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**VERTEX COLOURING OF CIRCULANT
GRAPHS: A COMBINATORIAL APPROACH**

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Abstract

Consider three integers n, a, b such that $n > 0$, $a \bmod n \neq 0$, $b \bmod n \neq 0$, and $a \bmod n \neq \pm b \bmod n$. The graph $C_n(a, b) = (V, E)$, where $V = \{v_0, v_1, \dots, v_{n-1}\}$ and $E = \{(v_i, v_{(i+a) \bmod n}), (v_i, v_{(i+b) \bmod n}), \text{ for } i = 0, \dots, n-1\}$ is called *circulant graph*. In this paper we propose exact linear algorithms for VERTEX-COLOURING: given an arbitrary circulant graph $C_n(a, b)$, find an assignment of colours to its vertices such that adjacent vertices receive different colours and the number of colours is minimized. The approach is purely combinatorial and new for the topic: it is based on a suitable matrix representation of an arbitrary $C_n(a, b)$, which highlights the strong topological properties of its cycles, and allows the design of very simple colouring algorithms.

Key words: vertex-colouring, circulant graph, chromatic number

1. Introduction

Consider three integers n, a, b such that $n > 0$, $a \bmod n \neq 0$, $b \bmod n \neq 0$, and $a \bmod n \neq \pm b \bmod n$. By $C_n(a, b) = (V, E)$ we denote the (simple undirected) *circulant graph* where $V = \{v_0, v_1, \dots, v_{n-1}\}$ and $E = \{(v_i, v_{(i+a) \bmod n}), (v_i, v_{(i+b) \bmod n}), \text{ for } i = 0, \dots, n-1\}$ (see Fig. 1). We say that $v_x, v_y \in V$ are *a-adjacent* (*b-adjacent*, resp.) and that $(v_x, v_y) \in E$ is an *a-edge* (*b-edge*, resp.) if $(x \pm a) \bmod n = y$ ($(x \pm b) \bmod n = y$, resp.). The adjacency matrix of such a circulant graph is symmetric and circulant.

Throughout the paper we shall assume, w.l.o.g., that $a, b \in \{1, \dots, n-1\}$: under this condition, $a \bmod n \neq b \bmod n$ is equivalent to the pair $a \neq b$ and $a + b \neq n$. As a consequence, we shall always have $n \geq 4$. Notice that the $C_n(a, b)$'s considered in the paper are either 3- or 4-regular. In fact, a $C_n(a, b)$ is 1-regular iff $a = b = \frac{n}{2}$, it is 2-regular iff $a = b$ or $a + b = n$ (but all these cases are excluded), it is 3-regular iff either a or b is equal to $\frac{n}{2}$, and it is 4-regular otherwise. The circulant graphs we deal with, are a subclass of the more general class of circulant graphs $C_n(a_1, a_2, \dots, a_k)$ when $k = 2$ (actually, we denote a_1 by a and a_2 by b). Circulant graphs $C_n(a_1, a_2, \dots, a_k)$ are defined on n vertices, and each vertex v_i is adjacent to vertices $v_{(i+a_j) \bmod n}$, for $j = 1, \dots, k$. They are Cayley graphs over the cyclic group \mathbb{Z}_n . If $a_1 = 1$ and $k > 2$ they are also called chordal rings or multiple-loops [16, 22, 29]; if $a_1 = 1$ and $k = 2$ they are called double loops [13, 18, 19], cyclic graphs [13], or 2-jumps [6].

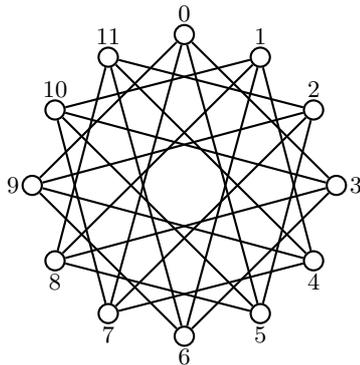


Figure 1: The circulant graph $C_{12}(3, 5)$.

