} = \gamma_u$ for any $u \in V_{B_\emptyset}$.*

Lemma 3.18. *Let $G = (H, B_Y, e)$ be a geared graph with $Y = \{u_{11}\}$, and let (β, β_0) be a facet defining inequality for $STAB(G)$ that has full support on B_Y. If S_1, S_2, and at least two configurations among $\{S_{20}, S_{21}, S_{22}\}$ are tight, then (β, β_0) is obtained by lifting a proper g-lifted inequality that is facet defining for $STAB(G_\emptyset)$ with the node u_{11}.*

Proof. The tightness of S_1 and S_2 implies the following inequalities: $\beta_a \geq \beta_{d_i}$, $\beta_c \geq \beta_{b_i}$, for $i = 1, 2$, $\beta_a + \beta_c \geq \beta_{u_{11}} + \beta_{b_1}$ and $\beta_a + \beta_c \geq \beta_{u_{11}} + \beta_{d_2}$. By Lemma 3.13, S_3 and S_4 are both not tight, and $S_{23} \setminus \{u_{12}\}$ is not tight, since otherwise $\beta_{b_2} \geq \beta_c + \beta_{d_2} \geq \beta_{b_2} + \beta_{d_2}$ would yield a contradiction. By hypothesis at least one between S_{21} and S_{22} is tight, and so, by Lemma 3.14, S_{15}, S_{16}, S_{17}, and S_{18} are not tight for (β, β_0).

We first consider the case S_{21} and S_{22} tight. The tightness of S_{21} implies that

$$\beta_{u_{11}} \geq \beta_{h_2}, \qquad \beta_{u_{11}} \geq \beta_a, \qquad \beta_{u_{11}} + \beta_{b_1} \geq \beta_{h_2} + \beta_{d_1}, \qquad \beta_{b_1} + \beta_{u_{11}} = \beta_a + \beta_c.$$

and the tightness of S_{22} implies that

$$\beta_{u_{11}} \geq \beta_{h_1}, \qquad \beta_{u_{11}} \geq \beta_c, \qquad \beta_{u_{11}} + \beta_{d_2} \geq \beta_{h_1} + \beta_{b_2}, \qquad \beta_{d_2} + \beta_{u_{11}} = \beta_a + \beta_c.$$

Suppose now that $\beta_a > \beta_{d_1}$; then S_{12} is not tight. If S_{14} is tight then $\beta_c + \beta_{d_1} \geq \beta_{b_1} + \beta_{u_{11}} = \beta_a + \beta_c$, that is $\beta_{d_1} \geq \beta_a$, a contradiction. Hence, S_{14} is not tight and S_8 is tight (otherwise the node d_1 would not be contained in any tight solution). Thus $\beta_{h_2} > \beta_c$ (as S_{12} is not tight) and, since $\beta_{u_{11}} \geq \beta_{h_2}$, S_9 is not tight. It follows that the clique $\{a, h_1, h_2, u_{11}\}$ is not missed by any tight solution of (β, β_0), a contradiction. Hence $\beta_a = \beta_{d_1}$ and, as S_2 is tight, S_{12} is also tight. As a consequence, $\beta_c + \beta_{d_1} = \beta_{b_1} + \beta_{u_{11}}$. Using diagonally symmetric arguments it can be proved that $\beta_c = \beta_{b_2}$, S_{10} is tight, and $\beta_a + \beta_{b_2} = \beta_{u_{11}} + \beta_{d_2}$.

As there must be at least one tight solution containing h_1, then S_5 or S_6 is tight. Suppose that S_6 is tight and S_5 is not tight, then $\beta_{d_2} > \beta_{b_2}$ and, as $\beta_a + \beta_{b_2} = \beta_{u_{11}} + \beta_{d_2}$, we have $\beta_a > \beta_{u_{11}}$, a contradiction. Suppose now that S_5 is tight and S_6 is not tight, then $\beta_{b_2} > \beta_{d_2}$. Moreover S_{11} is tight, because it is the only configuration that misses the clique $\{c, b_2, h_2, u_{11}\}$ and so $\beta_a = \beta_{u_{11}}$. Since S_5 and S_{10} are tight, it follows that $\beta_{h_1} = \beta_a$ and, since $\beta_{u_{11}} + \beta_{d_2} \geq \beta_{h_1} + \beta_{b_2}$ and $\beta_{b_2} > \beta_{d_2}$, it follows that $\beta_{u_{11}} > \beta_{h_1} = \beta_a$, a contradiction.

Thus S_5 and S_6 are both tight. Since the clique $\{a, d_1, h_1, u_{11}\}$ must be missed by at least one tight solution for (β, β_0), then S_7 or S_9 is tight. Hence, the hypotheses of Proposition 3.16 are satisfied and so the face defined by the inequality $(\beta_{G_\emptyset}, \beta_0)$, that is valid for $STAB(G_\emptyset)$, may be contained only in facets defined by inequalities (γ, γ_0) that are proper g-lifted inequalities or by inequalities with $\gamma_{B_\emptyset} = \mathbb{0}$.

Then the coefficients β_u have the same value for all $u \in V_{B_\emptyset}$. As (β, β_0) has full support on B_Y, $\beta_u > 0$ for each $u \in V_{B_Y}$; this implies that β_u is equal to a positive constant for each $u \in V_{B_\emptyset}$ and that there exists at least one facet defining inequality (γ, γ_0) for $STAB(G_\emptyset)$ which is a proper g-lifted inequality and contains the face defined by $(\beta_{G_\emptyset}, \beta_0)$.

We denote by $(\gamma, \gamma_{u_{11}}, \gamma_0)$ the inequality $\gamma_{G_0} x_{G_0} + \gamma_{u_{11}} x_{u_{11}} \leq \gamma_0$, obtained by lifting (γ, γ_0) with u_{11}. By Lemma 3.17, $(\gamma, \gamma_{u_{11}}, \gamma_0)$ is facet defining for $STAB(G)$ and $\gamma_{u_{11}} = \gamma_u$ for any $u \in V_{B_\emptyset}$. Clearly, every stable set S that is tight for (β, β_0) such that $u_{11} \notin S$ is also tight for $(\gamma, \gamma_{u_{11}}, \gamma_0)$. So we consider stable sets containing u_{11}. As S_6 is tight for (β, β_0), it is also tight for (γ, γ_0) and thus $S_{22} = S_6 \setminus \{h_1\} \cup \{u_{11}\}$ is tight for $(\gamma, \gamma_{u_{11}}, \gamma_0)$. As S_2 is tight for (β, β_0), it is also tight for (γ, γ_0) and thus $S_{21} = S_2 \setminus \{a, c\} \cup \{u_{11}, b_1\}$ is also tight for $(\gamma, \gamma_{u_{11}}, \gamma_0)$. Since S_{12} is tight for (β, β_0), if S_{20} is also tight for (β, β_0) then S_{14} is tight for (β, β_0) too. This implies that S_{14} is tight for (γ, γ_0) and, as $S_{14} = S_{20} \setminus \{u_{11}, b_1\} \cup \{a, d_1\}$, then S_{20} is also tight for $(\gamma, \gamma_{u_{11}}, \gamma_0)$.

We just showed that all the solutions that are tight for (β, β_0) are also tight for the proper g-lifted inequality $(\gamma, \gamma_{u_{11}}, \gamma_0)$. As a consequence, the facet defining inequalities (β, β_0) and $(\gamma, \gamma_{u_{11}}, \gamma_0)$ are equal up to a positive scalar factor. So, if S_{21} and S_{22} are both tight then (β, β_0) is the sequential lifting of a proper g-lifted inequality.

To prove the remaining cases suppose that S_{20} is tight, then

$$\beta_{u_{11}} + \beta_{b_1} \geq \beta_c + \beta_{d_1}, \qquad \beta_{u_{11}} + \beta_{d_2} \geq \beta_a + \beta_{b_2}.$$

If S_{21} is tight, we can assume that S_{22} is not tight, otherwise we match the previous case; thus S_{10} is not tight. As the clique $\{c, b_1, h_1\}$ is missed only by S_8, this configuration must be tight. Therefore, $\beta_{d_1} + \beta_{h_2} = \beta_{b_1} + \beta_{u_{11}}$ and, as $\beta_{u_{11}} \geq \beta_{h_2}$ (as S_{21} is tight), then $\beta_{d_1} \geq \beta_{b_1}$. Since $\beta_{u_{11}} + \beta_{b_1} \geq \beta_c + \beta_{d_1}$, it follows that $\beta_{u_{11}} \geq \beta_c$ and, as S_{22} is not tight, then S_9 is also not tight. Finally, observe that the lifted 5-wheel inequality on $W_1 \cup \{u_{11}\}$ is not missed by any tight solution of (β, β_0), a contradiction.

The case S_{20} and S_{22} tight yields a contradiction by using diagonally symmetric arguments. Thus the lemma follows. ∎

We can now prove the Step b.1) of Theorem 3.3. In particular, we prove that any given set of tight configurations of a facet defining inequality of $STAB(G_{11})$ with full support on B_Y is also tight for a facet defining inequality of $STAB(G_\emptyset)$ lifted with the node u_{11}.

Theorem 3.19. *Let $G = (H, B_Y, e)$ be a geared graph with $Y = \{u_{11}\}$ and let (β, β_0) be a facet defining inequality for $STAB(G)$. If (β, β_0) has full support on B_Y, then it is obtained by lifting with the node u_{11} either a proper g-lifted inequality or a proper geared inequality that is facet defining for $STAB(G_\emptyset)$.*

Proof. Suppose on the contrary that (β, β_0) is not a sequential lifting of any proper g-lifted or proper geared inequality that is facet defining for $STAB(G_\emptyset)$. Then, by Lemma 3.15, at least two configurations containing the node u_{11}, i.e. in the set $\{S_{20}, S_{21}, S_{22}\}$, must be tight for (β, β_0).

By Lemma 3.12, T_1, T_2, and T_3 are not tight, and, by Lemma 3.14, S_{15}, S_{16}, S_{17}, S_{18} are not tight. Therefore S_2 must be tight, because it is the only configuration that it is not tight for the clique $K_1 \cup \{b_1, d_1\}$. The same applies to S_1 and the clique $K_2 \cup \{b_2, d_2\}$. By Lemma 3.18 it follows that (β, β_0) is a sequential lifting of a proper g-lifted inequality, a contradiction. ∎

An analogous theorem holds for G_{12} and can be proved using symmetric arguments.

Theorem 3.20. *Let $G = (H, B_Y, e)$ be a geared graph with $Y = \{u_{12}\}$ and let (β, β_0) be a facet defining inequality for $STAB(G)$. If (β, β_0) has full support on B_Y, then it is obtained by lifting with the node u_{12} either a proper g-lifted inequality or a proper geared inequality that is facet defining for $STAB(G_\emptyset)$.*

We complete this subsection by stating the main result of this paper concerning the structure of the facet defining inequalities for $STAB(G)$ when $G = (H, B_Y, e)$ is a geared graph and $Y = \{u_{11}, u_{12}\}$. The proof is again very technical but follows the same scheme as the one for $Y = \{u_{11}\}$; so, the details of the proof and all the intermediate results have been moved to Appendix A.3.
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Theorem 3.21. *Let $G = (H, B_Y, e)$ be a geared graph with $Y = \{u_{11}, u_{12}\}$ and let (β, β_0) be a facet defining inequality for $STAB(G)$ that has full support on B_Y. Then (β, β_0) is obtained by lifting with u_{11} and u_{12} either a proper g-lifted inequality or a proper geared inequality that is facet defining for $STAB(G_\emptyset)$.*

4. XX-graphs are $\mathcal{G}$-perfect

In [9] we established that the XX-strip composition defined by Chudnovsky and Seymour [1] is equivalent to the extended gear composition applied only along simplicial edges that have the following property: $N(K_1 \cap K_2) \subseteq N(v_1) \cup N(v_2)$. We named these edges *super simplicial*.

