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THE STABLE SET POLYTOPE OF CLAW-FREE GRAPHS I: XX-STRIP COMPOSITION VERSUS GEAR COMPOSITION

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Abstract

The *gear composition* builds a new graph G by substituting a suitable edge of a given graph H with a fixed graph named *gear*. Here we extend this definition to obtain an operation that is suitable to handle claw-free graphs. We call *geared (fuzzy) line graphs* the graphs obtained from (fuzzy) line graphs by repeated applications of the extended gear composition and we prove that these graphs form a significant subclass of claw-free graphs with stability number at least four. The proof is based on the decomposition theorem of Chudnovsky and Seymour [1, 2].

We also show how the *extended gear composition* generates facet defining inequalities for the stable set polytope of a geared graph G. In a sequel we prove that these facet defining inequalities yield the complete linear description of the stable set polytope of geared (fuzzy) line graphs.

Key words: stable set polytope, graph composition, claw-free graphs.

1. Introduction

Given a graph $G = (V, E)$ and a vector $w \in \mathbb{Q}_+^V$ of node weights, the *stable set problem* is the problem of finding a set of pairwise nonadjacent nodes *(stable set)* of maximum weight. Let $\alpha(G, w)$ denote the maximum weight of a stable set of G; we refer to $\alpha(G) = \alpha(G, \mathbb{1})$ ($\mathbb{1}$ being the vector of all ones) as the *stability number* of G. The *stable set polytope*, denoted by $STAB(G)$, is the convex hull of the incidence vectors of the stable sets of G. Since the stable set problem is NP-hard, it is unlikely to find a defining linear system of $STAB(G)$ for general graphs. Nevertheless the study of the stable set polytope of claw-free graphs attracts the scientific curiosity since decades.

The interest for the stable set polytope of claw-free graphs lies back to 70's when the result of Edmonds on the matching polytope [4] was translated for the stable set polytope of line graphs (a line graph $L(G)$ of a graph G is obtained by considering the edges of G as nodes of $L(G)$ and two nodes of $L(G)$ are adjacent if and only if the corresponding edges of G have a common endnode).

After that fundamental result, the attention moved towards classes of graphs that properly contain line graphs: in particular, the class of *claw-free graphs*, i.e., the graphs such that the neighborhood of each node has stability number at most two, and the class of *quasi-line graphs*, i.e., the graphs such that the neighborhood of each node can be partitioned into two cliques. Notice that the class of claw-free graphs properly contains the class of quasi-line graphs. A number of conjectures were posed on the inequalities that are facet defining for $STAB(G)$ when G is claw-free or quasi-line but none of them was able to capture their structure [9, 20].

The interest for the study of the stable set polytope of claw-free graphs revived in late 80's after Grötschel, Lovàsz and Schrijver proved the equivalence of the separation and the optimization problems over polyhedra. Then it was clear that the case of claw-free graphs constituted an anomaly as they also remarked in [10]: in fact, for all classes of graphs for which it is known a polynomial time algorithm to solve the weighted stable set problem, it is also known a complete linear description of the stable set polytope (see bipartite graphs, line graphs [4], series-parallel graphs [12] or perfect graphs), apart from claw-free graphs. It is worth noticing that a polynomial time algorithm to solve the weighted stable set problem for claw-free graphs was known since 1980 [13, 14] (see also [18]).

We had to wait for the decomposition theorem of claw-free graphs of Chudnovsky and Seymour [1, 2] in 2004 to start to understand the structure of their stable set polytope and to find answers to so many questions. This decomposition theorem states that the class of claw-free graphs is the disjoint union of different classes of graphs having very specific features. In particular, Chudnovsky and Seymour proved that every claw-free graph that does not admit a 1-join either has stability number at most 3 or it is fuzzy circular interval or it can be obtained by composing three types of graphs, called *strips*: fuzzy linear interval strips, fuzzy XX-strips and fuzzy antihat strips. Moreover, they proved that quasi-line graphs are either composition of fuzzy linear interval graphs or fuzzy circular interval graphs [1]. Let $\mathcal{Q}^\ell$ denote the set of quasi-line graphs that are composition of fuzzy linear interval strips and $\mathcal{Q}^c$ denote the set of quasi-line graphs that are fuzzy circular interval. We call *striped graphs* the claw-free graphs that are obtained by composing fuzzy linear interval strips, fuzzy XX-strips and fuzzy antihat strips, and we denote the class of striped graphs by $\mathcal{C}^s$. Thus the Chudnovsky-Seymour decomposition states that every claw-free graph with stability number at least 4 and without 1-joins belongs either to $\mathcal{Q}^c$ or to $\mathcal{C}^s$. This result partially explains why it was so hard to treat with $STAB(G)$ as a whole polytope and suggests that, in order to find a linear description of $STAB(G)$ for claw-free graphs, it is convenient to study the facet defining inequalities separately for each of the subclasses of claw-free graphs they identify.

A defining linear system for $STAB(G)$ was given by Chudnovsky and Seymour [1] when $G \in \mathcal{Q}^\ell$ and by Eisenbrand et al. [5] when $G \in \mathcal{Q}^c$. It remains open the problem of finding a linear description for $STAB(G)$ when $G \in \mathcal{C}^s$ or G has stability number equal to 3. The case with $\alpha(G) = 2$ was solved by Cook (see [19]), while for the case $\alpha(G) = 3$ the roots of the facet defining inequalities of $STAB(G)$ have been studied in [17].

In this paper we consider a subclass of the striped graphs, namely those graphs that are composition of only two types of strips: fuzzy linear interval strips and XX-strips. We call these graphs *XX-graphs*. For this class of graphs we provide a decomposition that is alternative to the one defined by Chudnovsky and Seymour. This new decomposition is based on an extension of the *gear composition* introduced in [8] and it builds a graph G, called *geared graph*, by substituting a suitable edge of a given graph H with the fixed graph (*extended gear*) shown in Fig. 5.

We also provide a list of inequalities that are generated by the *extended gear composition* and that are facet defining for the stable set polytope of a geared graph. These inequalities turn out to be crucial in defining the facial structure of the stable set polytope of XX-graphs. In fact, they constitute the building blocks of a larger class of inequalities, called *multiple geared inequalities*, that has to be added to rank inequalities and lifted 5-wheel inequalities in order to have a complete linear description of $STAB(G)$ when G is an XX-graph. This linear system represents the first step towards the solution of the longstanding open question of finding a defining linear system for the stable set polytope of *genuine* claw-free graphs, namely claw-free graphs that are not quasi-line.

In Section 2 we recall the results in [7]. In Section 3, we define the *extended gear composition* and we prove that any graph XX-graph can be built from a graph in $\mathcal{Q}^\ell$ by iteratively applying the extended gear composition instead of the XX-strip composition defined by Chudnovsky and Seymour [1] (for other recent decompositions of claw-free graphs with stability number at least 4 see [15, 11]). Finally, in Section 4, we provide a list of inequalities that are facet defining for the stable set polytope of a geared graph. These inequalities constitute the base step of the recursive proof that will provide a linear description of $STAB(G)$ when G is an XX-graph (this result will appear in a companion paper).

2. Preliminaries

We now introduce some notations and basic definitions. We denote by G any finite, connected graph with node set V_G and edge set E_G. An edge $e \in E_G$ with endnodes u and v will be denoted by uv. Given a vector $\beta \in \mathbb{R}^m$ and a subset $S \subseteq \{1, \ldots, m\}$, define $\beta_S \in \mathbb{R}^{|S|}$ as the subvector of β restricted on the indices of S and $x^S \in \mathbb{R}^m$ as the incidence vector of S. Moreover let $\beta(S) = \sum_{i \in S} \beta_i$. A linear inequality $\sum_{j \in V_G} \pi_j x_j \leq \pi_0$ is *valid* for $STAB(G)$ if it holds for all $x \in STAB(G)$. For short, we also denote a linear inequality $\pi^T x \leq \pi_0$ as (π, π_0). A valid inequality for $STAB(G)$ *defines* a facet of $STAB(G)$ if and only if it is satisfied as an equality by $|V_G|$ affinely independent incidence vectors of stable sets of G (called *roots* or *tight solutions*). We also say that a stable set S is tight for (β, β_0) if x^S is a tight solution of (β, β_0).

We denote by $\delta(v)$ the set of edges of G having v as endnode and by $N(v)$ the set of nodes of V_G adjacent to v. We also denote by $G \setminus A$ the subgraph of G induced by $V_G \setminus A$ where $A \subseteq V_G$ and by $G \setminus e$ ($G + e$) the subgraph of G obtained by removing (adding) the edge e. A clique-cutset of G is a complete subgraph whose removal disconnects G.