In this paper we investigate the following problem:

VERTEX-COLOURING

- Given* a circulant graph $C_n(a, b)$
Find an assignment of colours to its vertices
Such That adjacent vertices receive different colours and the number of colours is minimized.

When adjacent vertices receive different colours, the colouring is *feasible*; $C_n(a, b)$ is *k-colourable* if it admits a feasible *k-colouring*, that is, a feasible vertex colouring with k colours; the smallest k such that $C_n(a, b)$ is *k-colourable* is the *chromatic number* $\chi(C_n(a, b))$; a feasible *k-colouring* is *optimal* if $k = \chi(C_n(a, b))$.

The vertex colouring problem on arbitrary graphs is known to be \mathcal{NP} -hard [17]. Two classes of graphs where it is solvable in polynomial time are the series-parallel graphs [26] and the perfect graphs [14]. In [8], the vertex colouring problem on circulant graphs $C_n(a_1, a_2, \dots, a_k)$ is proved

to be \mathcal{NP} -hard and not approximable better than a certain factor. The same authors, using spectral techniques, also show that $\lceil \log p \rceil$ eigenvectors are necessary and sufficient to feasibly colour p -chromatic circulant graphs of degree less than 5.

VERTEX-COLOURING is studied in [13, 15, 28]. In particular, in [13] the authors propose colouring algorithms for some 3-colourable $C_n(1, b)$'s. In [28] the authors prove a conjecture by [9] which characterizes all the 3-colourable $C_n(1, b)$'s. In [15] optimal colouring algorithms are proposed for all $C_n(a, b)$'s: many different subcases are identified, depending on the value of some parameters, a few of which related to the Hermite Normal Form associated to the given $C_n(a, b)$, and an optimal colouring algorithm is proposed for each subcase. We remark that the method by [15] is extended in [21], resulting in the characterization of the chromatic number of 5-regular $C_n(a, b, \frac{n}{2})$'s and in optimal colouring algorithms for such graphs. However, as pointed out in [21], the ‘‘proof is fairly complicated’’ and ‘‘it does not appear feasible to determine the chromatic number of 6-valent circulants’’ $C_n(a, b, c)$'s by generalizing the method of [15].

For the sake of completeness, we remark that [28] characterize 4-colourable 6-regular $C_n(1, b, b+1)$'s proving a conjecture by [9], that the chromatic number of $C_n(a, b, a+b)$'s is studied in [20], where optimal colouring algorithms are also proposed, and that in [3] the chromatic number of all circulant graphs $C_n(a, b, c)$ with $a < b < c$ and $n \geq 4bc$ is computed. Other results in [2, 27]. In this paper we propose a pure combinatorial approach for VERTEX-COLOURING, which is new for this problem. The approach takes advantage from the strong topological properties of the given $C_n(a, b)$, and is based on a matrix representation of it. This representation suitably organizes the adjacencies between vertices, so that simple, linear, optimal colouring algorithms for arbitrary $C_n(a, b)$'s, depending on easy-to-evaluate conditions on n, a, b and $\gcd(n, a), \gcd(n, b)$, can be derived.

The combinatorial nature of our approach stimulated the research for devising exact, linear, simple colouring algorithms for the graphs $C_n(a_1, a_2, \dots, a_k)$ such that every a_i , for $i \geq 3$, is a weighted combination of a and b , and the sum of the weights is odd and not ‘‘too large’’ [24]. The proposed approach also applies to the vertex colouring problem on similar graph classes, such as Toeplitz and integer distance graphs [25].

The paper is organized as follows: some preliminary conditions are discussed in Section 2; the matrix representation of the graph is introduced in Section 3; the chromatic number of an arbitrary $C_n(a, b)$ is characterized in Section 4; simple algorithms are proposed in Section 5.1 for the $C_n(1, 2)$'s and the $C_n(1, \frac{n-1}{2})$'s, in Section 5.2 for the $C_n(1, b)$'s with $2 < b \leq \frac{n}{2}$, and in Section 5.3 for the $C_n(a, b)$'s with $\gcd(n, a), \gcd(n, b) > 1$; Section 6 concludes.