We called *fuzzy line graphs* the graphs that are composition of fuzzy line interval strips (for more details on the decomposition of quasi-line graphs see [1, 2]) and we denoted this class of graphs as $\mathcal{Q}^\ell$. Then we defined a *geared fuzzy line graph* as the graph obtained by repeated applications of the extended gear composition and the edge subdivision along super simplicial edges of a fuzzy line graph:

Definition 4.1. *Let H be a (fuzzy) line graph. Let Γ_H be the set of the super simplicial edges of H and let a* g-operation *on $e \in \Gamma_H$ be either a gear composition or an edge subdivision applied along e. A graph $G \in \mathcal{G}^*_H$ if and only if*

either $G = H$,

or $G = (L, B_Y, e)$*, where $L \in \mathcal{G}^*_H$, B_Y is an extended gear, and $e \in \Gamma_H \cap E_L$ (i.e., e is a super simplicial edge of both L and H),*

or $G = L^e$*, where $L \in \mathcal{G}^*_H$ and $e \in \Gamma_H \cap E_L$.*

The graphs in $\bigcup_{H \in \mathcal{Q}^\ell} \mathcal{G}^*_H$ are called geared (fuzzy) line graphs.

The graphs in $\mathcal{G}^*_H$ are not quasi-line, since each B_Y contain two 5-wheels, but they are claw-free.

The XX*-graphs* are those graphs that can be decomposed by the Chudnovsky-Seymour strip decomposition into fuzzy linear interval strips and XX-strips. We proved that:

Theorem 4.2. *[9] The class of geared fuzzy line graphs is equivalent to the class of XX-graphs.*

If $STAB(H)$ and $STAB(H^e)$ are described only by rank inequalities then, by definitions 2.4 and 2.5 and Theorem 3.3, the geared inequalities generated by a single application of the extended gear composition will have at least a pair of coefficients equal to 2 corresponding to the hubs of an extended gear while the g-lifted inequalities will have all coefficients equal to 1. If we apply once more an extended gear composition we might obtain a g-lifted inequality associated with a geared inequality; so, it is not true that every g-lifted inequality is a rank inequality. Nevertheless, we can say that the inequalities generated by any extended gear composition (apart from lifted 5-wheel inequalities) are only of two types: either they contain pairs of hubs of a gear with coefficients 2 and have all the remaining coefficients equal to 1, or they have all the coefficients equal to 1. We call the former ones *multiple geared inequalities* and we refer to the others simply as rank inequalities. It can also be shown that a g-lifted inequality associated with a geared inequality may also be constructed as a geared inequality associated with a g-lifted inequality. As a consequence, if we assume that $STAB(H)$ and $STAB(H^e)$ are described only by rank inequalities, then the inequalities produced by repeated applications of the extended gear composition are (the sequential liftings of) either multiple geared inequalities or rank-inequalities or 5-wheel inequalities. More formally we define the following family of inequalities:

Definition 4.3. *A facet defining inequality (γ, γ_0) belongs to $\mathcal{G}$ if and only if (γ, γ_0) is (the sequential lifting of)*

either a rank inequality,

or a 5*-wheel inequality,*

or a multiple geared inequality, i.e., a geared inequality or a g-lifted inequality associated with an inequality in $\mathcal{G}$.

Consider now the polyhedron $\mathcal{G}STAB(G) = \{x \in \mathbb{R}^V_+ | \; x \; satisfies \; \mathcal{G}\}$. In [8, 9] it was proved that if G is a geared graph, then $STAB(G) \subseteq \mathcal{G}STAB(G)$; moreover, a graph G is said to be $\mathcal{G}$-perfect if the equality holds. Theorem 3.3 states that, if $G = (H, B_Y, e)$ is a geared graph, then a defining linear system for $STAB(G)$ can be easily provided once we know a defining linear system for $STAB(H)$ and $STAB(H^e)$. In fact, we have that:

Corollary 4.4. *Let $G = (H, B_Y, e)$ be a geared graph. If $STAB(H)$ and $STAB(H^e)$ are described only by inequalities in $\mathcal{G}$, i.e., H and H^e are $\mathcal{G}$-perfect, then G is $\mathcal{G}$-perfect.*

To extend the above theorem to the class of geared fuzzy line graphs, we need the following:

Theorem 4.5. *Let H be a graph and Γ_H the set of its super simplicial edges. Let H^F be the graph obtained from H by subdividing all the edges in $F \subseteq \Gamma_H$. If H and H^F are $\mathcal{G}$-perfect for any $F \subseteq \Gamma_H$, then every graph $G \in \mathcal{G}^*_H$ is $\mathcal{G}$-perfect.*

Proof. Let G_i denote the graph obtained from H by performing i g-operations along the edges e_j, $j = 1, \dots, i$, of Γ_H. We prove the theorem by induction on the number of g-operations. So, let $G = G_k$. If $k = 1$ the theorem follows by Corollary 4.4. If $k > 1$, then, by the inductive hypothesis, the theorem holds for every graph $L \in \mathcal{G}^*_H$ obtained by performing at most $k-1$ g-operations. Suppose by contradiction that G_k is not $\mathcal{G}$-perfect. If $G_k = (G_{k-1}, B_Y, e_k)$ then, by Corollary 4.4, at least one between G_{k-1} and $G^{e_k}_{k-1}$ is not $\mathcal{G}$-perfect. Since G_{k-1} is $\mathcal{G}$-perfect by inductive hypothesis, it follows that $G^{e_k}_{k-1}$ is not $\mathcal{G}$-perfect.

On the other hand, if G_k is obtained from G_{k-1} by subdivision of the edge e_k, then again we have that $G^{e_k}_{k-1}$ is not $\mathcal{G}$-perfect. So, by repeating the above arguments, $G^{e_k}_{k-1}$ is not $\mathcal{G}$-perfect only if $G^{\{e_k, e_{k-1}\}}_{k-2}$ is not $\mathcal{G}$-perfect. Thus iteratively we get that if G_k is not $\mathcal{G}$-perfect then $G^{\{e_1, e_2, \dots, e_k\}}_0 = H^F$ is not $\mathcal{G}$-perfect, a contradiction. ∎

We now prove that XX-graphs are $\mathcal{G}$-perfect.

Theorem 4.6. *If G is an XX-graph, then $STAB(G)$ is defined by the (sequential lifting of the) following inequalities:*

- *nonnegativity inequalities,*
- *rank inequalities,*
- 5*-wheel inequalities,*

- *multiple geared inequalities.*

Proof. By Theorem 4.2, the graph G is a geared fuzzy line graph, namely it belongs to the family $\mathcal{G}_H^*$ for some H fuzzy line graph. The class of fuzzy line graphs is closed under the operations of deleting and subdividing super simplicial edges (see Lemma 4 in [9]). This implies that the graph H^F is fuzzy line for any subset F of super simplicial edges of H.

Since $STAB(H)$ and $STAB(H^F)$ are described only by rank inequalities [1], it follows that H and H^F are $\mathcal{G}$-perfect. By Theorem 4.5, every graph in $\mathcal{G}_H^*$ is $\mathcal{G}$-perfect. Hence, $STAB(G)$ is completely described by inequalities in $\mathcal{G}$ and the theorem follows. ∎

5. Concluding remarks

In [9] we called *striped graphs* the claw-free graphs that are obtained by composing three kinds of strips: fuzzy linear interval strips, fuzzy XX-strips and fuzzy antihat strips. Chudnovsky and Seymour proved that claw-free, not quasi-line graphs with stability number at least 4 and without 1-joins are exactly the striped graphs without 1-joins. This implies that the XX-graphs (with $\alpha(G) \geq 4$) constitute a very large subclass of claw-free graphs since they correspond to all the striped graphs obtained without the use of the antihat strip [1, 2].

In [7] we conjectured that the striped graphs are $\mathcal{G}$-perfect. The validity of this conjecture would imply that striped graphs are not so far from the line graphs of 2-connected hypomatchable graphs, i.e., the graphs supporting the nontrivial rank facet inequalities of the stable set polytope of line graphs. Roughly speaking, our conjecture would imply that, for these graphs, every facet defining inequality of the stable set polytope (up to sequential lifting) is supported by either a clique or a 5-wheel or the line graph of a 2-connected hypomatchable graph where some “special” edges have been replaced by “gears”.

In this paper we prove this conjecture for XX-graphs (see Theorem 4.6), and we completely identify the coefficients of the facet defining inequalities of $STAB(G)$ when G is an XX-graph.

A. Appendix

A.1. Case generation for the non-full support inequalities

Let $G = (H, B_Y, e)$ be a geared graph with $Y = \{u_{11}\}$. In Lemma 3.4 we prove that only three subgraphs of B_Y may support a facet defining inequality (β, β_0) of $STAB(G)$ that has not full support on B_Y, it has $\beta_{u_{11}} > 0$, and it is not a clique or a lifted 5-wheel inequality.

The proof is by enumeration of all the possible 2^8 supports and shows that all the supports that are different from the ones listed in the thesis cannot be associated with a facet defining inequality.

First observe that any supporting graph of a facet defining inequality that is neither a clique inequality nor a 5-wheel inequality must contain a path between K_1 and K_2 whose internal nodes are contained in B_Y, otherwise these cliques are clique-cutsets and, by Theorem 3.1, $G \setminus A$ is not the supporting graph of a facet defining inequality. This means that A cannot separate K_1 from K_2. In particular, A contains neither $\{b_1, d_1\}$ nor $\{b_2, d_2\}$ and therefore $A \cap \{b_1, d_1, b_2, d_2\}$ is one of the following sets:

a) $\{d_1, d_2\}$, b) $\{b_1, d_2\}$, c) $\{d_1, b_2\}$, d) $\{b_1, b_2\}$,
e) $\{b_1\}$, f) $\{d_1\}$, g) $\{d_2\}$, h) $\{b_2\}$,
j) $\emptyset$.

Notice that cases (d), (g), (h) are diagonally symmetric with cases (a), (e), (f), respectively.

If $|A| = 6$, then $\{a, c, h_1, h_2\} \subset A$ and only cases from (a) to (d) may occur. In case (a), $A = \{d_1, d_2, a, c, h_1, h_2\}$ and $G \setminus A$ is disconnected; case (d) is diagonally symmetric with (a); indeed, if $A = \{b_1, b_2, c, a, h_2, h_1\}$ then $G \setminus A$ is also disconnected; case (c) produces a disconnected graph $G \setminus A$ too. In case (b), $A = \{b_1, d_2, a, c, h_1, h_2\}$ and $G \setminus A$ is admissible as considered in the proof of Lemma 3.4.

If $|A| = 5$, then A may be obtained as the union of a subset of type (a), (b), (c) and a subset of 3 nodes in $\{a, c, h_1, h_2\}$, or as the union of a subset of type (e), (f) and $\{a, c, h_1, h_2\}$ as shown in Table 1. All the other sets are diagonally symmetric the the previous ones.

Subset of $\{b_1, b_2, d_1, d_2\}$	Subset of $\{a, c, h_1, h_2\}$	Status of $G \setminus A$
$\{d_1, d_2\}$	$\{a, c, h_1\}$	disconnected
	$\{a, c, h_2\}$	*admissible*
	$\{a, h_1, h_2\}$	clique-cutset
	$\{c, h_1, h_2\}$	disconnected
$\{b_1, d_2\}$	$\{a, c, h_1\}$	clique-cutset
	$\{a, c, h_2\} = \sigma_d(\{a, c, h_1\})$	clique-cutset
	$\{a, h_1, h_2\}$	clique-cutset
	$\{c, h_1, h_2\} = \sigma_d(\{a, h_1, h_2\})$	clique-cutset
$\{d_1, b_2\}$	$\{a, c, h_1\}$	disconnected
	$\{a, c, h_2\} = \sigma_d(\{a, c, h_1\})$	disconnected
	$\{a, h_1, h_2\}$	disconnected
	$\{c, h_1, h_2\} = \sigma_d(\{a, h_1, h_2\})$	disconnected
$\{b_1\}$	$\{a, c, h_1, h_2\}$	clique-cutset
$\{d_1\}$	$\{a, c, h_1, h_2\}$	disconnected

Table 1: Enumeration for $|A| = 5$ and admissibility of $G \setminus A$

It turns out that $A = \{d_1, d_2, a, c, h_2\}$ and its symmetric version $A = \{b_1, b_2, a, c, h_1\}$ are the only two cases such that $G \setminus A$ is admissible, as stated in the proof of Lemma 3.4.