We say that G admits a 1-join if V_G can be partitioned into four sets A_1, B_1, A_2, B_2 such that $A_1 \cup A_2$ is a clique, B_1 and B_2 are nonempty, and the only edges between $A_1 \cup B_1$ and $A_2 \cup B_2$ are those between A_1 and A_2. Clearly, if G admits a 1-join then $A_1 \cup A_2$ is a clique-cutset of G.

A *k-hole* $C_k = (v_1, v_2, \ldots, v_k)$ is a chordless cycle of length k; a *k-antihole* $\overline{C}_k$ is the complement of a k-hole. A *k-antiwheel* $W = (h : \overline{C}_k)$ is a graph consisting of a k-antihole $\overline{C}_k$ and a node h (*hub* of W) adjacent to every node of $\overline{C}_k$. If $k = 3$, the 3-antiwheel is called *claw* and denoted by $(y : w_1, w_2, w_3)$, where y is the center of the claw. If $k = 5$, then $\overline{C}_5$ is isomorphic to C_5 and we refer to W as a *5-wheel*. We recall here a result of Fouquet on claw-free graphs that will be used in the following:

Theorem 2.1. *[6] Let G be a connected claw-free graph with stability number at least four. Then G contains no odd-antiwheel of length greater than five.*

It is not difficult to see that a claw-free graph is quasi-line if and only if the neighborhood of each node does not contain an odd-antihole. Hence, the above theorem implies that a claw-free graph with stability number at least four is quasi-line if and only if it does not contain a 5-wheel W. Notice also that the inequality $\sum_{i=1}^{5} x_{v_i} + 2x_h \leq 2$ is facet defining for $STAB(W)$ and it is called *5-wheel inequality*.

A *gear* B is a graph of eight nodes $\{a, b_1, b_2, c, d_1, d_2, h_1, h_2\}$ such that $W_1 = (h_1 : a, d_1, b_1, c, h_2)$ and $W_2 = (h_2 : a, d_2, b_2, c, h_1)$ are 5-wheels (see Fig. 1); moreover, the edges of these wheels are the only edges of B.

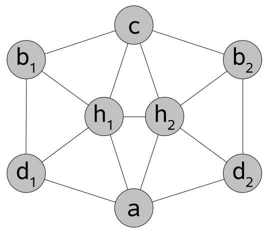

Figure 1: The gear with nodes $h_1, h_2, a, b_1, b_2, c, d_1, d_2$.

In the remaining of this section we recall the definition of *gear composition* given in [8] and some of the polyhedral properties of this composition [7].

A node v of a graph H is said to be *simplicial* if its neighborhood induces a clique of H. An edge v_1v_2 of a graph H is said to be *simplicial* if $K_1 = N(v_1) \setminus \{v_2\}$ and $K_2 = N(v_2) \setminus \{v_1\}$ are cliques of H. Notice that K_1 and K_2 might have nonempty intersection.

Definition 2.2. *Let $H = (V_H, E_H)$ be a graph with a simplicial edge $e = v_1v_2$ and let $B = (V_B, E_B)$ be a gear. The* gear composition *of H and B along e generates a new graph G such that:*

$V_G = V_H \setminus \{v_1, v_2\} \cup V_B,$
$E_G = E_H \setminus (\delta(v_1) \cup \delta(v_2)) \cup E_B \cup F_1 \cup F_2,$ *where* $F_i = \{d_iu | u \in K_i\} \cup \{b_iu | u \in K_i\}$ *for* $i = 1, 2.$

The graph G is called the geared graph *generated by H and B along e and denoted by $G = (H, B, e)$.*

A sketch of how the gear composition works is shown in Fig. 2.

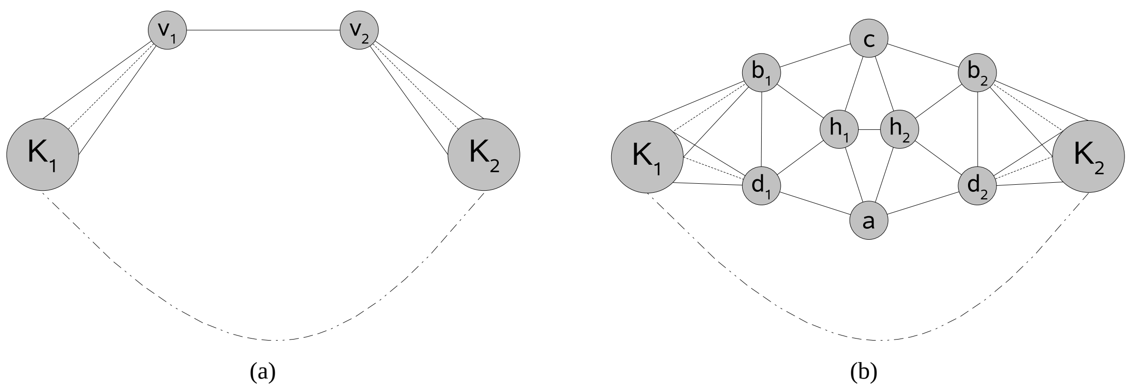

Figure 2: (a) A graph H with a simplicial edge v_1v_2; (b) The geared graph $G = (H, B, v_1v_2)$.

Definition 2.3. *Let H be a graph with a simplicial edge $e = v_1v_2$. Let H^e be the graph obtained from H by subdividing the edge e with a new node t.*

An inequality (π, π_0) *which is valid for* $STAB(H)$ *is said to be* g-extendable *(with respect to* e*) if* $\pi_{v_1} = \pi_{v_2} = \lambda > 0$ *and it is not the inequality* $x_{v_1} + x_{v_2} \leq 1$.

An inequality (π, π_0) *which is valid for* $STAB(H^e)$ *is said to be* g-liftable *(with respect to* e*) if* $\pi_{v_1} = \pi_{v_2} = \pi_t = \lambda > 0$.

Definition 2.4. *Let* $H = (V_H, E_H)$ *be a graph containing the simplicial edge* $e = v_1v_2$*, let* $B = (V_B, E_B)$ *be a gear and let* (π, π_0) *be a valid inequality for* $STAB(H)$ *that is g-extendable with respect to* e*. Then the inequalities*

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B \setminus \{h_1, h_2\}} x_i + 2\lambda(x_{h_1} + x_{h_2}) \leq \pi_0 + 2\lambda \tag{1}$$

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B \setminus A} x_i \leq \pi_0 + \lambda \tag{2}$$

where $A \in \{\{b_1, c\}, \{b_2, c\}, \{d_1, a\}, \{d_2, a\}, \{a, c\}\}$ *is the set of nodes of* V_B *with zero coefficient*

are called geared inequalities *associated with* (π, π_0)*. The unique geared inequality that has full support on* V_B *is (1) and it is called* proper geared inequality.

In [8, 7] it was proved that:

Theorem 2.5. *Let* $G = (H, B, e)$ *be a geared graph and let* (π, π_0) *be a g-extendable inequality of* $STAB(H)$*. If* (π, π_0) *is facet defining for* $STAB(H)$*, then the geared inequalities (1) and (2) associated with* (π, π_0) *are facet defining for* $STAB(G)$*.*

Example 2.1. Consider a 5-hole C_5, a gear B and the geared graph G obtained as the gear composition of C_5 and B along the simplicial edge $e = v_1v_2$ (see Fig. 3). Thus, we write $G = (C_5, B, e)$.