2. Some preliminary conditions

This section is devoted to review some conditions under which the $C_n(a, b)$'s are connected or isomorphic. Isomorphic graphs, in particular, are important because it is sometimes convenient to replace the given graph by an isomorphic one, in order to better describe the colouring algorithms.

Proposition 2.1. [6] *The graph $C_n(a, b)$ is connected if and only if $\gcd(n, a, b) = 1$.*

Notice that the condition $\gcd(n, a, b) = 1$ implies either $\gcd(n, a) = \gcd(n, b) = 1$ or $\gcd(n, a) \neq \gcd(n, b)$, and allows for writing $n = H \gcd(n, a) \gcd(n, b)$, where $H \in \mathbb{Z}^+$. In the sequel, we shall often use this decomposition for n .

We now shortly summarize a few results on isomorphism of $C_n(a, b)$'s.

Given two graphs $C_n(a, b) = (V, E)$ and $C_{n'}(a', b') = (V', E')$ we shall say that they are *isomorphic*, denoted by $C_n(a, b) \simeq C_{n'}(a', b')$, if there exists a mapping function $f : V \rightarrow V'$ such that $(f(u), f(v)) \in E'$ if and only if $(u, v) \in E$ (of course, $n = |V| = |V'| = n'$ and $|E| = |E'|$). It is easy to see that the following graphs are isomorphic to $C_n(a, b)$: $C_n(n - a, b)$, $C_n(a, n - b)$, $C_n(n - a, n - b)$, $C_n(b, a)$, $C_n(b, n - a)$, $C_n(n - b, a)$, $C_n(n - b, n - a)$. For example, the graph $C_{12}(3, 5)$ in Fig. 1 is isomorphic to the graphs $C_{12}(9, 5)$, $C_{12}(3, 7)$, $C_{12}(9, 7)$, $C_{12}(5, 3)$, $C_{12}(5, 9)$, $C_{12}(7, 3)$, $C_{12}(7, 9)$. A simple way to check if $C_n(a, b) \simeq C_{n'}(a', b')$ or not, can be found in [23].

Proposition 2.2. *Any circulant graph $C_n(x, y)$ admits an isomorphic $C_n(a, b)$ with $a, b \leq \frac{n}{2}$.*

Proof. It follows from the fact that $\min\{z, n - z\} \leq \frac{n}{2}$ for any $z \in \{1, \dots, n - 1\}$. ■

The family of all the graphs isomorphic to a given $C_n(a, b)$ is described by the following theorem, which proves a conjecture by Ádám [1]:

Theorem 2.3. [11] *$C_n(a, b) \simeq C_n(a', b')$ if and only if there exists a $\mu \in \{1, \dots, n - 1\}$ such that $\gcd(n, \mu) = 1$ and $\{a', b'\} = \{\pm\mu a \bmod n, \pm\mu b \bmod n\}$.*

As a corollary we have:

Corollary 2.4. [15] *Let n be odd. Then, $C_n(1, 2) \simeq C_n(1, \frac{n-1}{2})$.*

In fact one gets $C_n(1, 2)$ from $C_n(1, \frac{n-1}{2})$ for $\mu = 2$, and $C_n(1, \frac{n-1}{2})$ from $C_n(1, 2)$ for $\mu = \frac{n-1}{2}$. We shall also make use of the following two corollaries to Theorem 2.3:

Corollary 2.5. [15] *Let n, a verify $\gcd(n, a) = 1$, and let t be an integer such that $(ta) \bmod n = 1$. Then $C_n(a, b) \simeq C_n(1, (tb) \bmod n)$.*

The conditions $\gcd(n, a) = 1$ and $(ta) \bmod n = 1$ imply that $\gcd(n, t) = 1$, showing that the corollary is an application of Theorem 2.3 for $\mu = t$.