The cases with $|A| \leq 4$ are obtained using similar arguments. Observe that a further property can be used to reduce the number of cases to check. For instance, in Table 1 the set $\{b_1, d_2, a, c, h_1\}$ is diagonally symmetric with $\{b_1, d_2, a, c, h_2\}$. The former set is generated as the union of $\{b_1, d_2\}$ which is symmetric with itself and $\{a, c, h_1\}$ which is symmetric with $\{a, c, h_2\}$. Thus it is sufficient to list only one of them.

A.2. Proof of Theorem 3.9

Theorem 4. *Let $G = (H, B_Y, e)$ be a geared graph with $Y = \{u_{11}, u_{12}\}$ and let (β, β_0) be a facet defining inequality for $STAB(G)$ that does not have full support on V_{B_Y}. If $\beta_{u_{11}}$ and $\beta_{u_{12}}$ are both positive then (β, β_0) is a lifted 5-wheel inequality.*

Proof. Let $A = \{u \in V_{B_Y} : \beta_u = 0\}$ and $G \setminus A$ be the supporting graph of the inequality (β, β_0). Observe that if S is a tight stable set of G, then $S \setminus A$ is also tight. So, in the following, also if not explicitely remarked, we will always refer to tight stable sets in $G \setminus A$. The proof is by enumeration of all nonsymmetric sets A such that $G \setminus A$ is admissible (i.e. connected and without clique cutsets). Note that all the three symmetries (horizonthal, vertical, diagonal) can now be applied. Moreover, in order to prune the number of possible cases to consider, we will make a strong use of the following three claims.

Claim 1. $h_1, h_2 \notin A$.

Suppose $h_1 \in A$; then by Observation 4 there exists a tight stable set S_{h_1} in $G \setminus (A \cup N(h_1))$. It is not difficult to see that S_{h_1} contains either d_2 or b_2. But then either $S_{h_1} \cup \{u_{11}\}$ or $S_{h_1} \cup \{u_{12}\}$ violates (β, β_0), a contradiction. By vertical symmetry also $h_2 \notin A$. *(End of Claim 1)*

Claim 2. $\{b_1, d_1\} \nsubseteq A$ and $\{b_2, d_2\} \nsubseteq A$.

Otherwise K_2 or K_1 is a clique-cutset of $G \setminus A$, respectively. *(End of Claim 2)*

Claim 3. A cannot contain any of the following sets: $\{d_1, a\}, \{d_2, a\}, \{b_1, c\}, \{b_2, c\}$.

We will only show that $\{d_1, a\}$ cannot be contained in A: the other cases will follow by symmetric arguments. Suppose by contradiction that $\beta_{d_1} = \beta_a = 0$. First observe that, by Claim 2, $\beta_{b_1} > 0$. Now let S be a tight stable set missing the clique $\{c, b_2, h_2, u_{11}\}$. If $h_1 \notin S$ then $S \cup \{u_{11}\}$ is feasible and violates (β, β_0). Hence $h_1 \in S$, $\beta_{h_1} \geq \beta_c$, and $\beta_{h_1} \geq \beta_{u_{11}} > 0$. By Observation 4 there exists a tight stable set S_{d_1} in $G \setminus (A \cup N(d_1))$. Then either $c \in S_{d_1}$ or $u_{12} \in S_{d_1}$, since otherwise $S_{d_1} \cup \{b_1\}$ would violate (β, β_0). Suppose first that $c \in S_{d_1}$ and consider a tight stable set S missing the clique $\{c, h_1, h_2, u_{11}\}$. Observe that either $b_1 \in S$ or $u_{12} \in S$ since otherwise $S \cup \{h_1\}$ would violate (β, β_0). If $b_1 \in S$ then $\beta_{b_1} \geq \beta_{h_1} \geq \beta_c$ and so $S_{d_1} \setminus \{c\} \cup \{b_1, u_{11}\}$ would violate (β, β_0), a contradiction. Hence $u_{12} \in S$; consequently $\beta_{u_{12}} \geq \beta_{h_1}$. Let now S_a be a tight stable set in $G \setminus (A \cup N(a))$: clearly, either $c \in S_a$ or $\{b_1, b_2\} \subseteq S_a$. If $c \in S_a$ then $S_a \setminus \{c\} \cup \{u_{11}, u_{12}\}$ violates (β, β_0), a contradiction. Then $\{b_1, b_2\} \subseteq S_a$. Since $S_a \setminus \{b_1\} \cup \{h_1\}$ is feasible, we have that $\beta_{b_1} \geq \beta_{h_1} \geq \beta_c$. But then $S_{d_1} \setminus \{c\} \cup \{b_1, u_{11}\}$ violates (β, β_0), a contradiction.

Finally consider the case with $c \notin S_{d_1}$ and consequently $u_{12} \in S_{d_1}$; hence, $b_2 \in S_{d_1}$ (since otherwise $S_{d_1} \cup \{u_{11}\}$ would violate (β, β_0)) and then $\beta_{u_{12}} + \beta_{b_2} > \beta_c + \beta_{d_2}$ (otherwise we are in the previous case). Let T be a tight stable set containing c. Then $d_2 \notin T$, else $T \setminus \{c, d_2\} \cup \{u_{12}, b_2\}$ violates (β, β_0). Hence, $\beta_c \geq \beta_{u_{11}} + \beta_{u_{12}}$ and $\beta_c \geq \beta_{h_1}$. Thus, $\beta_{h_1} = \beta_c > \beta_{u_{12}}$ and $S_{d_1} \setminus \{u_{12}\} \cup \{h_1\}$ violates (β, β_0), a contradiction. *(End of Claim 3)*

It is not difficult to check that, in order to satisfy the Claims 1, 2, and 3, we can only build sets A of cardinality less than or equal to 3.

In particular, if $|A| = 3$ then, up to horizonthal symmetry, the only case occurring is $A = \{d_1, c, d_2\}$. In this case, let S_c be a tight stable set in $G \setminus (A \cup N(c))$. Then $a \in S_c$ and the following inequality holds: $\beta_a \geq \beta_{u_{11}} + \beta_{u_{12}}$ and so $\beta_a > \beta_{u_{11}}$. Let T be a tight stable set missing $\{a, h_1, h_2, u_{12}\}$. Then $b_1 \in T$ (otherwise $T \cup \{u_{12}\}$ would violate (β, β_0)) and $u_{11} \in T$ (otherwise $T \cup \{a\}$ would violate (β, β_0)). Hence, $\beta_{u_{11}} \geq \beta_a$, a contradiction.

Then, if $|A| = 2$, it follows that A is one of the following nonsymmetric configurations: $\{d_1, b_2\}$, $\{c, d_2\}$, $\{a, c\}$, or $\{d_1, d_2\}$. If $A = \{d_1, b_2\}$, consider the tight stable set S_{d_1} in $G \setminus (A \cup N(d_1))$.

Then $u_{12} \notin S_{d_1}$ since otherwise $S_{d_1} \cup \{u_{11}\}$ would violate (β, β_0). Hence, $S_{d_1} \supseteq \{c, d_2\}$, and so $\beta_c \geq \beta_{b_1} + \beta_{u_{11}}$. By diagonal symmetry it can be shown that $u_{12} \notin S_{b_2}$ and so $S_{b_2} \supseteq \{a, b_1\}$, thus implying that $\beta_{b_1} \geq \beta_c$, a contradiction.

The case with $A = \{c, d_2\}$ does not occur for the same considerations used to prove (viii) in case $|A| = 2$ of Lemma 3.4. Consider the case $A = \{a, c\}$: there exists two tight stable sets S_a in $G \setminus (A \cup N(a))$ and S_c in $G \setminus (A \cup N(c))$. It is not difficult to see that $S_a \supseteq \{b_1, b_2\}$ and $S_c \supseteq \{d_1, d_2\}$. It follows that $\beta_{d_i} = \beta_{b_i}$ for $i = 1, 2$. But then $S_a \setminus \{b_1\} \cup \{d_1, u_{12}\}$ violates (β, β_0), a contradiction.

Consider now the case $A = \{d_1, d_2\}$ and two tight stable sets S_{d_i}, $i = 1, 2$, in $G \setminus (A \cup N(d_i))$, respectively. Then $S_{d_1} \cap \{c, u_{12}\} \neq \emptyset$ and $S_{d_2} \cap \{c, u_{11}\} \neq \emptyset$ (otherwise $S_{d_i} \cup \{b_i\}$ would violate (β, β_0)). Now, if $c \in S_{d_i}$ for some i then $S_{d_i} \cup \{a\}$ would violate (β, β_0). As a consequence, b_2 and u_{12} belong to S_{d_1}, and $\beta_{b_2} \geq \beta_{u_{11}}$. Moreover, $u_{11} \in S_{d_2}$ and so $\beta_{u_{11}} \geq \beta_{b_2} + \beta_a \geq \beta_{u_{11}} + \beta_a$, a contradiction.

If $|A| = 1$ then there are two nonsymmetric cases to be considered: $A = \{b_1\}$ and $A = \{c\}$.

Case 1. $A = \{b_1\}$.

By Observation 4, there exists a tight stable set S_{b_1} in $G \setminus (A \cup N(b_1))$ and it is not difficult to see that $\{a, b_2\} \subseteq S_{b_1}$ or $u_{11} \in S_{b_1}$.

Suppose that there exists a tight stable set S containing u_{12} and containing neither b_2 nor u_{11}. Then $d_1 \in S$ (otherwise $S \cup \{u_{11}\}$ would violate (β, β_0)), $\beta_{d_1} \geq \beta_{u_{11}}$ and $\beta_{d_1} + \beta_{u_{12}} \geq \beta_a + \beta_c$. If $\{a, b_2\} \subseteq S_{b_1}$ then $\beta_a + \beta_{b_2} \geq \beta_{d_1} + \beta_{u_{12}} + \beta_{b_2} \geq \beta_a + \beta_c + \beta_{b_2}$, a contradiction. If $u_{11} \in S_{b_1}$ then $\beta_{u_{11}} \geq \beta_c + \beta_{d_1}$ and so $\beta_{u_{11}} > \beta_{d_1}$, a contradiction. Hence, every tight stable set containing u_{12} contains either b_2 or u_{11}.

Let T be a tight stable set missing the clique $\{c, b_2, h_2, u_{11}\}$. Then $u_{12} \notin T$ as neither b_2 nor u_{11} belongs to T, and $h_1 \in T$ otherwise $T \cup \{c\}$ would violate (β, β_0). It follows that $\beta_{h_1} \geq \beta_{u_{11}}$.

Subcase 1a. $\{a, b_2\} \subseteq S_{b_1}$. The arguments used in the proof of Subcase 1a) in Lemma 3.4 can be easily adapted by adding u_{12} when needed (i.e., every tight stable set containing b_2 also contains a or h_1 or u_{12} and every tight stable set containing u_{11} also contains d_2 or u_{12}) to prove that every tight stable set of (β, β_0) is tight for the 5-wheel inequality associated with W_2 lifted with $\{u_{11}, u_{12}\}$, as claimed.