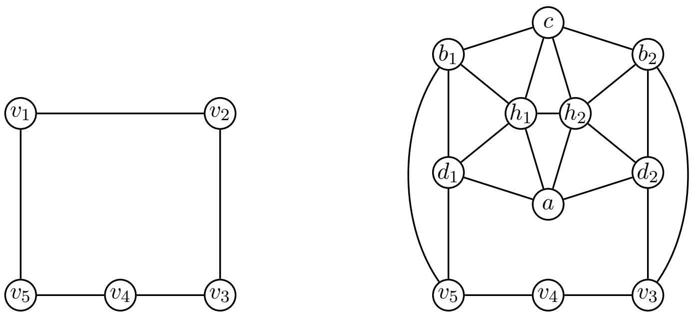

Figure 3: A 5-hole C_5 and a geared 5-hole G

As the 5-hole inequality $x(V_{C_5}) \leq 2$ is valid for $STAB(C_5)$ and it is g-extendable with respect to e, the following inequality

$$x(V_G \setminus \{h_1, h_2\}) + 2x_{h_1} + 2x_{h_2} \leq 4$$

is a proper geared inequality associated with $x(V_{C_5}) \leq 2$. Since $x(V_{C_5}) \leq 2$ is facet defining for $STAB(C_5)$, it follows that the above inequality is facet defining for $STAB(G)$ by Theorem 2.5. Furthermore, the following five inequalities

$$x(V_G \setminus A) \leq 3, \text{ where } A \in \{\{b_1, c\}, \{b_2, c\}, \{d_1, a\}, \{d_2, a\}, \{a, c\}\},$$

are non-proper geared inequality associated with $x(V_{C_5}) \leq 2$ and they are all facet defining for $STAB(G)$ by Theorem 2.5. □

Definition 2.6. *Let $H = (V_H, E_H)$ be a graph containing the simplicial edge $e = v_1v_2$, let $B = (V_B, E_B)$ be a gear and let (π, π_0) be a valid inequality for $STAB(H^e)$ that is g-liftable with respect to e. Then the inequalities*

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B} x_i \leq \pi_0 + \lambda, \tag{3}$$

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B \setminus A} x_i \leq \pi_0 \tag{4}$$

where $A \in \{\{b_1, c, b_2, h_1, h_2\}, \{d_1, a, d_2, h_1, h_2\}\}$ is the set of nodes of V_B with zero coefficient

are called g-lifted inequalities *associated with (π, π_0). The unique g-lifted inequality that is full support on V_B is (3) and it is called* proper g-lifted inequality.

A result analogous to Theorem 2.5 was proved for the g-lifted inequalities [7]:

Theorem 2.7. *Let $G = (H, B, e)$ be a geared graph and let (π, π_0) be a g-liftable inequality of $STAB(H^e)$. If (π, π_0) is facet defining for $STAB(H^e)$, then the g-lifted inequalities (3) and (4) associated with (π, π_0) are facet defining for $STAB(G)$.*

Example 2.2. Consider a 4-hole C_4, a gear B and the geared graph G obtained as the gear composition of C_4 and B along the simplicial edge $e = v_1v_2$ (see Fig. 4). Thus, we write $G = (C_4, B, e)$.

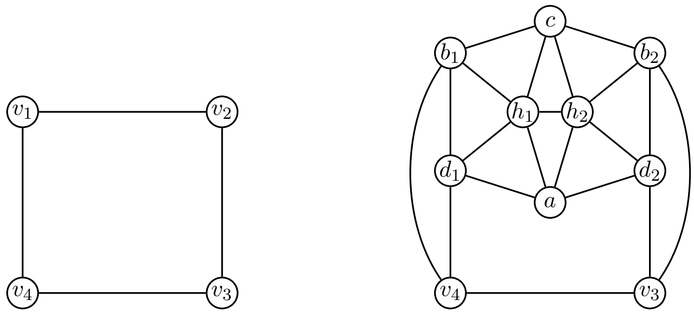

Figure 4: A 4-hole C_4 and a geared 4-hole G

The subdivision of the simplicial edge $e = v_1v_2$ with a new node t generates a 5-hole C_4^e. Since $x(V_{C_4^e}) \leq 2$ is valid for $STAB(C_4^e)$ and it is g-liftable with respect to e, the following inequality

$$x(V_G) \leq 3$$

is a proper g-lifted inequality associated with $x(V_{C_4^e}) \leq 2$. Since $x(V_{C_4^e}) \leq 2$ is facet defining for $STAB(C_4^e)$, it follows that the above inequality is facet defining for $STAB(G)$ by Theorem 2.7. Moreover, the following two inequalities

$$x(V_G \setminus A) \leq 3 \text{ where } A \in \{\{b_1, c, b_2, h_1, h_2\}, \{d_1, a, d_2, h_1, h_2\}\}.$$

are non-proper g-lifted inequalities associated with $x(V_{C_4^e}) \leq 2$ and they are both facet defining for $STAB(G)$ by Theorem 2.7. □

In [7] we showed that, if G is a geared graph, the linear description of $STAB(G)$ is completely determined by the linear description of $STAB(H)$ and $STAB(H^e)$. Indeed we proved the following:

Theorem 2.8. *[7] Let $G = (H, B, e)$ be a geared graph generated by H and B along a simplicial edge e. Then the stable set polytope $STAB(G)$ is described by the following linear inequalities:*

- *nonnegativity inequalities,*
- *clique inequalities,*
- *(lifted) 5-wheel inequalities,*
- *geared inequalities of type (1) or (2) associated with facet defining inequalities of $STAB(H)$ having nonzero coefficients on the endnodes of e,*
- *g-lifted inequalities of type (3) or (4) associated with facet defining inequalities of $STAB(H^e)$ having nonzero coefficients on the endnodes of e,*
- *facet defining inequality of $STAB(H)$ having zero coefficients on the endnodes of e.*

This result implies that, if H and H^e "behave well" with respect to the stable set problem (meaning that there exists a defining linear system for their stable set polytopes) then the geared graph obtained from H does the same. In the following we extend the polyhedral properties of the gear composition to a more general operation to prove that a large subclass of claw-free graphs behave well with respect to the stable set problem.

3. Geared (fuzzy) line graphs

Chudnovsky and Seymour [1] proved that claw-free, not quasi-line graphs with stability number at least 4 that do not admit a 1-join are *striped graphs*, meaning that they are obtained by composing only three kinds of graphs, called *strips*. One kind of strips, the *linear interval strips*, is used to generate quasi-line graphs; the other two kinds of strips, the *XX-strips* and the *antihat-strips*, contain 5-wheels and so they are necessary to generate claw-free graphs which are not quasi-line.

This structure suggests the idea that striped graphs are not so distant from line graphs in terms of polyhedral description of their stable set polytopes. In this paper and in its sequel we give an evidence of this fact by showing that a defining linear system for the stable set polytope of a large subclass of striped graphs is built starting from the defining linear system of the stable set polytope of line graphs using only the gear composition. We actually conjectured that this holds for all striped graphs [7].

Before stating the decomposition theorem of Chudnovsky and Seymour we recall some of their definitions that will be used in the following (see [1] for further details):

Definition 3.1. *A* strip (G, a, b) *consists of a claw-free graph G together with two designated simplicial vertices a, b called the* ends *of the strip. Two strips can be composed as follows: let A and B be the nodes of $G \setminus \{a, b\}$ adjacent in G to a and b respectively, and define A' and B' similarly. Take the disjoint union of $G \setminus \{a, b\}$ and $G' \setminus \{a', b'\}$; and let H be the graph obtained from this by adding all possible edges between A and A' and between B and B'.*

Definition 3.2. *Let T be a graph with node set $\{u_1, \ldots, u_{13}\}$ and with adjacency as follows. $(u_1, \ldots, u_6)$ is a hole of G of length 6. Next, u_7 is adjacent to u_1, u_2; u_8 is adjacent to u_4, u_5; u_9 is adjacent to u_6, u_1, u_2, u_3; u_{10} is adjacent to u_3, u_4, u_5, u_6, u_9; u_{11} is adjacent to $u_3, u_4, u_6, u_1, u_9, u_{10}$; u_{12} is adjacent to $u_2, u_3, u_5, u_6, u_9, u_{10}$; u_{13} is adjacent to $u_1, u_2, u_4, u_5, u_7, u_8$. Let $X \subseteq \{u_{11}, u_{12}, u_{13}\}$; then the strip $(T \setminus X, u_7, u_8)$ is called an XX*-strip.

Definition 3.3. *A* homogeneous pair of cliques *in G is a pair (A, B) such that:*

- *A and B are cliques in G and $A \cap B = \emptyset$,*
- *$|A| \geq 2$ or $|B| \geq 2$,*
- *no node of $G \setminus (A \cup B)$ has both a neighbour and a non-neighbour in A, and the same in B.*

We can now state the decomposition theorem of Chudnovsky and Seymour; the decomposition involves the antihat strips, but we omit their definition since they will never be used in the following.

Theorem 3.4. *[1] For every claw-free graph G with $\alpha(G) \geq 4$, if G does not admit a 1-join and there is no homogeneous pair of cliques in G, then either G is a circular interval graph, or G is a composition of linear interval strips, XX-strips, and antihat strips.*

Since graphs containing homogeneous pairs of cliques cannot be represented with the above strips, Chudnovsky and Seymour introduced the concept of *fuzziness* and gave a "fuzzy" version of the above theorem where all the strips are actually fuzzy strips. In the following the attention will be focused on fuzzy linear interval graphs and fuzzy circular interval graphs; so, we recall an alternative definition of fuzziness valid only for these two classes of graphs (for further details on this definition we refer the interested reader to [1]).