If $\gcd(n, a) > \gcd(n, b) = 1$, swap a and b , and the above corollary applies again. Moreover, if both $\gcd(n, a) = 1$ and $\gcd(n, b) = 1$ hold, the above proposition applies twice. For example: $C_{31}(3, 2) \simeq C_{31}(1, 11) \simeq C_{31}(17, 1)$.

Corollary 2.6. *Any circulant graph $C_n(x, y)$ with $\min\{\gcd(n, x), \gcd(n, y)\} = 1$ admits an isomorphic $C_n(1, b)$ with $b \leq \frac{n}{2}$.*

Proof. Apply Corollary 2.5 to $C_n(x, y)$, possibly swapping x and y , and obtain a suitable $C_n(1, z)$ isomorphic to it, then apply Proposition 2.2. ■

The last corollary suggests the following

Definition 2.7. *A circulant graph $C_n(a, b)$ is well-defined if it is connected, and if either $\min\{\gcd(n, a), \gcd(n, b)\} \geq 2$ or $\min\{\gcd(n, a), \gcd(n, b)\} = 1$, $a = 1$, and $b \leq \frac{n}{2}$.*

By the above definition, any $C_n(a, b)$ is either well-defined itself, or it admits an isomorphic circulant graph which is well-defined. Thus, from now on, w.l.o.g., we shall consider well-defined graphs, only. As an example, the graph $C_{18}(3, 14)$ is well-defined. The graph $C_{16}(2, 5)$ is not well-defined: we have to consider $C_{16}(1, 6)$ which is isomorphic to it, and well-defined. Neither the graph $C_{31}(3, 2)$ is well-defined: we can arbitrarily consider one among $C_{31}(1, 11)$ and $C_{31}(1, 14)$, both isomorphic to it and well-defined.

3. Representations for $C_n(a, b)$

In the present section we describe the infinite matrix $M_n^*(a, b)$ representing (infinitely many copies of) $C_n(a, b)$, and two suitable finite submatrices of it, namely the matrix $M_n(a, b)$ and the “pseudo”-matrix $S_n(1, b)$, both with n elements. $M_n(a, b)$ will be used to describe the colouring algorithms for the (well-defined) graphs $C_n(a, b)$ with $\min\{\gcd(n, a), \gcd(n, b)\} \geq 2$, while $S_n(1, b)$ will be used to describe the colouring algorithms for the well-defined graphs $C_n(1, b)$.

3.1. Matrix $M_n^*(a, b)$

Matrix $M_n^*(a, b)$ is a matrix with an infinite number of rows and columns, which can be defined for any connected circulant graph $C_n(a, b)$. Each element corresponds to a vertex of the graph (see Fig. 2, where the value of an element is the index of the corresponding vertex). On the contrary, every vertex of the graph is represented by infinitely many (regularly placed) elements of $M_n^*(a, b)$. Consider an arbitrary element $m_{i,j}^*$ corresponding to vertex v_x . Then, elements $m_{i,j-1}^*$, $m_{i,j+1}^*$ correspond to vertices $v_{(x-a) \bmod n}$, $v_{(x+a) \bmod n}$, respectively, which are both a -adjacent to v_x , and elements $m_{i-1,j}^*$, $m_{i+1,j}^*$ correspond to vertices $v_{(x-b) \bmod n}$, $v_{(x+b) \bmod n}$, respectively, which are both b -adjacent to v_x . Thus, traversing an arbitrary row (column, resp.) of $M_n^*(a, b)$ corresponds to infinitely cycling over the corresponding a -cycle (b -cycle, resp.). The definition of a - and b -adjacency is extended to matrix elements, and shows that, if the given graph is not connected, not all vertices can be represented in $M_n^*(a, b)$. We remark that similar structures are defined in [5, 7, 12, 29].