Subcase 1b. $u_{11} \in S_{b_1}$. It follows that $\beta_{u_{11}} \geq \beta_c + \beta_{d_1}$ and, as $\beta_{h_1} \geq \beta_{u_{11}}$, $\beta_{h_1} > \beta_{d_1}$. Assume that $\{a, b_2\}$ is not contained in any tight stable set. Let S' be a tight stable set missing the clique $\{c, h_1, h_2, u_{11}\}$. If $u_{12} \in S'$, then also $b_2 \in S'$. It follows that $\beta_{u_{12}} + \beta_{b_2} \geq \beta_c + \beta_{d_2}$. Let S'' be a tight stable set missing the clique $\{c, h_1, h_2, u_{12}\}$. As $\{a, b_2\}$ is not contained in any tight stable set, $\{u_{11}, d_2\} \subseteq S''$ and so $\beta_{u_{11}} + \beta_{d_2} > \beta_a + \beta_{b_2}$. Summing the last two inequalities, we get $\beta_{u_{11}} + \beta_{u_{12}} > \beta_a + \beta_c$, thus implying that a does not belong to any tight stable set, a contradiction. Hence $u_{12} \notin S'$; but then $d_1 \in S'$ and, consequently, $\beta_{d_1} \geq \beta_{h_1}$, a contradiction.

Hence there exists a tight stable set S containing $\{a, b_2\}$. If $S \subseteq V_G \setminus (N(b_1) \cup \{b_1\})$, then the Subcase 1a) applies to S. So let us assume that $S \cap K_1 \neq \emptyset$; as a consequence, $\beta_a + \beta_{b_2} \geq \beta_{u_{11}} + \beta_{d_2}$ and $\beta_a \geq \beta_{h_1}$. If $d_2 \in S_{b_1}$, then $S_{b_1} \setminus \{u_{11}, d_2\} \cup \{a, b_2\}$ leads to the Subcase 1a). Thus consider the case $d_2 \notin S_{b_1}$ and consequently $S_{b_1} \cap K_2 \neq \emptyset$; but then $\beta_{h_1} = \beta_{u_{11}} \geq \beta_a + \beta_c \geq \beta_{h_1} + \beta_c$, a contradiction. *(End of Case 1)*

Case 2. $A = \{c\}$.

By Observation 4, there exists a tight stable set S_c in $G \setminus (A \cup N(c))$ and it is not difficult to see that S_c contains $\{d_1, d_2\}$ or $\{a\}$.

Suppose first that $\{d_1, d_2\} \subseteq S_c$. Hence $\beta_{d_i} \geq \beta_{h_i}$ for $i = 1, 2$. Since $\beta_{d_1} + \beta_{d_2} \geq \beta_{b_1} + \beta_{u_{11}} + \beta_{d_2}$, it follows that $\beta_{d_1} > \beta_{b_1}$ and $\beta_{d_1} > \beta_{u_{11}}$. Similarly, the inequality $\beta_{d_1} + \beta_{d_2} \geq \beta_{d_1} + \beta_{u_{12}} + \beta_{b_2}$ implies that $\beta_{d_2} > \beta_{b_2}$ and $\beta_{d_2} > \beta_{u_{12}}$.

Consider now a tight stable set S_i missing the clique $K_i \cup \{b_i, d_i\}$, $i = 1, 2$. We have that S_i contains h_i, $i = 1, 2$. Indeed, if $\{u_{11}, u_{12}\} \subseteq S_i$, for some i, we would have that $S_1 \setminus \{u_{11}\} \cup \{d_1\}$ or $S_2 \setminus \{u_{12}\} \cup \{d_2\}$ violates (β, β_0), a contradiction. Hence $\beta_{h_i} \geq \beta_{d_i}$, and so $\beta_{h_i} = \beta_{d_i}$ for $i = 1, 2$.

Since $\beta_{d_1} \geq \beta_{b_1} + \beta_{u_{11}}$, $\beta_{d_2} \geq \beta_{b_2} + \beta_{u_{12}}$, and $\beta_{d_i} = \beta_{h_i}$ for $i = 1, 2$, it follows that $\beta_{u_{11}} + \beta_{u_{12}} < \beta_{h_1} + \beta_{d_2}$, $\beta_{u_{11}} + \beta_{u_{12}} < \beta_{d_1} + \beta_{h_2}$, $\beta_{u_{11}} + \beta_{d_2} < \beta_{h_1} + \beta_{d_2}$, and $\beta_{d_1} + \beta_{u_{12}} < \beta_{d_1} + \beta_{h_2}$. The above inequalities imply that every tight stable set S containing u_{11} or u_{12} satisfy one of the following conditions: $S \supseteq \{b_1, d_2, u_{11}\}$, $S \supseteq \{d_1, b_2, u_{12}\}$, or $S \supseteq \{u_{11}, u_{12}\}$ and $S \cap K_i \neq \emptyset$ for $i = 1, 2$.

Let us consider a tight stable set T_1 missing $\{a, d_1, h_1, u_{11}\}$ and a tight stable set T_2 missing $\{a, d_2, h_2, u_{12}\}$. Since $\beta_{d_i} > \beta_{b_i}$, $i = 1, 2$, we have that $b_i \notin T_i$ (otherwise $T_i \setminus \{b_i\} \cup \{d_i\}$ would violate (β, β_0)). It follows that T_1 contains either u_{12} or h_2 and T_2 contains either u_{11} or h_1. We actually prove that $h_2 \in T_1$ and $h_1 \in T_2$ is the only feasible configuration. Indeed, if $u_{12} \in T_1$ then $\beta_{u_{12}} \geq \beta_{h_1}$ and, since $\beta_{h_2} = \beta_{d_2} > \beta_{u_{12}}$, it follows that $\beta_{h_2} > \beta_{h_1}$. Hence $h_1 \notin T_2$ (otherwise $T_2 \setminus \{h_1\} \cup \{h_2\}$ would violate (β, β_0)), and so $u_{11} \in T_2$. As above we have that $\beta_{u_{11}} \geq \beta_{h_2}$ and, since $\beta_{h_1} = \beta_{d_1} > \beta_{u_{11}}$, it follows that $\beta_{h_1} > \beta_{h_2}$, a contradiction. Thus $h_2 \in T_1$ and simmetrically $h_1 \in T_2$, implying that $\beta_{h_1} = \beta_{h_2} \geq \beta_a$.

Hence every tight stable set S containing b_1 also contains a or u_{11} (otherwise $S \setminus \{b_1\} \cup \{d_1\}$ would violate (β, β_0)) and every tight stable set S' containing b_2 also contains a or u_{12} (otherwise $S' \setminus \{b_2\} \cup \{d_2\}$ would violate (β, β_0)). Moreover every tight stable set $\tilde{S}$ containing $\{b_1, a\}$ contains also b_2, otherwise $\tilde{S} \setminus \{b_1, a\} \cup \{d_1, h_2\}$ would violate (β, β_0). Similarly we can prove that every tight stable set containing $\{a, b_2\}$ contains b_1.

As a consequence the tight solutions of (β, β_0) are not linearly independent. In fact, consider the matrix $M \in \{0, 1\}^{|V_G| \times |V_G|}$ whose columns are the incidence vectors of the roots of (β, β_0): the sum of rows indexed by the nodes b_1 and u_{12} equals the sum of the rows indexed by b_2 and u_{11}, contradicting the nonsingularity of M and consequently the fact that (β, β_0) is facet defining for $STAB(G)$.

Suppose now that no tight stable set contains both d_1 and d_2. Hence, $a \in S_c$ and so $\beta_a \geq \beta_{h_i}$ $(i = 1, 2)$ and $\beta_a \geq \beta_{u_{11}} + \beta_{u_{12}}$. Thus, $\beta_a > \beta_{u_{11}}$ and $\beta_a > \beta_{u_{12}}$.

Let us consider a tight stable set T_1 missing $\{a, d_1, h_1, u_{11}\}$. Clearly, $u_{12} \notin T_1$ (otherwise $T_1 \setminus \{u_{12}\} \cup \{a\}$ would violate (β, β_0)) and $d_2 \notin T_1$ (otherwise $T_1 \cup \{u_{11}\}$ would violate (β, β_0)). It follows that $h_2 \in T_1$ and so $\beta_{h_2} = \beta_a$. Simmetrically, it can be proved that h_1 belongs to a tight stable set T_2 missing $\{a, d_2, h_2, u_{12}\}$ and so $\beta_{h_1} = \beta_a$.

Let R_1 be a tight stable set missing $\{a, h_1, h_2, u_{11}\}$. If $u_{12} \notin R_1$ then R_1 has to contain $\{d_1, b_2\}$ or $\{b_1, d_2\}$ (since $\{d_1, d_2\}$ cannot be tight by hypothesis), and so $R_1 \cup \{u_{12}\}$ or $R_1 \cup \{u_{11}\}$, respectively, violates (β, β_0), a contradiction. Hence, $u_{12} \in R_1$. Moreover, $d_1 \in R_1$ (otherwise $R_1 \setminus \{u_{12}\} \cup \{a\}$ would violate (β, β_0)) and $b_2 \in R_1$ (otherwise $R_1 \setminus \{u_{12}\} \cup \{h_2\}$ would violate (β, β_0)). Similarly we can prove that the tight stable set R_2 missing $\{a, h_1, h_2, u_{12}\}$ contains $\{b_1, u_{11}, d_2\}$.

Since $\{d_1, d_2\}$ is contained in no tight stable set and $\{a, b_1, b_2\}$ is a feasible stable set, it follows that $\beta_{b_1} + \beta_{u_{11}} + \beta_{d_2} + \beta_{d_1} + \beta_{u_{12}} + \beta_{b_2} > \beta_a + \beta_{b_1} + \beta_{b_2} + \beta_{d_1} + \beta_{d_2}$, thus implying that $\beta_{u_{11}} + \beta_{u_{12}} > \beta_a$, a contradiction. *(End of Case 2)*

Thus the theorem follows. ∎

A.3. Proof of Theorem 3.21

In this appendix we give a detailed proof of Theorem 3.21. This proof requires some intermediate results to establish which combinations of configurations can be tight for a facet defining inequality of $STAB(G_\Omega)$ having full support on B_Ω. We start with two preliminary results on $STAB(G_{11})$.

Proposition A.1. *Let (γ, γ_0) be a facet defining inequality for $STAB(G_{11})$ with $\gamma_{B_{11}} \neq \mathbb{0}$. If S_3, S_4, S_{13}, S_{14}, S_{15}, S_{16}, S_{17}, S_{18}, S_{19} are tight, then (γ, γ_0) is obtained by lifting a proper geared inequality that is facet defining for $STAB(G_\emptyset)$ with the node u_{11}.*

Proof. As S_3 is tight, then $\gamma_{h_1} \geq \gamma_a + \gamma_c$ and $\gamma_{h_1} \geq \gamma_{u_{11}}$. Thus (γ, γ_0) is neither a (proper or non-proper) g-lifted inequality of type (3), (4) lifted with u_{11}, nor an inequality of type (5). Moreover, (γ, γ_0)

is neither a clique inequality supported by any of the cliques $\{c, b_2, h_2, u_{11}\}$, $\{a, d_2, h_2\}$, $\{b_2, d_2, h_2\}$, nor the 5-wheel inequality supported by W_2 lifted with u_{11}. Similarly, as S_4 is tight, then (γ, γ_0) is neither a clique inequality supported by any of the cliques $\{a, d_1, h_1, u_{11}\}$, $\{c, b_1, h_1\}$, $\{b_1, d_1, h_1\}$, nor the 5-wheel inequality on W_1 lifted with u_{11}.

As S_{13} and S_{14} are tight, then (γ, γ_0) is not the clique inequality supported by $\{c, h_1, h_2, u_{11}\}$ or $\{a, h_1, h_2, u_{11}\}$. As S_{15} and S_{17} are tight, then (γ, γ_0) is not the clique inequality supported by $K_1 \cup \{b_1, d_1\}$ or $K_2 \cup \{b_2, d_2\}$.