Definition 3.5. *A graph G is said to be fuzzy circular interval if:*

1. *there is a map ϕ from V_G to a circle C, and*
2. *there is a family $\mathcal{I}$ of intervals of C (none including another) such that no point of C is an end of more than one interval, so that*
3. *for $u, v \in G$, if $uv \in E_G$ then $\{\phi(u), \phi(v)\}$ is a subset of one interval of $\mathcal{I}$, and if $uv \notin E_G$ then $\phi(u)$ and $\phi(v)$ are both ends of any interval of $\mathcal{I}$ containing both of them (and in particular, if $\phi(u) = \phi(v)$ then u and v are adjacent).*

Fuzzy linear interval graphs are defined analogously with the circle replaced by a line L.

Hence, if $uv \notin E_G$ two possibilities may occur: either $\{\phi(u), \phi(v)\}$ is not a subset of any interval of $\mathcal{I}$ or $[\phi(u), \phi(v)]$ is an interval of $\mathcal{I}$. Moreover, if x and y are both ends of an interval of $\mathcal{I}$ and one of the sets $\phi^{-1}(x)$, $\phi^{-1}(y)$ has at least two nodes of V_G, then the pair $(\phi^{-1}(x), \phi^{-1}(y))$ is a homogeneous pair of cliques.

Chudnovsky and Seymour proved the following structure theorem for quasi-line graphs:

Theorem 3.6. *[1] Let G be a connected quasi-line graph. Then G is either a fuzzy circular interval graph or a strip composition of fuzzy linear interval strips.*

In this paper we are interested in claw-free graphs with stability number at least 4 that do not admit a 1-join. Observe that the requirement that G does not admit a 1-join is not at all restrictive for our purposes. In fact, when looking for facet defining inequalities of the stable set polytope, one can always restrict to study graphs that do not contain clique-cutsets since a well-known result of Chvátal states that:

Theorem 3.7. *[3] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Let $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ and $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$. If $G_1 \cap G_2$ is a complete graph, then the defining linear system of $STAB(G_1 \cup G_2)$ is given by the union of the defining linear systems of $STAB(G_1)$ and $STAB(G_2)$.*

So, we denote by $\mathcal{Q}^c$ the class of fuzzy circular interval graphs and by $\mathcal{Q}^\ell$ the class of graphs that are a composition of fuzzy linear interval strips. In the following we shall refer to graphs in $\mathcal{Q}^\ell$ as *fuzzy line graphs*. Hence Theorem 3.6 implies that the set of quasi-line graphs is the union of $\mathcal{Q}^c$ and $\mathcal{Q}^\ell$.

It is worth noticing that the fuzziness does not seem to have much relevance from the polyhedral point of view. This was already noticed by Chudnovsky and Seymour who proved that:

Theorem 3.8. *[1] If G is a fuzzy line graph, then $STAB(G)$ is described by the Edmonds' inequalities.*

And it was further confirmed by the work of Eisenbrand et al. on fuzzy circular interval graphs.

Lemma 3.9. *[5] Let F be a facet of $STAB(G)$ where G is a fuzzy circular interval graph. Then F is also a facet of $STAB(G')$, where G' is a circular interval graph obtained from G by removing some edges.*

Thus, the "fuzzy" version of Theorem 3.4 together with Theorem 3.6 and Theorem 3.7 imply that: finding a linear description of $STAB(G)$ for claw-free, not quasi-line graphs with stability number at least 4 and without 1-joins amounts to finding a linear description of $STAB(G)$ for graphs that are composition of fuzzy linear interval strips, fuzzy XX-strips and fuzzy antihat strips. Here, we focused on graphs that are

composition of XX-strips and fuzzy linear interval strips.

We call these graphs *XX-graphs* and denote their family as $\mathcal{XX}$.

In the following we show that any XX-graph can be obtained by iteratively applying an extended version of the gear composition defined in Section 2 to a (fuzzy) line graph, namely to a graph in $\mathcal{Q}^\ell$. We write the word fuzzy inside parenthesis to mean that our results hold also without the fuzziness. We start by showing the connection between the XX-strip composition and the gear composition.

Lemma 3.10. *The graph obtained by composing a strip (G, v_1, v_2) with the XX-strip $(T\setminus\{u_{11}, u_{12}, u_{13}\}, u_7, u_8)$ is a geared graph.*

Proof. Rename the nodes $\{u_6, u_2, u_4, u_3, u_1, u_5, u_9, u_{10}\}$ of the XX-strip $(T\setminus\{u_{11}, u_{12}, u_{13}\}, u_7, u_8\})$ as the nodes $\{a, b_1, b_2, c, d_1, d_2, h_1, h_2\}$ of a gear B. Thus, the strip composition of (G, v_1, v_2) and the XX-strip $(T \setminus \{u_{11}, u_{12}, u_{13}\}, u_7, u_8\})$, as defined in Definition 3.1, corresponds to the gear composition of $G' = (V_G, E_G \cup \{v_1v_2\})$ and the gear B along the edge v_1v_2. In fact, being the nodes v_1 and v_2 simplicial, we have that the edge v_1v_2 of G' is simplicial. The graph obtained by applying the above strip composition is precisely the geared graph (G', B, v_1v_2). ∎

As a consequence of the above lemma and Definition 3.2 we have that an XX-strip composition produces a geared graph $G = (H, B, e)$ plus an extra set of nodes contained in $\{u_{11}, u_{12}, u_{13}\}$ which are suitably adjacent to B. This, together with Theorem 3.4, implies that a large number of claw-free graphs can be seen as geared graphs. We now prove that we can restrict ourselves to consider only XX-strips not containing node u_{13} since this node can be added using an appropriate linear interval strip.

Lemma 3.11. *The class of XX-graphs coincides with the subclass of claw-free graphs that are compositions of XX-strips of type $(T \setminus \{u_{13}\}, u_7, u_8)$ and (fuzzy) linear interval strips.*

Proof. Let G be an XX-graph obtained by composing a strip (L, v_1, v_2) and an XX-strip $(T \setminus X, u_7, u_8)$. Suppose that $u_{13} \notin X$. Consider the linear interval strip (L', a_0, b_0) such that $V_{L'} = \{v_1', v_2', u_{13}, a_0, b_0\}$ and $E_{L'}$ consists of the edges of the two triangles (v_1', u_{13}, a_0) and (v_2', u_{13}, b_0). It is trivial to see that the graph L'' obtained by composing the strip (L, v_1, v_2) with the linear interval strip (L', a_0, b_0) has v_1' and v_2' as simplicial nodes. Thus (L'', v_1', v_2') is a strip and its composition with the XX-strip $(T \setminus (X \cup \{u_{13}\}), u_7, u_8)$ yields the graph G. Thus the lemma follows. ∎

From the above lemmas it follows that the gear composition may be viewed as an XX-strip composition where $X = \{u_{11}, u_{12}\}$. To consider also the case when $X \subset \{u_{11}, u_{12}\}$, we need to extend our definition of gear composition as follows:

Definition 3.12. *Let $B = (V_B, E_B)$ be a gear and let u_{11} and u_{12} be two new nodes such that u_{11} is adjacent to $\{d_1, a, h_1, h_2, c, b_2\}$ and u_{12} is adjacent to $\{d_2, a, h_1, h_2, c, b_1\}$. Let $Y \subseteq \{u_{11}, u_{12}\}$ and $\delta(Y) = \bigcup_{u \in Y} \delta(u)$.*
An extended gear B_Y is a graph with node set $V_B \cup Y$ and edge set $E_B \cup \delta(Y)$ (see Fig. 5).

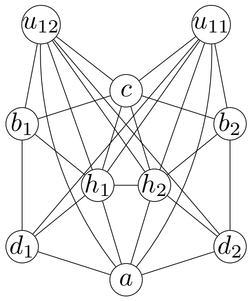

Figure 5: The extended gear B_Y with $Y = \{u_{11}, u_{12}\}$.

Let $H = (V_H, E_H)$ be a graph with a simplicial edge $e = v_1v_2$ and let B_Y be an extended gear where $Y \subseteq \{u_{11}, u_{12}\}$.
The extended gear composition *of H and B_Y along v_1v_2 is a gear composition where the gear B is replaced by B_Y. The graph G generated by the extended gear composition of H and B_Y along e will still be called* geared graph *and denoted by (H, B_Y, e).*

In order to prove that the extended gear composition is equivalent to the XX-strip composition, we have to guarantee that the removal of a simplicial edge preserves the property of a graph of being (fuzzy) line. To do so, we consider only simplicial edges that have the following property:

Definition 3.13. *A simplicial edge $e = v_1v_2$ is* super simplicial *if $N(K_1 \cap K_2) \subseteq N(v_1) \cup N(v_2)$.*

Lemma 3.14. *Let $e = v_1v_2$ be a super simplicial edge of H. Then H is a (fuzzy) line graph if and only if $(H \setminus e, v_1, v_2)$ is a strip and $H \setminus e$ is a (fuzzy) line graph. Moreover, if H is (fuzzy) line then H^e is (fuzzy) line.*

Proof. First we prove the "if" direction. It suffices to observe that H is obtained by composing the strip $(H \setminus e, v_1, v_2)$ with a linear interval strip (P, a_0, b_0) consisting of a path $P = (a_0, u_1, u_2, b_0)$ and then renaming the nodes u_i as v_i, $i = 1, 2$.

To prove the "only if" direction observe that any (fuzzy) line graph is a quasi-line graph. Hence to prove that $H \setminus e$ is a (fuzzy) line graph we must first show that it contains neither a claw nor an odd-antiwheel. Suppose by contradiction that $H \setminus e$ contains a claw C. Since the only edge which was removed from H is $e = v_1v_2$, we have that C contains both v_1 and v_2. So, $C = (y : v_1, v_2, w)$ with $y \in K_1 \cap K_2$ and $w \in N(y) \setminus (N(v_1) \cup N(v_2))$, contradicting the hypothesis that e is super simplicial.

Suppose now by contradiction that $H \setminus e$ contains a 5-wheel W. Since H is quasi-line, it follows that v_1 and v_2 must be two non adjacent nodes of the rim, i.e., $W = (y : v_1, w_1, v_2, w_2, w_3)$. Thus v_1v_2 is not a simplicial edge of H as $N(v_1) \setminus \{v_2\}$ contains w_1 and w_3 which are not adjacent, a contradiction. A similar argument shows that if $H \setminus e$ contains an odd-antiwheel of length greater than or equal to 7 then the edge e was not simplicial in H, a contradiction.

Thus $H \setminus e$ is quasi-line and, by Theorem 3.6, it belongs either to $\mathcal{Q}^\ell$ or to $\mathcal{Q}^c$.

Claim. If $H \setminus e \in \mathcal{Q}^c$, then $H \in \mathcal{Q}^c$.

Since $H \setminus e \in \mathcal{Q}^c$, there exists a map ϕ that assigns an ordering to the nodes of $K_1 \cup K_2 \cup \{v_1, v_2\}$ on the circle C and a family of intervals $\mathcal{I}$ of C satisfying 2) and 3) of Definition 3.5. Since $K_i \cup \{v_i\}$ $(i = 1, 2)$ are cliques there exist intervals $I_i \in \mathcal{I}$ containing all points $\phi(u)$ with $u \in K_i \cup \{v_i\}$.

Notice that every node in $K_1 \cap K_2$ (if exists) is adjacent only to nodes in $K_1 \cup K_2 \cup \{v_1, v_2\}$ since e is super simplicial. This implies that it is possible to find a new function ϕ' that maps the elements in $K_1 \cap K_2$ on the circle C so that $\phi'(v_1) < x_r < x_{r+1} < \ldots < x_s < \phi'(v_2)$ where $x_i = \phi'(u_i)$ and $K_1 \cap K_2 = \{u_i | i = r, \ldots, s\}$ while all the nodes of $K_1 \setminus K_2$ ($K_2 \setminus K_1$) are mapped before $\phi'(v_1)$ (after $\phi'(v_2)$, respectively) along the circle C.

Since v_1 and v_2 are not adjacent in $H \setminus e$, one of the following two possibilities may occur: either there does not exist an interval containing both $\phi'(v_1)$ and $\phi'(v_2)$ or any interval containing both $\phi'(v_1)$ and $\phi'(v_2)$ has $\phi'(v_1)$ and $\phi'(v_2)$ as its ends. In both cases it suffices to add to $\mathcal{I}$ an interval containing both $\phi'(v_1)$ and $\phi'(v_2)$ in order to have a feasible circular representation of H, and the claim follows.

(End of Claim)

Since H is (fuzzy) line by hypothesis, i.e., $H \in \mathcal{Q}^\ell$, the above Claim implies that $H \setminus e$ is (fuzzy) line. Moreover, since e is a super simplicial edge of H its endnodes are simplicial nodes of $H \setminus e$; hence, $H \setminus e$ is a strip.

Finally, to prove that H^e is a (fuzzy) line graph one just needs to compose the strip $(H \setminus e, v_1, v_2)$ with the strip (P, a_0, b_0) where P is the path (a_0, u_1, t, u_2, b_0) and then rename u_i as v_i, $i = 1, 2$. This completes the proof of the lemma. ∎

We now show that the XX-graphs admit a decomposition different from the strip decomposition: they can be obtained by repeated applications of the extended gear composition to a (fuzzy) line graph. To prove this, we first define the geared (fuzzy) line graphs, namely the graphs obtained from a (fuzzy) line graph H by iteratively applying extended gear compositions and edge-subdivisions along super simplicial edges of H.

Definition 3.15. *Let H be a (fuzzy) line graph. Let Γ_H be the set of the super simplicial edges of H and let a* g-operation *on $e \in \Gamma_H$ be either a gear composition or an edge subdivision applied along e. A graph $G \in \mathcal{G}^*_H$ if and only if*

> either $G = H$,
>
> or $G = (L, B_Y, e)$, *where $L \in \mathcal{G}^*_H$, B_Y is an extended gear, and $e \in \Gamma_H \cap E_L$ (i.e., e is a super simplicial edge of both L and H),*
>
> or $G = L^e$, *where $L \in \mathcal{G}^*_H$ and $e \in \Gamma_H \cap E_L$.*

The graphs in $\bigcup_{H \in \mathcal{Q}^\ell} \mathcal{G}^*_H$ are called geared (fuzzy) line graphs.

Notice that in Definition 3.15 the extended gear compositions and the edge-subdivisions are performed only along super simplicial edges of L that are also super simplicial in the original graph H. This implies that in order to generate graphs in $\mathcal{G}^*_H$ we are not allowed to use any of the edges created by an earlier application of the two operations: in particular, the edges $v_1 t$ and $t v_2$, created by an edge-subdivision of $e = v_1 v_2 \in \Gamma_H$, cannot be used to perform any extended gear composition or edge-subdivision. Indeed, they do not belong to Γ_H even though they have the property of being super simplicial. It follows that any graph in $\mathcal{G}^*_H$ is obtained by performing the above operations at most $|\Gamma_H|$ times, thus implying that, for any fixed graph H, the family $\mathcal{G}^*_H$ contains a finite number of graphs.

The following theorem establishes the equivalence of the extended gear composition and the XX-strip composition.

Theorem 3.16. *The geared (fuzzy) line graphs are the XX-graphs, i.e.,* $\bigcup_{H\in\mathcal{Q}^\ell}\mathcal{G}^*_H=\mathcal{XX}$.

Proof. From Definition 3.12 and Lemma 3.10, it is not difficult to see that the extended gear composition is equivalent to an XX-strip composition where X always contains u_{13} (see Definition 3.2). In fact, an extended gear B_Y becomes an XX-strip by simply adding to V_{B_Y} two new nodes, say k_1 and k_2, such that $N(k_i)=\{b_i,d_i\}$ for $i=1,2$. These nodes correspond to the simplicial nodes u_7 and u_8 of the XX-strip (B_Y,k_1,k_2). Thus, the XX-strip composition of (B_Y,k_1,k_2) and $(H\setminus e,v_1,v_2)$ produces the geared graph (H,B_Y,e) where $e=v_1v_2$.

Since, by Lemma 3.11, the XX-strip composition with $u_{13}\in X$ suffices to obtain all XX-graphs, it follows that XX-graphs can be obtained by repeated applications of the extended gear composition to a (fuzzy) line graph $H\in\mathcal{Q}^\ell$, i.e., a graph that is a composition of (fuzzy) linear interval strips.

To prove the opposite, suppose by contradiction that there exists a graph G in $\mathcal{G}^*_H$ for some $H\in\mathcal{Q}^\ell$ which is not an XX-graph. Without loss of generality we assume that G is obtained by performing the smallest number of g-operations. If $G=H$ then, by definition, H is an XX-graph. Hence, either $G=L^e$ or $G=(L,B_Y,e)$ where $e=v_1v_2$ is a super simplicial edge of H.