Finally, (γ, γ_0) is not a non-proper geared inequality of type (2) with $A \in \{\{b_2, c, u_{11}\}, \{d_2, a\}, \{b_1, c\}, \{d_1, a, u_{11}\}, \{a, c, u_{11}\}\}$ as S_{15}, S_{16}, S_{17}, S_{18}, S_{19} are tight, respectively. Thus the only facet defining inequality that admits the given set of configurations as tight configurations is a proper geared inequality of $STAB(G_\emptyset$ lifted with node u_{11} and the proposition follows. ∎

Proposition A.2. *Let (γ, γ_0) be a facet defining inequality for $STAB(G_{11})$ with $\gamma_{B_{11}} \neq \mathbb{0}$. If S_1, S_2, S_{21}, and S_{22} are tight configurations and one of the following conditions occurs:*

a) S_{11} and S_9 are tight,

b) S_{11}, S_7, and S_{14} are tight,

c) S_{11}, S_7, and S_{12} are tight,

d) S_9, S_6, and S_{13} are tight,

e) S_9, S_6, and S_{10} are tight,

then (γ, γ_0) is obtained by lifting a proper g-lifted inequality that is facet defining for $STAB(G_\emptyset)$ with the node u_{11}.

Proof. As S_1 and S_2 are tight, then (γ, γ_0) is neither a (proper or non-proper) lifted geared inequality of $STAB(G_\emptyset)$ lifted with the node u_{11} nor a non-proper g-lifted inequality of type (5). Moreover, S_1 and S_2 miss the cliques $K_1 \cup \{b_1, d_1\}$, $K_2 \cup \{d_2, b_2\}$, $\{b_1, d_1, h_1\}$, and $\{b_2, d_2, h_2\}$, and so (γ, γ_0) is not a clique inequality supported by any of these cliques. As S_{21} and S_{22} are tight then (γ, γ_0) is neither a non-proper g-lifted inequality (4) nor a lifted 5-wheel inequality (6).

Moreover, the configuration S_{22} misses the clique $\{c, b_1, h_1\}$ and the configuration S_{21} misses the clique $\{a, d_2, h_2\}$; thus, (γ, γ_0) is not a clique inequality supported by any of these cliques.

If S_{11} is tight then (γ, γ_0) is neither the clique inequality supported by $\{c, h_1, h_2, u_{11}\}$ nor the clique inequality supported by $\{c, b_2, h_2, u_{11}\}$. Finally if S_9 is tight or S_7 and one between $\{S_{12}, S_{14}\}$ are tight, then (γ, γ_0) is neither the clique inequality supported by $\{a, d_1, h_1, u_{11}\}$ nor the clique inequality supported by $\{a, h_1, h_2, u_{11}\}$. As a consequence the only inequality that admits the given set of configurations as tight configurations is a proper g-lifted inequality of $STAB(G_\emptyset$ lifted with node u_{11}.

The cases d) and e) of the proposition follow from the cases b) and c) by diagonal symmetry. ∎

Lemma A.3. *[9] Let $G = (H, B_Y, e)$ be a geared graph with $Y = \{u_{11}, u_{12}\}$ and $e = v_1v_2$. Let (γ, γ_0) be a proper geared inequality (1) associated with a g-extendable facet defining inequality (π, π_0) of $STAB(H)$ and then lifted with the node u_{11}. Then the inequality $\gamma_{G_{11}} x_{G_{11}} + \gamma_{u_{12}} x_{u_{12}} \leq \gamma_0$ obtained by lifting the node u_{12} is facet defining for $STAB(G)$ and the lifting coefficient of u_{12} is $\gamma_{u_{12}} = \pi_{v_1} = \gamma_u$ for any $u \in V_{B_{11}} \setminus \{h_1, h_2\}$.*

Lemma A.4. *[9] Let $G = (H, B_Y, e)$ be a geared graph with $Y = \{u_{11}, u_{12}\}$ and $e = v_1v_2$. Let (γ, γ_0) be a proper g-lifted inequality (3) associated with a g-liftable facet defining inequality (π, π_0) of $STAB(H^e)$ and then lifted with the node u_{11}. Then the inequality $\gamma_{G_{11}} x_{G_{11}} + \gamma_{u_{12}} x_{u_{12}} \leq \gamma_0$ obtained by lifting the node u_{12} is facet defining for $STAB(G)$ and the lifting coefficient of u_{12} is $\gamma_{u_{12}} = \pi_{v_1} = \gamma_u$ for any $u \in V_{B_{11}}$.*

Lemma A.5. *Let $G = (H, B_Y, e)$ be a geared graph with $Y = \{u_{11}, u_{12}\}$, and let (β, β_0) be a facet defining inequality for $STAB(G)$ that has full support on B_Y and admits S_1, S_2, and S_{21} as tight configurations. If S_{22} or S_{25} is tight, then (β, β_0) is obtained by lifting a proper g-lifted inequality that is facet defining for $STAB(G_\emptyset)$ with the nodes u_{11} and u_{12}.*

Proof. By hypothesis, $\beta_a + \beta_c = \beta_{u_{11}} + \beta_{b_1}$ and $\beta_a + \beta_c \geq \beta_{u_{11}} + \beta_{u_{12}}$. By Lemma 3.13 and Lemma 3.14, the configurations S_3, S_4, S_{15}, S_{16}, S_{17}, and S_{18} are not tight for (β, β_0).

Case 1. S_{22} tight.

If S_{11} and S_9 are tight then, by Proposition A.2 a), the face defined by inequality $(\beta_{G_{11}}, \beta_0)$ is contained in facets defined by only two types of inequalities: (i) those that are obtained by lifting with the node u_{11} a proper g-lifted inequality that is facet defining for $STAB(G_\emptyset)$, or (ii) those that have zero coefficient on every node of B_{11}. As $\beta_{B_Y} > \mathbb{0}$, at least one such inequality, say (γ, γ_0), is of type (i). This implies that the coefficients β_u have the same value for all $u \in V_{B_{11}}$; we prove that $\beta_{u_{12}} = \beta_u$ for $u \in V_{B_{11}}$ as well. In fact, if $\beta_{u_{12}} > \beta_u$, then S_{24} violates (β, β_0) (as S_{21} is tight), a contradiction. If $\beta_{u_{12}} < \beta_u$, then S_{24}, S_{25}, S_{26}, S_{27} are not tight. If S_{23} is tight, then S_{20} violates (β, β_0) and if S_{28} is tight, then S_{19} violates (β, β_0). Then no tight configuration contains u_{12}, a contradiction.

Observe now that, by Lemma A.4, the inequality $(\gamma, \gamma_{u_{12}}, \gamma_0)$ obtained by lifting (γ, γ_0) with node u_{12} has $\gamma_{u_{12}} = \gamma_u$ for any $u \in V_{B_{11}}$ and it is facet defining for $STAB(G)$. Clearly, every stable set S that is tight for (β, β_0) such that $u_{12} \notin S$, it is also tight for $(\gamma, \gamma_{u_{12}}, \gamma_0)$. It is not difficult to check that this property can also be extended to the configurations containing node u_{12} so that if a configuration is tight for (β, β_0), then it is also tight for $(\gamma, \gamma_{u_{12}}, \gamma_0)$, i.e. the two inequalities are equivalent.

Suppose now that S_{11} is tight and S_9 is not tight, i.e., $\beta_{u_{11}} = \beta_a$ (as S_{21} is tight) and $\beta_{u_{11}} > \beta_c$ (as S_{22} is tight). Since $\beta_a + \beta_c \geq \beta_{u_{11}} + \beta_{u_{12}}$ and $\beta_{u_{11}} > \beta_c$, it follows that $\beta_a > \beta_{u_{12}}$, and so S_{25} is not tight (since otherwise $S_{25} \setminus \{u_{12}\} \cup \{a\}$ would violate (β, β_0)). Since there exists a tight stable set missing the clique $\{a, d_1, h_1, u_{11}\}$, and S_4, S_{17}, and S_{25} are not tight, it follows that S_7 is tight for (β, β_0). If S_{14} is tight then, by Proposition A.2b), the face defined by $(\beta_{G_{11}}, \beta_0)$ is contained in a facet defined by a proper g-lifted inequality lifted with node u_{11} and, by using similar arguments as before, it can be shown that (β, β_0) is the sequential lifting of a proper g-lifted inequality of $STAB(G_\emptyset)$, as claimed. Hence, suppose that S_{14} is not tight for (β, β_0). If S_{12} is tight then case (c) of Proposition A.2 can be applied and we are done. Thus, suppose that S_{12} is not tight and so S_{23} or S_{24} must be tight since otherwise the clique $\{a, h_1, h_2, u_{11}\}$ would not be missed by any tight solution of (β, β_0). If S_{24} is tight then $\beta_{u_{12}} > \beta_c$ (since S_{12} is not tight), and so $\beta_{u_{11}} + \beta_{u_{12}} > \beta_a + \beta_c$, contradicting the hypothesis. It follows that S_{24} is not tight and S_{23} is tight for (β, β_0), and so $\beta_{d_1} + \beta_{u_{12}} + \beta_{b_2} > \beta_{d_1} + \beta_c + \beta_{d_2}$ (as S_{14} is not tight). Since $\beta_c \geq \beta_{u_{12}}$ ($\beta_a + \beta_c \geq \beta_{u_{11}} + \beta_{u_{12}}$ and $\beta_a = \beta_{u_{11}}$), it follows that $\beta_{b_2} > \beta_{d_2}$. But then $S_{22} \setminus \{d_2, u_{11}\} \cup \{b_2, a\}$ violates (β, β_0), a contradiction.

The case when S_9 is tight and S_{11} is not tight can be proved using the diagonal symmetry.

Finally, suppose that neither S_{11} nor S_9 is tight and so $\beta_{u_{11}} > \beta_a$ (as S_{21} is tight) and $\beta_{u_{11}} > \beta_c$ (as S_{22} is tight). Since $\beta_a + \beta_c \geq \beta_{u_{11}} + \beta_{u_{12}}$, we have that $\beta_a > \beta_{u_{12}}$ and $\beta_c > \beta_{u_{12}}$ and so S_{24} and S_{25} are not tight. Then S_6 and S_7 are both tight since they are the only configurations missing the cliques $\{c, b_2, h_2, u_{11}\}$ and $\{a, d_1, h_1, u_{11}\}$, respectively. Thus $\beta_{h_1} = \beta_{h_2} = \beta_{u_{11}}$, and then S_{10} and S_{12} are not tight ($\beta_{u_{11}} > \beta_a$ and $\beta_{u_{11}} > \beta_c$). As S_6 and S_7 are tight, we also have that $\beta_{b_1} \geq \beta_{d_1}$, $\beta_{d_2} \geq \beta_{b_2}$; then S_{23} is not tight (as $\beta_{u_{11}} > \beta_{u_{12}}$). Moreover, S_{13} and S_{14} are both tight since they are the only configurations missing the cliques $\{c, h_1, h_2, u_{11}\}$ and $\{a, h_1, h_2, u_{11}\}$; therefore, $\beta(S_{13}) + \beta(S_{14}) > \beta(S_{20}) + \beta(S_{23})$ implies that $\beta_a + \beta_c > \beta_{u_{11}} + \beta_{u_{12}}$. Hence, S_{26}, S_{27} and S_{28} are not tight and u_{12} belongs to no tight configuration, a contradiction.

Case 2. S_{25} tight.

We can assume, without loss of generality, that S_{22} is not tight (otherwise we match the previous case) and S_{24} is not tight (otherwise the thesis follows from the previous case by horizonthal symmetry).

Notice that at least one configuration between S_5 and S_6 must be tight for (β, β_0) (otherwise there would be no tight configuration containing the node h_1). So, if S_5 is tight then $\beta_{h_1} = \beta_{u_{12}}$ (as S_{25} is tight); if S_6 is tight then $\beta_{h_1} > \beta_{u_{11}}$ (as S_{22} is not tight).

Consider first the case when S_5 and S_6 are both tight. Since S_{21} is tight, then $\beta_{u_{11}} \geq \beta_{h_2}$ and this implies that $\beta_{u_{12}} = \beta_{h_1} > \beta_{u_{11}} \geq \beta_{h_2}$. As a consequence, S_8 is not tight and then S_{10} must be tight, because it is the only remaining configuration missing the clique $\{c, b_1, h_1, u_{12}\}$. If follows that $\beta_a = \beta_{u_{12}} = \beta_{h_1}$ (as S_{25} is tight); but then $S_{21} \setminus \{u_{11}\} \cup \{a\}$ violates (β, β_0), a contradiction.