Suppose first that $G=L^e$. By the minimality of G we know that L is an XX-graph. Since e is a super simplicial edge in $\Gamma_H\cap E_L$ it does not belong to any XX-strip of L. So, we can build a new graph $\tilde{L}$ from L by replacing each XX-strip $(T\setminus X,v_1,v_2)$ with the linear interval strip consisting of the simple path (a_0,v_1,v_2,b_0). It follows that $\tilde{L}$ is a (fuzzy) line graph and, by Lemma 3.14, $\tilde{L}\setminus e$ is also a (fuzzy) line graph. Now we reconstruct $L\setminus e$ from $\tilde{L}\setminus e$ by replacing the simple paths previously introduced with the corresponding XX-strips. Thus $L\setminus e$ is obtained as a composition of XX-strips and (fuzzy) linear interval strips, and so, it is an XX-graph. Since G is obtained by composing the strip $(L\setminus e,v_1,v_2)$ and (a,v_1',t,v_2',b) and renaming v_i' as v_i, $i=1,2$, we have that G is an XX-graph, as claimed.

Consider now the case $G=(L,B_Y,e)$. As above, we can prove that $L\setminus e$ is an XX-graph. If we add to B_Y two nodes k_1 and k_2 adjacent to b_1,d_1 and b_2,d_2, respectively, we have that G is obtained by composing the strips $(L\setminus e,v_1,v_2)$ and the XX-strip $(B_Y\cup\{k_1,k_2\},k_1,k_2)$. Thus the thesis follows. ∎

From the above results it follows that the XX-graphs can be built in two different ways: either using the strip composition defined by Chudnovsky and Seymour in [1] or using the extended gear composition. This result allows us to exploit the polyhedral properties of the gear composition to find a linear description for the stable set polytope of XX-graphs. This will be discussed in the next section.

4. New facet defining inequalities for the stable set polytope of geared graphs

In [7], we showed how the gear composition affects the stable set polytope of a geared graph $G=(H,B,e)$ by proving that the linear description of $STAB(G)$ is completely determined by the linear description of $STAB(H)$ and $STAB(H^e)$. Our aim is to prove a theorem analogous to Theorem 2.8 when the gear composition is replaced by the extended gear composition or, in other words, when $Y\neq\emptyset$. We start by showing that geared inequalities and g-lifted inequalities can be "lifted" to the higher dimensional space containing u_{11} and u_{12} using the *sequential lifting* procedure defined in [16].

Let $\mathscr{S}(G)$ denote the set of stable sets of G. If $\sum_{j\in V_G\setminus\{v\}}\pi_jx_j\leq\pi_0$ is a facet defining inequality of $STAB(G\setminus\{v\})$, then the inequality

$$\sum_{j\in V_G\setminus\{v\}}\pi_jx_j+\pi_vx_v\leq\pi_0\ \ with\ \ \pi_v=\pi_0-\max_{S\in\mathscr{S}(G\setminus(N(v)\cup\{v\}))}\pi(S)$$

is facet defining for $STAB(G)$. This inequality will be called *sequential lifting of* (π,π_0) and π_v will be called the *lifting coefficient of* v. This procedure can be iterated to generate facet defining inequalities, simply called *lifted inequalities*, in a higher dimensional space.

In the rest of the paper we denote as $G_\emptyset, G_{11}$, and G_{12} the graphs generated by the extended gear composition of H and B_Y along e when Y equals the sets $\emptyset, \{u_{11}\}$ and $\{u_{12}\}$, respectively. We start finding the lifting coefficient of u_{11} and u_{12} for inequalities (1), (2), (3), and (4).

Lemma 4.1. *Let (β, β_0) be a proper geared inequality of type (1) that is facet defining for $STAB(G_\emptyset)$. Then the node u_{11} is lifted with coefficient $\beta_{u_{11}} = \lambda$.*

Proof. Let (π_H, π_0) be the facet defining inequality for $STAB(H)$ that generates (β, β_0); then $\beta_0 = \pi_0 + 2\lambda$. By definition of sequential lifting

$$\beta_{u_{11}} = \pi_0 + 2\lambda - \max_{S \in \mathscr{S}(G_\emptyset \setminus N(u_{11}))} \beta(S).$$

There exists a tight stable set T for (π_H, π_0) such that $T \cap (K_2 \cup \{v_2\}) = \emptyset$, since (π_H, π_0) is not the clique inequality supported by $K_2 \cup \{v_2\}$. Thus, $v_1 \in T$ (since otherwise $T \cup \{v_2\}$ would violate (π_H, π_0)) and $\overline{S} = T \setminus \{v_1\} \cup \{b_1, d_2\}$ is a stable set in $G_\emptyset \setminus N(u_{11})$ such that $\beta(\overline{S}) = \pi_H(T \setminus \{v_1\}) + 2\lambda = \pi_0 - \lambda + 2\lambda = \pi_0 + \lambda$. Therefore, $\beta_{u_{11}} \leq \lambda$.

Now, suppose that $\beta_{u_{11}} < \lambda$; then there exists a stable set $S' \in \mathscr{S}(G_\emptyset \setminus N(u_{11}))$ such that $\beta(S') > \pi_0 + \lambda$; then b_1 and d_2 belong to S' (otherwise $S' \cup \{h_1\}$ or $S' \cup \{h_2\}$ would be stable sets violating (β, β_0)). Therefore, $S'' = S' \setminus \{b_1\} \cup \{d_1, c\}$ is a stable set of G_{11} such that $\beta(S'') = \beta(S') + \lambda > \pi_0 + 2\lambda = \beta_0$, a contradiction. Hence, $\beta_{u_{11}} = \lambda$ and the thesis follows. ∎
Similarly, we prove the following two lemmas:

Lemma 4.2. *Let (β, β_0) be a geared inequality of type (2) that is facet defining for $STAB(G_\emptyset)$. Then the node u_{11} is lifted with coefficient $\beta_{u_{11}} = \lambda$ if $A = \{b_1, c\}$ or $A = \{d_2, a\}$, and $\beta_{u_{11}} = 0$ otherwise.*

Proof. Let (π_H, π_0) be the facet defining inequality for $STAB(H)$ generating (β, β_0); then $\beta_0 = \pi_0 + \lambda$. There exists a tight stable set T for (π_H, π_0) such that $T \cap (K_2 \cup \{v_2\}) = \emptyset$. Thus, $v_1 \in T$ and $\overline{S} = T \setminus \{v_1\} \cup \{b_1, d_2\}$ is a stable set in $G_\emptyset \setminus N(u_{11})$. If $A = \{d_1, a\}$, $A = \{b_2, c\}$, or $A = \{a, c\}$, then $\beta(\overline{S}) = \pi_0 + \lambda$ and therefore $\beta_{u_{11}} = 0$. If $A = \{b_1, c\}$ or $A = \{d_2, a\}$, then $\beta(\overline{S}) = \pi_H(T \setminus \{v_1\}) + \lambda = \pi_0 - \lambda + \lambda = \pi_0$. Therefore, $\beta_{u_{11}} \leq \lambda$.

Suppose that $\beta_{u_{11}} < \lambda$; then there exists a stable set $S' \in \mathscr{S}(G_\emptyset \setminus N(u_{11}))$ such that $\beta(S') > \pi_0$; then $b_1, d_2 \in S$ (otherwise $S' \cup \{h_1\}$ or $S' \cup \{h_2\}$ would be stable sets violating (β, β_0)). Therefore $S'' = S' \setminus \{b_1\} \cup \{d_1, c\}$ is a stable set that violates (β, β_0), a contradiction. Thus $\max_{S \in \mathscr{S}(G_\emptyset \setminus N(u_{11}))} \beta(S) = \pi_0$ and $\beta_{u_{11}} = \lambda$. ∎

Lemma 4.3. *Let (β, β_0) be a proper g-lifted inequality (3) that is facet defining for $STAB(G_\emptyset)$. Then the node u_{11} is lifted with coefficient $\beta_{u_{11}} = \lambda$. Moreover, if (β, β_0) is a g-lifted inequality of type (4) that is facet defining for $STAB(G_\emptyset)$, then the node u_{11} is lifted with coefficient $\beta_{u_{11}} = 0$.*

Proof. First, we consider the case of inequality (3). Let (π_{H^e}, π_0) be the facet defining inequality for $STAB(H^e)$ that generates (β, β_0); then $\beta_0 = \pi_0 + \lambda$. There exists a tight stable set T for (π_{H^e}, π_0) such that $T \cap \{t, v_2\} = \emptyset$. Thus, $v_1 \in T$ and $\overline{S} = T \setminus \{v_1\} \cup \{b_1\}$ is a stable set in $G_\emptyset \setminus N(u_{11})$ such that $\beta(\overline{S}) = \pi_{H^e}(T \setminus \{v_1\}) + \lambda = \pi_0$. Therefore, $\beta_{u_{11}} \leq \lambda$.