Let us now consider the case when S_5 is tight and S_6 is not tight; then $\beta_{b_2} > \beta_{d_2}$ and S_{11} is tight because it is the only remaining configuration missing the clique $\{c, b_2, h_2, u_{11}\}$. This implies that $\beta_a = \beta_{u_{11}}$ (as S_{21} is tight), $\beta_a \geq \beta_{h_2}$, and $\beta_c = \beta_{b_1}$ (as S_2 is tight). It follows that S_{20} is not tight (otherwise $S_{20} \setminus \{d_2, u_{11}\} \cup \{b_2, a\}$ would violate (β, β_0)). Furthermore, $\beta_a \geq \beta_{d_1}$ (as S_2 is tight) and, consequently, S_{14} cannot be tight (otherwise $S_{14} \setminus \{c, d_1, d_2\} \cup \{a, b_1, b_2\}$ would violate (β, β_0)). Then S_9 is tight (otherwise no tight configuration would contain the node d_2), and so $\beta_c > \beta_{u_{12}}$ (as S_{25} is tight and $\beta_{b_2} > \beta_{d_2}$). Together with $\beta_a = \beta_{u_{11}}$, this implies that S_{26}, S_{27}, and S_{28} are not tight and u_{11} belongs to the unique tight solution S_{21}. Hence, by Lemma 3.15, (β, β_0) is the sequential lifting of a facet defining inequality for $STAB(G \setminus \{u_{11}\})$. Notice that the only facet defining inequalities for $STAB(G \setminus \{u_{11}\}) = STAB(G_{12})$ that have full support on B_{12} are proper geared or g-lifted inequalities for $STAB(G_\emptyset)$ lifted with the node u_{12} (see Theorem 3.20). Since S_1 is tight for (β, β_0) and is not tight for any lifted proper geared inequality, we have that (β, β_0) is the sequential lifting of a proper g-lifted inequality.

Finally, consider the case when S_6 is tight and S_5 is not tight. Since S_{25} is tight and S_5 is not, then $\beta_{u_{12}} > \beta_{h_1}$; since S_6 is tight, then $\beta_{h_1} \geq \beta_c$ and, as a consequence, S_{12} is not tight. Since S_6 is tight and S_{22} is not, we have that $\beta_{h_1} > \beta_{u_{11}}$ and then $\beta_{u_{12}} > \beta_{u_{11}}$. Moreover, since $\beta_{u_{11}} \geq \beta_a$ (as S_{21} is tight), it follows that $\beta_{u_{12}} > \beta_a$. Thus S_{10} cannot be tight and S_8 must be tight, since it is the only remaining configuration missing the clique $\{c, b_1, h_1, u_{12}\}$. This implies that $\beta_{d_1} \geq \beta_{b_1}$; but then $S_{21} \setminus \{b_1, u_{11}\} \cup \{d_1, u_{12}\}$ violates (β, β_0), a contradiction. ∎

Lemma A.6. *Let $G = (H, B_Y, e)$ be a geared graph with $Y = \{u_{11}, u_{12}\}$ and let (β, β_0) be a facet defining inequality for $STAB(G)$ that has full support on B_Y. If S_3 or S_4 are tight configurations for (β, β_0), then (β, β_0) is obtained by lifting a proper geared inequality that is facet defining for $STAB(G_\emptyset)$ with the nodes u_{11} and u_{12}.*

Proof. If S_3 is tight then $\beta_{h_1} \geq \beta_{u_{11}} + \beta_{u_{12}}$ and $\beta_{h_1} \geq \beta_a + \beta_c$. This implies that $\beta_{h_1} > \beta_a$, $\beta_{h_1} > \beta_c$, $\beta_{h_1} > \beta_{u_{11}}$, $\beta_{h_1} > \beta_{u_{12}}$ and so S_9, S_{10}, S_{22}, S_{25}, and S_{27} are not tight configurations for (β, β_0). Since T_2 is not tight by Lemma 3.12, it follows that S_4 is tight for (β, β_0). In fact, if not, every tight solution of (β, β_0) would be also tight for the lifted 5-wheel inequality supported by $W_1 \cup \{u_{11}, u_{12}\}$, a contradiction.

By vertical symmetry, if S_4 is tight then $\beta_{h_2} \geq \beta_{u_{11}} + \beta_{u_{12}}$ and $\beta_{h_2} \geq \beta_a + \beta_c$. This implies that $\beta_{h_2} > \beta_a$, $\beta_{h_2} > \beta_c$, $\beta_{h_2} > \beta_{u_{11}}$, $\beta_{h_2} > \beta_{u_{12}}$ and so S_{11}, S_{12}, S_{21}, S_{24}, and S_{26} are not tight configurations for (β, β_0). Since T_1 is not tight by Lemma 3.12, it follows that S_3 is tight for (β, β_0). In fact, if not, every tight solution of (β, β_0) would be also tight for the lifted 5-wheel inequality supported by $W_2 \cup \{u_{11}, u_{12}\}$, a contradiction.

Hence, S_3 and S_4 are both tight and the above derived conditions always hold; moreover, $\beta_{h_1} = \beta_{h_2}$ and, by Lemma 3.13, S_1 and S_2 are not tight.

If S_{20}, S_{23} and S_{28} are not all tight for (β, β_0) then u_{11} or u_{12} belongs to at most one tight solution and, by Lemma 3.15, (β, β_0) is the sequential lifting of a facet defining inequality of a lower dimensional polytope, say $STAB(G')$ with $G' = (H, B_{Y'}, e)$ and $Y' \subset \{u_{11}, u_{12}\}$. By theorems 3.19 and 3.20, this implies that (β, β_0) is obtained by lifting either a proper geared or a proper g-lifted inequality that is facet defining inequality of $STAB(G_\emptyset)$ with the nodes u_{11} and u_{12}. More precisely, since S_3 and S_4 are tight, the former case must occur.

Hence, we may assume that S_{20}, S_{23} and S_{28} are tight for (β, β_0) and so $\beta_{h_i} = \beta_{u_{11}} + \beta_{u_{12}} \geq \beta_a + \beta_c$. If $\beta_{u_{11}} + \beta_{u_{12}} > \beta_a + \beta_c$, then S_{19} is not tight and at most one between S_{13} and S_{14} is tight for (β, β_0) since $\beta(S_{20}) + \beta(S_{23}) > \beta(S_{13}) + \beta(S_{14})$ holds. As a consequence, at least one node between a and c is not contained in any tight solution, a contradiction. It follows that $\beta_{u_{11}} + \beta_{u_{12}} = \beta_a + \beta_c$, and so S_{13}, S_{14} and S_{19} are tight for (β, β_0).

Since (β, β_0) is not a clique inequality supported by $K_i \cup \{b_i, d_i\}$, $i = 1, 2$, there exist tight configurations of (β, β_0) that do not intersect these cliques. These tight solutions belong to the set $\{S_{15}, S_{16}, S_{17}, S_{18}\}$. If the four solutions are all tight then, by Proposition A.1, the tight configurations for $(\beta_{G_{11}}, \beta_0)$ are also tight for a proper geared inequality of $STAB(G_\emptyset)$ lifted with the node u_{11}. Hence, by using Lemma A.3, it is not difficult to check that (β, β_0) is the sequential lifting of $(\beta_{G_{11}}, \beta_0)$ lifted with the node u_{12}, as claimed.

We consider the cases when exactly one configuration in $\{S_{15}, S_{16}\}$ and one configuration in $\{S_{17}, S_{18}\}$ are tight. If S_{15}, S_{17} are tight, and S_{16}, S_{18} are not tight, then $\beta_{b_1} > \beta_{d_1}$ and $\beta_{b_2} > \beta_{d_2}$. Since S_6 and S_8 cannot be tight, S_5 and S_7 are tight because we are left with only 12 possible tight configurations, i.e., S_3, S_4, S_5, S_7, S_{13}, S_{14}, S_{15}, S_{17}, S_{19}, S_{20}, S_{23}, and S_{28}. But it is not difficult to check that these roots are not linearly independent, a contradiction. The case when S_{16} and S_{18} are tight is horizonthally symmetric with this one.

If S_{15}, S_{18} are tight, and S_{16}, S_{17} are not tight, then $\beta_{d_1} > \beta_{b_1}$ and $\beta_{b_2} > \beta_{d_2}$. Since S_6 and S_7 cannot be tight, S_5 and S_8 are tight since we are left with only 12 possible tight configurations, i.e., S_3, S_4, S_5, S_8, S_{13}, S_{14}, S_{15}, S_{18}, S_{19}, S_{20}, S_{23}, and S_{28}. But these roots are not linearly independent, a contradiction. The case when S_{16} and S_{17} are tight is horizonthally symmetric with this one.

We are left with the case that exactly one among $\{S_{15}, S_{16}, S_{17}, S_{18}\}$ is not tight. Suppose first that S_{18} is not tight. Since S_{15} and S_{16} are tight it follows that $\beta_{b_2} = \beta_{d_2}$; since S_{17} is tight and S_{18} is not, it follows that $\beta_{b_1} > \beta_{d_1}$. This implies that $\beta_c > \beta_a$ since $\beta_{b_1} + \beta_{b_2} + \beta_a = \beta_{d_1} + \beta_{d_2} + \beta_c$ (as S_{13} and S_{14} are tight). Moreover, $\beta_a = \beta_{u_{11}}$ (since S_{20} is tight, i.e. $\beta_{b_1} + \beta_{b_2} + \beta_a = \beta_{b_1} + \beta_{d_2} + \beta_{u_{11}}$), $\beta_c = \beta_{u_{12}}$ (since S_{23} is tight, i.e. $\beta_{d_1} + \beta_{d_2} + \beta_c = \beta_{d_1} + \beta_{b_2} + \beta_{u_{12}}$), $\beta_{b_1} = \beta_c$ (since S_{19}, S_3, S_{15}, and S_{13} are tight, i.e. $\beta_a + \beta_c = \beta_{h_1} = \beta_{b_1} + \beta_a$), and $\beta_{d_1} = \beta_a$ (since also S_{16} and S_{14} are tight, i.e. $\beta_a + \beta_c = \beta_{h_1} = \beta_{d_1} + \beta_c$).

Finally, since S_{15} and S_{17} are tight and $\beta_{h_1} = \beta_{h_2}$, it follows that $\beta_{b_1} = \beta_{b_2}$. To summarize we have that the coefficients of β_{B_Y} are the following:

$$\beta_a = \beta_{d_1} = \beta_{u_{11}} = p, \quad \beta_c = \beta_{b_1} = \beta_{b_2} = \beta_{d_2} = \beta_{u_{12}} = q, \quad \beta_{h_1} = \beta_{h_2} = p + q$$

where $q > p > 0$. Hence the inequality (β, β_0) can be written as the following convex combination: $\mu(\beta_{G \setminus B_Y}, \gamma'_{B_Y}, \gamma'_0) + (1 - \mu)(\beta_{G \setminus B_Y}, \gamma''_{B_Y}, \gamma''_0)$ where $\mu = \frac{p}{q}$ and

$$\begin{array}{llll} \gamma'_u = q \quad \forall u \in V_{B_Y} \setminus \{h_1, h_2\}, & \gamma'_{h_1} = \gamma'_{h_2} = 2q, & \text{and} & \gamma'_0 = \beta_0 + q - p \\ \gamma''_u = q \quad \forall u \in V_{B_Y} \setminus \{d_1, a, u_{11}\}, & \gamma''_{d_1} = \gamma''_a = \gamma''_{u_{11}} = 0, & \text{and} & \gamma''_0 = \beta_0 - p. \end{array} \tag{8}$$

We now prove that the inequalities $(\beta_{G \setminus B_Y}, \gamma'_{B_Y}, \gamma'_0)$ and $(\beta_{G \setminus B_Y}, \gamma''_{B_Y}, \gamma''_0)$ are valid for $STAB(G)$. The maximal stable sets S of G can be partitioned into four classes depending on their intersection with the cliques K_i, $i = 1, 2$:

1) $S \cap K_1 \neq \emptyset$ and $S \cap K_2 = \emptyset$;
2) $S \cap K_1 = \emptyset$ and $S \cap K_2 \neq \emptyset$;
3) $S \cap K_i \neq \emptyset$ for $i = 1, 2$;
4) $S \cap K_i = \emptyset$ for $i = 1, 2$.