Suppose now that $\beta_{u_{11}} < \lambda$, then there exists a stable set S' such that $S' \cap N(u_{11}) = \emptyset$, $\beta(S') > \pi_0$. It follows that $b_1, d_2 \in S'$, otherwise $S' \cup \{h_1\}$ or $S' \cup \{h_2\}$ are stable sets violating (β, β_0). Therefore, $S'' = S' \setminus \{b_1\} \cup \{d_1, c\}$ is a stable set and, consequently, $\beta(S'') = \beta(S') + \lambda > \pi_0 + \lambda$, a contradiction.

In the case of inequalities (4) with $A = \{b_1, c, b_2, h_1, h_2\}$, the graph $G_\emptyset \setminus A$ is isomorphic to H^e and there exists a tight stable set T in $G_\emptyset \setminus A$ having empty intersection with $\{d_1, a\}$; hence, T is also a tight stable set in $G_\emptyset \setminus N(u_{11})$ and, as $\beta_0 = \pi_0$, it follows that $\beta_{u_{11}} = 0$. ∎

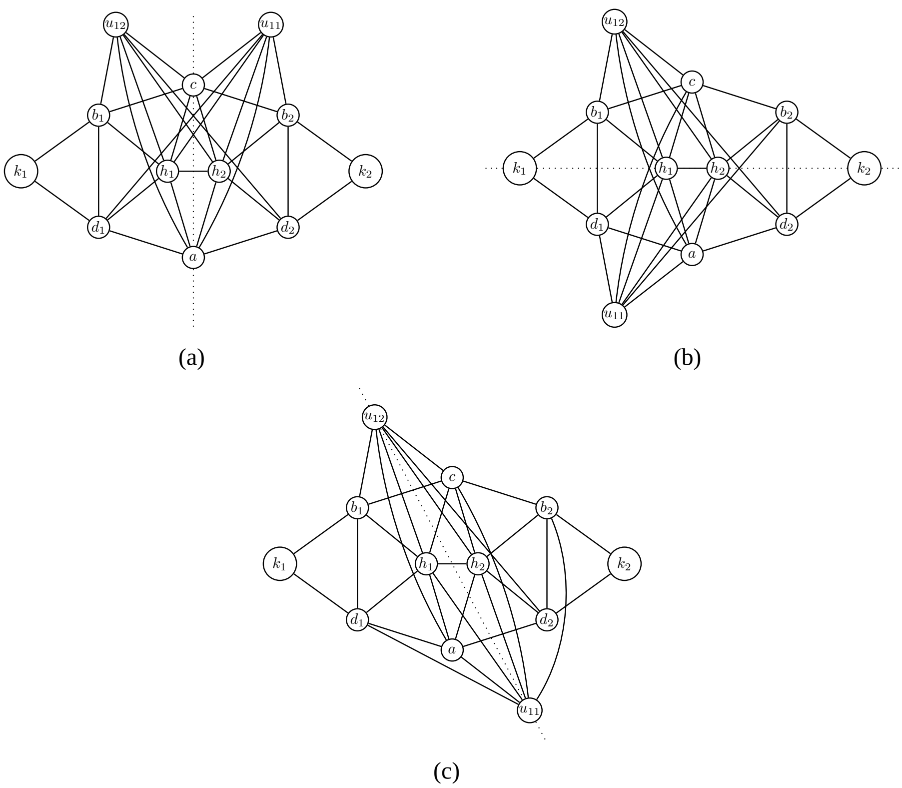

Figure 6: (a) vertical symmetry; (b) horizonthal symmetry; (c) diagonal symmetry.

Results symmetric to those given in lemmas 4.1, 4.2, 4.3 hold when the lifted node is u_{12} instead of u_{11}. These results can be proved taking advantage of the highly symmetric structure of an extended gear B_Y.

Given a graph $G = (V, E)$ we say that a permutation $\sigma : V \to V$ is a *symmetry* of G if and only if $\sigma(N(v)) = N(\sigma(v))$ for each $v \in V$ (where $\sigma(N(v)) = \{\sigma(u) | u \in N(v)\}$). To simplify the notation we shall write $\sigma(v_1, v_2, \ldots, v_k) = (u_1, u_2, ..., u_k)$ instead of $\sigma(v_1) = u_1$, $\sigma(v_2) = u_2$, …, $\sigma(v_k) = u_k$.

Consider an extended gear B_Y with $Y = \{u_{11}, u_{12}\}$ and let B^*_Y be the graph obtained from B_Y by adding two new nodes, say k_1 and k_2, such that $N(k_i) = \{b_i, d_i\}$, $i = 1, 2$. It is easy to verify that the following observation holds.

Observation 1. *The permutation functions*

$$\sigma_v(a, c, b_1, d_1, b_2, d_2, h_1, h_2, u_{11}, u_{12}, k_1, k_2) = (a, c, b_2, d_2, b_1, d_1, h_2, h_1, u_{12}, u_{11}, k_2, k_1)$$

$$\sigma_h(a, c, b_1, d_1, b_2, d_2, h_1, h_2, u_{11}, u_{12}, k_1, k_2) = (c, a, d_1, b_1, d_2, b_2, h_1, h_2, u_{12}, u_{11}, k_1, k_2)$$

$$\sigma_d(a, c, b_1, d_1, b_2, d_2, h_1, h_2, u_{11}, u_{12}, k_1, k_2) = (c, a, d_2, b_2, d_1, b_1, h_2, h_1, u_{11}, u_{12}, k_2, k_1)$$

are symmetries for the graph B^*_Y and will be referred to as vertical symmetry, horizonthal symmetry, *and* diagonal symmetry, *respectively.*

In Fig. 6 we give a figurative representation of the above symmetries: every symmetry maps each node on one side of the dotted line onto the corresponding node on the other side. Notice that $\sigma_v(u_{11}) = \sigma_h(u_{11}) = u_{12}$ and $\sigma_v(u_{12}) = \sigma_h(u_{12}) = u_{11}$, while $\sigma_d(u_{11}) = u_{11}$ and $\sigma_d(u_{12}) = u_{12}$. Therefore, when we delete both nodes u_{11} and u_{12}, the permutations σ_v, σ_h, and σ_d restricted to $V_{B^*_\emptyset}$ are still symmetries; if we delete only the node u_{12} (u_{11}), the permutation σ_d restricted to $V_{B^*_Y} \setminus \{u_{12}\}$ ($V_{B^*_Y} \setminus \{u_{11}\}$) is still a symmetry, while σ_v and σ_h are not defined and so, they are not symmetries.

The proofs of the results symmetric with those given in lemmas 4.1, 4.2, 4.3 with u_{12} as lifted node instead of u_{11} are then obtained via the vertical symmetry described above and summarized in the following lemma.

Lemma 4.4. *Let (β, β_0) be a facet defining inequality for $STAB(G_\emptyset)$.*

i) If (β, β_0) is a proper geared inequality then the node u_{12} is lifted with coefficient $\beta_{u_{12}} = \lambda$.

ii) If (β, β_0) is a non-proper geared inequality then the node u_{12} is lifted with coefficient $\beta_{u_{12}} = \lambda$ if $A = \{b_2, c\}$ or $A = \{d_1, a\}$, and $\beta_{u_{12}} = 0$ otherwise.

iii) If (β, β_0) is a proper g-lifted inequality then the node u_{12} is lifted with coefficient $\beta_{u_{12}} = \lambda$.

iv) If (β, β_0) is a non-proper g-lifted inequality then the node u_{12} is lifted with coefficient $\beta_{u_{12}} = 0$.

It remains to consider the lifting coefficient $\beta_{u_{11}}$ for geared and g-lifted inequalities after the node u_{12} has been lifted.

Lemma 4.5. *Let (β', β_0) be a facet defining inequality for $STAB(G_{12})$, obtained by lifting an inequality (β, β_0) of $STAB(G_\emptyset)$ of type (1) $\div$ (4) with the node u_{12}.*
If (β', β_0) is the lifting of either a proper geared inequality (1), or a proper g-lifted inequality (3), or a lifted geared inequality (2) with $A \in \{\{b_1, c\}, \{d_2, a\}\}$, then the node u_{11} is lifted with coefficient $\beta'_{u_{11}} = \lambda$.
In all other cases, the node u_{11} is lifted with coefficient $\beta'_{u_{11}} = 0$.