Since $(\beta_{G \setminus B_Y}, \gamma'_{B_Y}, \gamma'_0)$ and $(\beta_{G \setminus B_Y}, \gamma''_{B_Y}, \gamma''_0)$ differ only on V_{B_Y} and on the right hand side, it suffices to consider the values of $\beta(S \cap V_{B_Y})$ for maximal stable sets S of G. Let S be of type 1) and let

S^* be its associated configuration on B_Y^*. We have that $\beta_{B_Y}(S \cap V_{B_Y}) = 2q + p$ if $S^* \in \{S_5, S_6\}$, $\beta_{B_Y}(S \cap V_{B_Y}) = p + q$ if $S^* \in \{S_1, S_{10}, S_{22}, S_{27}, T_2\}$, and $\beta_{B_Y}(S \cap V_{B_Y}) = 2q$ if $S^* \in \{S_9, S_{25}\}$.

As for these solutions $\beta_{G \setminus B_Y}(S \setminus V_{B_Y})$ is constant and $\max\{\beta_{B_Y}(S \cap V_{B_Y}) : S \cap K_1 \neq \emptyset, S \cap K_2 = \emptyset\} = 2p + q$, the inequality $(\beta_{G\setminus B_Y}, \gamma'_{B_Y}, \gamma'_0)$ is satisfied by all solutions of type 1) because

$$\gamma'_{B_Y}(S \cap V_{B_Y}) \leq 3q = 2q + p + (q - p) = \max\{\beta_{B_Y}(S \cap V_{B_Y}) : S \cap K_1 \neq \emptyset, S \cap K_2 = \emptyset\} + (q - p),$$

i.e., the left hand side of $(\beta_{G\setminus B_Y}, \gamma'_{B_Y}, \gamma'_0)$ increases at most of $q - p$ with respect to β_0. Moreover, the inequality $(\beta_{G\setminus B_Y}, \gamma''_{B_Y}, \gamma''_0)$ is also satisfied by all solutions of type 1), because

$$\gamma''_{B_Y}(S \cap V_{B_Y}) \leq 2q = 2q + p - p = \max\{\beta_{B_Y}(S \cap V_{B_Y}) : S \cap K_1 \neq \emptyset, S \cap K_2 = \emptyset\} - p,$$

i.e, the left hand side of $(\beta_{G\setminus B_Y}, \gamma''_{B_Y}, \gamma''_0)$ decreases at least of p with respect to β_0.

We can repeat similar arguments for any maximal stable set S such that S^* is of type 2), 3), and 4), so proving that $(\beta_{G\setminus B_Y}, \gamma'_{B_Y}, \gamma'_0)$ and $(\beta_{G\setminus B_Y}, \gamma''_{B_Y}, \gamma''_0)$ are both valid for $STAB(G)$. Hence (β, β_0) is the convex combination of two valid inequalities, contradicting the hypothesis that it is facet defining for $STAB(G)$.

As the set of configurations S_3, S_4, S_{13}, S_{14}, S_{19}, S_{20}, S_{23}, S_{28} is closed under any of the three symmetries and the case S_{17} is not tight (S_{16} is not tight, S_{15} is not tight) is horizonthally (vertically, diagonally, respectively) symmetric with the case S_{18} is not tight, the thesis follows. ∎

Lemma A.7. *Let $G = (H, B_Y, e)$ be a geared graph with $Y = \{u_{11}, u_{12}\}$. Let (β, β_0) be a facet defining inequality for $STAB(G)$ that has full support on V_{B_Y}. If neither S_{21} nor S_{24} is tight for (β, β_0), then (β, β_0) is obtained by lifting with u_{11} and u_{12} either a proper g-lifted inequality or a proper geared inequality that is facet defining for $STAB(G_\emptyset)$.*

Proof. By Lemma 3.12, the solutions of type T_1, T_2, T_3, T_4 are not tight for (β, β_0). If S_3 or S_4 is tight the thesis follows by Lemma A.6. So, assume that S_3 and S_4 are not tight. Since S_{21} and S_{24} are not tight, it follows that S_{11} or S_{12} is tight for (β, β_0). In fact, if not, then every tight solution of (β, β_0) would be also tight for the lifted 5-wheel inequality supported by $W_2 \cup \{u_{11}, u_{12}\}$, a contradiction.

Suppose first that S_{11} is tight; then $\beta_a > \beta_{u_{11}}$ and $\beta_a + \beta_{b_1} > \beta_{d_1} + \beta_{u_{12}}$ (since S_{21} and S_{24} are not tight). It follows that S_{23} is not tight (since otherwise $S_{23} \setminus \{d_1, u_{12}\} \cup \{b_1, a\}$ would violate (β, β_0)).

If S_{12} is also tight then $\beta_c > \beta_{u_{12}}$ (as S_{24} is not tight), and so S_{26}, S_{27} and S_{28} are not tight (as they can be increased by substituting $\{u_{11}, u_{12}\}$ with $\{a, c\}$), thus implying that u_{12} belongs to a unique tight configuration, namely S_{25}. Thus, by Lemma 3.15, (β, β_0) is a sequential lifting of a facet defining inequality for $STAB(G_{11})$ and, by Theorem 3.19, the thesis follows.

Hence we may assume that S_{12} is not tight for (β, β_0). Now, if S_{20} or S_{22} is tight then $\beta_{d_2} > \beta_{b_2}$ (otherwise substituting $\{d_2, u_{11}\}$ with $\{b_2, a\}$ would lead to a violation of (β, β_0)). Hence, S_5 and S_{15} are not tight, and so every tight solution of (β, β_0) is tight for the inequality supported by the clique $\{a, d_2, h_2, u_{12}\}$, a contradiction. Hence, S_{20}, S_{21}, and S_{22} are not tight. Since there must exist a tight configuration of (β, β_0) missing the clique $\{a, d_2, h_2, u_{12}\}$ and S_{12} is not tight, we have that S_5 or S_{15} is tight. This implies that $\beta_{h_1} \geq \beta_{u_{12}}$, $\beta_{b_2} \geq \beta_{d_2}$, and, as S_{11} is tight and S_{12} is not tight, $\beta_a + \beta_{b_1} + \beta_{b_2} > \beta_c + \beta_{d_1} + \beta_{d_2}$, i.e., S_{14} is not tight. Since S_{12}, S_{20}, and S_{22} are also not tight and there must exist a tight solution of (β, β_0) missing the clique $\{a, h_1, h_2, u_{12}\}$, we have that S_9 is tight and so $\beta_c \geq \beta_{h_1} \geq \beta_{u_{12}}$. It follows that S_{26}, S_{27}, and S_{28} are not tight, since otherwise u_{11} and u_{12} could be replaced by a and c in all cases to provide stable sets that violate (β, β_0). Hence, u_{11} belongs to no tight solution of (β, β_0), a contradiction.

Since $\sigma_h(S_{21}) = S_{24}$ and $\sigma_h(S_{11}) = S_{12}$, the case when S_{12} is tight (and S_{11} is not tight) follows by horizonthal symmetry. ∎

Theorem 4. *Let $G = (H, B_Y, e)$ be a geared graph with $Y = \{u_{11}, u_{12}\}$ and let (β, β_0) be a facet defining inequality for $STAB(G)$ that has full support on B_Y. Then (β, β_0) is obtained by lifting with u_{11} and u_{12} either a proper g-lifted inequality or a proper geared inequality that is facet defining for $STAB(G_\emptyset)$.*

Proof. By Lemma A.6, if S_3 or S_4 is tight then the thesis follows. So, we can assume that neither S_3 nor S_4 is tight for (β, β_0). By Lemma A.7, if neither S_{21} nor S_{24} is tight then the thesis follows. By diagonal symmetry the same holds if neither S_{22} nor S_{25} is tight. To complete the proof it suffices to consider the following four cases that are symmetric in pairs: "both S_{21} and S_{22} tight" that is horizonthally symmetric to "both S_{24} and S_{25} tight"; and "both S_{21} and S_{25} tight" that is horizonthally symmetric to "both S_{22} and S_{24} tight". Thus the proof consists of two nonsymmetric cases:

1. both S_{21} and S_{22} are tight,
2. both S_{21} and S_{25} are tight.

However, since the proofs for these two cases are very similar, we will distinguish them only on the parts of the proof where different arguments are used. Therefore, we suppose that S_{21} is tight and at least one between S_{22} and S_{25} is tight.

By Lemma 3.12, T_1, T_2, T_3, and T_4 are not tight for (β, β_0), while S_{15}, S_{16}, S_{17}, and S_{18} are not tight by Lemma 3.14.

By Lemma A.5, it suffices to consider the case when S_1 and S_2 are not both tight for (β, β_0). Clearly, if S_1 is tight and S_2 is not tight then S_{26} must be tight for (β, β_0) (since otherwise the clique $K_1 \cup \{b_1, d_1\}$ would never be missed by a tight solution of (β, β_0)). Hence, $\beta_a + \beta_c < \beta_{u_{11}} + \beta_{u_{12}}$ and so $\beta(S_1 \setminus \{a, c\} \cup \{u_{11}, u_{12}\}) > \beta(S_1)$, a contradiction. A similar argument shows that the case S_1 not tight and S_2 tight cannot occur.

Hence, we are left with the case: neither S_1 nor S_2 is tight for (β, β_0). Since the cliques $K_1 \cup \{b_1, d_1\}$ and $K_2 \cup \{b_2, d_2\}$ must be missed by at least one tight solution of (β, β_0), we have that both S_{26} and S_{27} are tight for (β, β_0). Since S_{27} is tight and S_1 is not, $\beta_a + \beta_c < \beta_{u_{11}} + \beta_{u_{12}}$; then S_{19} is not tight because otherwise $S_{19} \setminus \{a, c\} \cup \{u_{11}, u_{12}\}$ would violate (β, β_0).

We now distinguish four cases accordingly with the tightness of S_{13} and S_{14}.

Case a. S_{13} and S_{14} are both tight. It follows that $\beta(S_{13}) + \beta(S_{14}) \geq \beta(S_{20}) + \beta(S_{23})$, i.e., $\beta_{b_1} + \beta_a + \beta_{b_2} + \beta_{d_1} + \beta_c + \beta_{d_2} \geq \beta_{b_1} + \beta_{u_{11}} + \beta_{d_2} + \beta_{d_1} + \beta_{u_{12}} + \beta_{b_2}$, implying that $\beta_a + \beta_c \geq \beta_{u_{11}} + \beta_{u_{12}}$, a contradiction.

Case b. S_{14} is tight and S_{13} is not tight. For S_{14} being tight, we have that $\beta_c + \beta_{d_1} \geq \beta_{u_{11}} + \beta_{b_1}$. If S_{11} is tight then, since S_2 is not, $\beta_{b_1} > \beta_c$ and consequently $\beta_{d_1} > \beta_{u_{11}}$. Then $S_{26} \setminus \{u_{11}\} \cup \{d_1\}$ violates (β, β_0). So S_{11} is not tight. This implies that S_{10} is tight, since it represents the only configuration containing the node a. As S_1 is not tight, we have that $\beta_{b_2} > \beta_c$ and then, since $\beta_{d_2} + \beta_c \geq \beta_{b_2} + \beta_{u_{12}}$ (as S_{14} is tight), $\beta_{d_2} > \beta_{u_{12}}$. Then $S_{27} \setminus \{u_{12}\} \cup \{d_2\}$ violates (β, β_0), a contradiction

Case c. S_{13} is tight and S_{14} is not tight. A contradiction follows from Case b) by diagonal symmetry.

The last case is:

Case d. Neither S_{13} nor S_{14} is tight. Since a and c must belong to a tight stable set, it follows that at least a configuration in $\{S_{10}, S_{11}\}$ and at least a configuration in $\{S_9, S_{12}\}$ must be tight for (β, β_0). Suppose first that S_{11} is tight and so $\beta_a = \beta_{u_{11}}$ (as S_{21} is tight). Since S_{21} and S_{26} are tight, and since $S_{21} \setminus \{u_{11}, b_1\} \cup \{a, c\} = S_2$ is not tight, it follows that $\beta_{b_1} = \beta_{u_{12}} > \beta_c$. Thus, S_{12} is not tight and so S_9 is tight.

Consider now the two cases S_{22} tight or S_{25} tight. If S_{22} is tight then $\beta_c = \beta_{u_{11}}$ and, consequently, $\beta_{u_{12}} > \beta_c = \beta_{u_{11}} = \beta_a$. Hence, S_{10} is not tight (otherwise $S_{10} \setminus \{a\} \cup \{u_{12}\}$ would violate (β, β_0)). Moreover, as $\beta_{u_{11}} + \beta_{d_2} > \beta_a + \beta_{b_2} = \beta_{u_{11}} + \beta_{b_2}$, then $\beta_{d_2} > \beta_{b_2}$ and S_5 is not tight. Now consider the case S_{25} tight, then $\beta_{u_{12}} + \beta_{b_2} = \beta_c + \beta_{d_2}$ (as S_9 is tight) and, as $\beta_{u_{12}} > \beta_c$, $\beta_{d_2} > \beta_{b_2}$ and S_5

is not tight; hence, S_{10} is not tight (otherwise $S_{10} \setminus \{a, b_2\} \cup \{u_{11}, d_2\}$ would violate (β, β_0)) and then $\beta_{u_{12}} > \beta_a = \beta_{u_{11}}$.

Hence, if S_{22} or S_{25} is tight then both S_5 and S_{10} are not tight and $\beta_{u_{12}} > \beta_{u_{11}}$. Moreover, since $\beta_{u_{11}} + \beta_{b_1} \geq \beta_{u_{12}} + \beta_{d_1}$ (as S_{21} is tight) and $\beta_{u_{12}} > \beta_{u_{11}}$, we have that $\beta_{b_1} > \beta_{d_1}$, thus implying that S_8 is not tight for (β, β_0). If S_{23} is tight, then $\beta_{d_1} + \beta_{u_{12}} + \beta_{b_2} > \beta_{b_1} + \beta_a + \beta_{b_2}$ (as S_{13} is not tight), i.e., $\beta_{d_1} + \beta_{u_{12}} > \beta_{b_1} + \beta_a = \beta_{b_1} + \beta_{u_{11}}$, a contradiction; thus S_{23} is not tight.

To summarize we are left with only 12 configurations which might be tight for (β, β_0): S_6, S_7, S_9, S_{11}, S_{20}, S_{21}, S_{22}, S_{24}, S_{25}, S_{26}, S_{27}, S_{28}. As (β, β_0) is a facet defining inequality, then the incidence vectors of the tight configurations must define a matrix with rank equal to 12 by Proposition 3.10. Therefore the above configurations are all tight for (β, β_0).

Claim 1. *(β, β_0) is a convex combination of a valid inequality $(\beta_{G \setminus B_Y}, \gamma'_{B_Y}, \gamma'_0)$ with $\gamma'_u = q$ for all $u \in V_{B_Y}$ and a valid inequality of type $(\beta_{G \setminus B_Y}, \gamma''_{B_Y}, \gamma''_0)$ with $\gamma''_u = q$ if $u \in \{b_1, d_2, u_{12}\}$ and $\gamma''_u = 0$ if $u \in V_{B_Y} \setminus \{b_1, d_2, u_{12}\}$, where $q > 0$.*

Notice that now S_{22} and S_9 are both tight, so from the above considerations $\beta_a = \beta_{u_{11}} = \beta_c$, $\beta_{b_1} = \beta_{u_{12}}$ (S_{21} and S_{26} tight), and $\beta_{u_{12}} > \beta_{u_{11}}$. Moreover, since S_6 and S_9 are both tight, then $\beta_{h_1} = \beta_c$; since S_{25} and S_{27} are both tight, then $\beta_{u_{11}} = \beta_{b_2}$; since S_7 and S_{11} are both tight, then $\beta_a = \beta_{h_2}$; since S_7 and S_{24} are both tight, then $\beta_{b_1} + \beta_{h_2} = \beta_{d_1} + \beta_{u_{12}}$, i.e., $\beta_{h_2} = \beta_{d_1}$; since S_9 and S_{25} are both tight, then $\beta_c + \beta_{d_2} = \beta_{b_2} + \beta_{u_{12}}$, i.e., $\beta_{u_{12}} = \beta_{d_2}$. Summing up, we have that

$$\beta_a = \beta_c = \beta_{u_{11}} = \beta_{h_1} = \beta_{h_2} = \beta_{b_2} = \beta_{d_1} = p, \qquad \beta_{u_{12}} = \beta_{b_1} = \beta_{d_2} = q,$$

where $q > p > 0$. Hence the inequality (β, β_0) can be written as the following convex combination: $\mu(\beta_{G \setminus B_Y}, \gamma'_{B_Y}, \gamma'_0) + (1 - \mu)(\beta_{G \setminus B_Y}, \gamma''_{B_Y}, \gamma''_0)$ where $\mu = \frac{p}{q}$ and

$$\begin{array}{llr} \gamma'_u = q \quad \forall u \in V_{B_Y} & \text{and} & \gamma'_0 = \beta_0 + q - p \\ \gamma''_u = q \quad \forall u \in \{b_1, d_2, u_{12}\}, \quad \gamma''_u = 0 \quad \forall u \in V_{B_Y} \setminus \{b_1, d_2, u_{12}\}, & \text{and} & \gamma''_0 = \beta_0 - p. \end{array} \tag{9}$$

We now prove that the inequalities $(\beta_{G \setminus B_Y}, \gamma'_{B_Y}, \gamma'_0)$ and $(\beta_{G \setminus B_Y}, \gamma''_{B_Y}, \gamma''_0)$ are valid for $STAB(G)$. The maximal stable sets S of G can be partitioned into four classes depending on their intersection with the cliques K_i, $i = 1, 2$:

1) $S \cap K_1 \neq \emptyset$ and $S \cap K_2 = \emptyset$;
2) $S \cap K_1 = \emptyset$ and $S \cap K_2 \neq \emptyset$;
3) $S \cap K_i \neq \emptyset$ for $i = 1, 2$;
4) $S \cap K_i = \emptyset$ for $i = 1, 2$.

Since $(\beta_{G \setminus B_Y}, \gamma'_{B_Y}, \gamma'_0)$ and $(\beta_{G \setminus B_Y}, \gamma''_{B_Y}, \gamma''_0)$ differ only on V_{B_Y} and on the right hand side, it suffices to consider the values of $\beta(S \cap V_{B_Y})$ for maximal stable sets S of G . Let S be of type 1) and let S^* be its associated configuration on B^*_Y. We have that $\beta_{B_Y}(S \cap V_{B_Y}) = q + p$ if $S^* \in \{S_6, S_9, S_{22}, S_{25}, S_{27}\}$, $\beta_{B_Y}(S \cap V_{B_Y}) = 2p$ if $S^* \in \{S_1, S_5, S_{10}\}$, and $\beta_{B_Y}(S \cap V_{B_Y}) = p$ if $S^* \in \{T_2\}$.

As for these solutions $\beta_{G \setminus B_Y}(S \setminus V_{B_Y})$ is constant and $\max\{\beta_{B_Y}(S \cap V_{B_Y}) : S \cap K_1 \neq \emptyset, S \cap K_2 = \emptyset\} = p + q$, the inequality $(\beta_{G \setminus B_Y}, \gamma'_{B_Y}, \gamma'_0)$ is satisfied by all solutions of type 1) because

$$\gamma'_{B_Y}(S \cap V_{B_Y}) \leq 2q = q + p + (q - p) = \max\{\beta_{B_Y}(S \cap V_{B_Y}) : S \cap K_1 \neq \emptyset, S \cap K_2 = \emptyset\} + (q - p),$$

i.e., the left hand side of $(\beta_{G \setminus B_Y}, \gamma'_{B_Y}, \gamma'_0)$ increases at most of $q - p$ with respect to β_0. Moreover, the inequality $(\beta_{G \setminus B_Y}, \gamma''_{B_Y}, \gamma''_0)$ is satisfied by all solutions of type 1), because

$$\gamma''_{B_Y}(S \cap V_{B_Y}) \leq q = q + p - p = \max\{\beta_{B_Y}(S \cap V_{B_Y}) : S \cap K_1 \neq \emptyset, S \cap K_2 = \emptyset\} - p,$$

i.e, the left hand side of $(\beta_{G \setminus B_Y}, \gamma''_{B_Y}, \gamma''_0)$ decreases at least of p with respect to β_0.

We can repeat similar arguments for any maximal stable set S such that S^* is of type 2), 3), and 4), so proving that $(\beta_{G\setminus B_Y}, \gamma'_{B_Y}, \gamma'_0)$ and $(\beta_{G\setminus B_Y}, \gamma''_{B_Y}, \gamma''_0)$ are both valid for $STAB(G)$. Hence (β, β_0) is the convex combination of two valid inequalities. *(End of Claim 1)*

As a consequence of the above claim we have that (β, β_0) is not facet defining, a contradiction.

Therefore S_{11} is not tight and S_{10} is tight. This implies that $\beta_a + \beta_{b_2} \geq \beta_{u_{12}} + \beta_{b_2}$, i.e., $\beta_a \geq \beta_{u_{12}}$. As S_{27} is tight and S_1 is not tight, then $\beta_{u_{11}} + \beta_{u_{12}} > \beta_a + \beta_c$; it follows that $\beta_{u_{11}} > \beta_c$, S_9 is not tight and consequently S_{12} is tight. Since S_{21} and S_{12} are tight, then $\beta_{d_1} + \beta_c = \beta_{b_1} + \beta_{u_{11}}$, thus $\beta_{d_1} > \beta_{b_1}$, i.e., S_7 is not tight and S_8 is tight because it is the unique solution that contains h_2. Moreover, $\beta_{b_1} + \beta_{u_{11}} + \beta_{d_2} = \beta_{d_1} + \beta_c + \beta_{d_2}$; then, as S_{14} is not tight, the same holds for S_{20}. Since S_{21} is tight and S_{11} is not tight, then $\beta_{u_{11}} > \beta_a$. As S_{10} is tight and $S_{10} \setminus \{a, b_2\} \cup \{u_{11}, d_2\}$ is feasible, then $\beta_a + \beta_{b_2} \geq \beta_{u_{11}} + \beta_{d_2}$; consequently $\beta_{b_2} > \beta_{d_2}$, S_6 is not tight and S_5 is tight (because it is the only solution containing h_1).

To summarize we are left with only 12 configurations which might be tight: S_5, S_8, S_{10}, S_{12}, S_{21}, S_{22}, S_{23}, S_{24}, S_{25}, S_{26}, S_{27}, S_{28}. As (β, β_0) is a facet defining inequality, then the incidence vectors of the tight configurations must define a matrix with rank equal to 12. Therefore the above configurations are all tight for (β, β_0).

Claim 2. (β, β_0) *is a convex combination of an inequality* $(\beta_{G\setminus B_Y}, \gamma'_{B_Y}, \gamma'_0)$ *with* $\gamma'_u = q$ *for all* $u \in V_{B_Y}$ *and an inequality of type* $(\beta_{G\setminus B_Y}, \gamma''_{B_Y}, \gamma''_0)$ *with* $\gamma''_u = q$ *if* $u \in \{d_1, b_2, u_{11}\}$ *and* $\gamma''_u = 0$ *if* $u \in V_{B_Y} \setminus \{d_1, b_2, u_{11}\}$*, where* $q > 0$.

The claim follows from Claim 1 by horizonthal symmetry. *(End of Claim 2)*

As a consequence of the previous claims, we have that also Case d) yields a contradiction. Thus the theorem follows. ∎

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