Proof. If $\beta'_{u_{12}} = 0$, then lemmas 4.2 and 4.3 imply that: $\beta'_{u_{11}} = \lambda$ if (β, β_0) is a geared inequality of type (2) with $A \in \{\{b_1, c\}, \{d_2, a\}\}$, and $\beta'_{u_{11}} = 0$ otherwise.

Therefore, we need to check the cases when $\beta'_{u_{12}} \neq 0$, namely the cases when (β', β_0) is the lifting of a proper geared inequality or a proper g-lifted inequality, or a geared inequality (2) with $A = \{b_2, c\}$ (or the diagonally symmetric case with $A = \{d_1, a\}$).

Consider first the cases when (β', β_0) is the lifting of either a proper geared inequality or a proper g-lifted inequality. We can prove that $\beta'_{u_{11}} \leq \lambda$ by simply replacing $G_\emptyset$ with G_{12} in the proofs of Lemma 4.1 and Lemma 4.3, respectively.

Now, suppose that $\beta'_{u_{11}} < \lambda$, then there exists a stable set $S' \in \mathscr{S}(G_{12} \setminus N(u_{11}))$ such that $\beta'(S') > \beta_0 - \lambda$. Then either $u_{12} \in S'$ or $\{b_1, d_2\} \subseteq S'$ (otherwise $S' \cup \{h_1\}$ or $S' \cup \{h_2\}$ would be stable sets violating (β', β_0)). If $u_{12} \in S'$, then $S'' = S' \setminus \{u_{12}\} \cup \{a, c\}$ is a stable set of G_Ω such that $\beta'(S'') = \beta'(S') + \lambda > (\beta_0 - \lambda) + \lambda = \beta_0$, a contradiction. If $\{b_1, d_2\} \subseteq S'$, then $S'' = S' \setminus \{b_1\} \cup \{d_1, c\}$ is a stable set of G_Ω such that $\beta'(S'') = \beta'(S') + \lambda > (\beta_0 - \lambda) + \lambda = \beta_0$, a contradiction. Hence, $\beta'_{u_{11}} = \lambda$.

Consider the third case where (β', β_0) is the lifting of a geared inequality (2) with $A = \{b_2, c\}$: let (π_H, π_0) be the facet defining inequality for $STAB(H)$ that generates (β', β_0). Clearly $\beta_0 = \pi_0 + \lambda$. There exists a tight stable set T for (π_H, π_0) such that $T \cap (K_2 \cup \{v_2\}) = \emptyset$. Thus, $v_1 \in T$ and $\overline{S} = T \setminus \{v_1\} \cup \{b_1, d_2\}$ is a stable set in $G_{12} \setminus N(u_{11})$. As $\beta'(\overline{S}) = \pi_H(T \setminus \{v_1\}) + 2\lambda = \pi_0 - \lambda + 2\lambda = \pi_0 + \lambda = \beta_0$, then $\beta'_{u_{11}} = 0$. ∎

By vertical symmetry, the results of Lemma 4.5 hold once we interchange the role of u_{11} and u_{12}.

However the extension of the gear B with the nodes u_{11} and u_{12} does not generate only inequalities that are (sequential liftings of) inequalities of type (1)$\div$ (4) in Definitions 2.4 and 2.6. Indeed, new facet defining inequalities are generated. More precisely,

Theorem 4.6. *Let $G = (H, B_Y, e)$ be the geared graph generated by H and B_Y along $e = v_1v_2$ where $Y = \{u_{11}, u_{12}\}$. Let (π, π_0) be a g-liftable facet defining inequality for $STAB(H^e)$, then*

$$\sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda(x_{d_1} + x_{u_{11}} + x_{b_2}) \leq \pi_0 \tag{5}$$

is facet defining for both $STAB(G_{11})$ and $STAB(G)$; moreover,

$$\sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda(x_{b_1} + x_{u_{12}} + x_{d_2}) \leq \pi_0 \tag{6}$$

is facet defining for both $STAB(G_{12})$ and $STAB(G)$.

Proof. We prove that (5) is facet defining for $STAB(G_{11})$. Observe that $G_{11} \setminus \{b_1, a, c, d_2, h_1, h_2\}$ is isomorphic to H^e by renaming $\{v_1, t, v_2\}$ as $\{d_1, u_{11}, b_2\}$. Thus (5) is facet defining for $STAB(G_{11} \setminus \{b_1, a, c, d_2, h_1, h_2\})$. It is not difficult to see that all the nodes in $\{b_1, a, c, d_2, h_1, h_2\}$ have zero-lifting coefficients; thus the inequality (5) is facet defining for $STAB(G_{11})$. Since (5) has a tight solution containing u_{11} and not containing b_1 and d_2, it follows that u_{12} has a zero-lifting coefficient. Therefore, the inequality (5) is also facet defining for $STAB(G)$.

The second part of the statement follows by diagonal symmetry. ∎

The above result implies that Theorem 2.8 cannot be generalized to geared graphs $G = (H, B_Y, e)$ with $Y \neq \emptyset$ by simply applying the sequential lifting procedure to the nodes in Y. In fact, at least two new facet defining inequalities arise when the nodes u_{11} and u_{12} are added to a gear B. The new inequalities (5) (6) have a structure that is very similar to the g-lifted inequalities listed in (4). In particular: the nodes of the gear associated with the nonzero coefficients of each inequality of type (4) induce the simple paths (d_1, a, d_2) and (b_1, c, b_2). The same holds for inequalities (5) and (6), where the simple paths are (d_1, u_{11}, b_2) and (b_1, u_{12}, d_2), respectively.

We summarize the results obtained so far and provide a list of those inequalities that are necessary for a linear description of $STAB(G)$ of a geared graph G generated by H and B_Y along e where $Y = \{u_{11}, u_{12}\}$.

Theorem 4.7. *Let $G = (H, B_Y, v_1v_2)$ be a geared graph generated by H and B_Y along $e = v_1v_2$ where $Y = \{u_{11}, u_{12}\}$. Consider the following inequalities:*

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_{B_Y} \setminus \{h_1, h_2\}} x_i + 2\lambda(x_{h_1} + x_{h_2}) \leq \pi_0 + 2\lambda \tag{7}$$

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_{B_Y} \setminus A} x_i \leq \pi_0 + \lambda \tag{8}$$

$$where \quad A \in \{\{b_1, c, u_{12}\}, \{b_2, c, u_{11}\}, \{d_1, a, u_{11}\}, \{d_2, a, u_{12}\}, \{a, c, u_{11}, u_{12}\}\}$$

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_{B_Y}} x_i \leq \pi_0 + \lambda, \tag{9}$$

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_{B_Y} \setminus A} x_i \leq \pi_0 \tag{10}$$

$$where \quad A \in \{\{b_1, c, b_2, h_1, h_2, u_{11}, u_{12}\}, \{d_1, a, d_2, h_1, h_2, u_{11}, u_{12}\}, \{b_1, a, c, d_2, h_1, h_2, u_{12}\}, \{d_1, a, c, b_2, h_1, h_2, u_{11}\}\}.$$

For each g-extendable inequality (π, π_0) *that is facet defining for* $STAB(H)$*, the inequalities (7) and (8) are facet defining for* $STAB(G)$*.*
For each g-liftable inequality (π, π_0) *that is facet defining for* $STAB(H^e)$*, the inequalities (9) and (10) are facet defining for* $STAB(G)$*.*

Theorem 4.7 follows from theorems 2.5 and 2.7 together with lemmas 4.1÷4.5, and Theorem 4.6. Note that the facet defining inequalities of $STAB(G_{11})$ (and $STAB(G_{12})$) can be derived from the list given above by setting $\beta_{u_{12}} = 0$ (and $\beta_{u_{11}} = 0$) in inequalities (7)÷(9) and then removing the inequality (10) corresponding to $A = \{b_1, a, c, d_2, h_1, h_2, u_{12}\}$ when $Y = \{u_{12}\}$ and corresponding to $A = \{d_1, a, c, b_2, h_1, h_2, u_{11}\}$ when $Y = \{u_{11}\}$.

We end this paper by observing that all the inequalities considered so far are necessary in a linear description for the stable set polytope of claw-free graphs. In a sequel of this paper we shall prove that they are also sufficient for the class of XX-graphs.

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