31	34	1	4	7	10	13	16	19	22	25	28	31	34	1	4	7	10
3	6	9	12	15	18	21	24	27	30	33	0	3	6	9	12	15	18
11	14	17	20	23	26	29	32	35	2	5	8	11	14	17	20	23	26
19	22	25	28	31	34	1	4	7	10	13	16	19	22	25	28	31	34
27	30	33	0	3	6	9	12	15	18	21	24	27	30	33	0	3	6
35	2	5	8	11	14	17	20	23	26	29	32	35	2	5	8	11	14
7	10	13	16	19	22	25	28	31	34	1	4	7	10	13	16	19	22
15	18	21	24	27	30	33	0	3	6	9	12	15	18	21	24	27	30
23	26	29	32	35	2	5	8	11	14	17	20	23	26	29	32	35	2
31	34	1	4	7	10	13	16	19	22	25	28	31	34	1	4	7	10
3	6	9	12	15	18	21	24	27	30	33	0	3	6	9	12	15	18

Figure 2: Part of the infinite matrix $M_{36}^*(3, 8)$. Each rectangle represents $M_{36}(3, 8)$.

3.2. Matrix $M_n(a, b)$

This section is devoted to introduce matrix $M_n(a, b)$ which is a convenient representation for the description of the colouring algorithms on well-defined $C_n(a, b)$'s with $\gcd(n, a), \gcd(n, b) \geq 2$.

The representative matrix $M_n(a, b) = [m_{i,j}]$ is an n -element rectangular submatrix of $R = \gcd(n, a)$ consecutive rows and $C = \frac{n}{R}$ consecutive columns of $M_n^*(a, b)$. Its elements are in one-to-one correspondence with the vertices of $C_n(a, b)$ (see any rectangle in Fig. 2). Without loss of generality, vertex v_0 matches to element $m_{1,1}$ in the upper left corner of $M_n(a, b)$. Thus an arbitrary element $m_{i,j}$ corresponds to vertex v_x where $x = ((i-1)b + (j-1)a) \bmod n$ for $i = 1, \dots, R$, and $j = 1, \dots, C$.

Since $M_n(a, b)$ is a submatrix of $M_n^*(a, b)$, two consecutive elements of a row are a -adjacent, as well as the first and last elements of a same row: that is to say, there is a one-to-one correspondence between rows of $M_n(a, b)$ and a -cycles.

As for the columns, two consecutive elements of a column are b -adjacent, but the first and last elements of a same column are not, generally speaking. Thus, no one-to-one correspondence can be set between the columns of $M_n(a, b)$ and the b -cycles, unless the number of vertices in a b -cycle equals the number of rows of $M_n(a, b)$, that is $\frac{n}{\gcd(n, b)} = R$: only in this case, in fact, the first and last elements of a same column correspond to b -adjacent vertices. On the contrary, if $\frac{n}{\gcd(n, b)} > R$, the number R of elements in a column of $M_n(a, b)$ is not sufficient to contain all the $\frac{n}{\gcd(n, b)}$ vertices of a b -cycle (the case $\frac{n}{\gcd(n, b)} < R$ never applies, as $n = H \gcd(n, a) \gcd(n, b)$ and $R = \gcd(n, a)$). However, it can be proved that

Lemma 3.1. [23] *Consider an arbitrary element in the last row of $M_n(a, b)$. Then, there exists a unique element in the first row of $M_n(a, b)$ which is b -adjacent to it.*

It follows that the $\frac{n}{\gcd(n, b)}$ vertices of each b -cycle are split onto $\frac{n}{\gcd(n, b)} \frac{1}{R} = \frac{n}{\gcd(n, a) \gcd(n, b)} = H$ columns. For example, consider element $m_{3,1}$ of $M_{36}(3, 8)$ (see Fig. 3). Then, in the first row of $M_{36}(3, 8)$, its only b -adjacent element is $m_{1,9}$ (corresponding to v_{24}). Now consider one of its four b -cycles, say the one whose vertices are $v_0, v_8, v_{16}, v_{24}, v_{32}, v_4, v_{12}, v_{20}, v_{28}$: the first three vertices are found in the first column of $M_{36}(3, 8)$, the second three in the ninth column, and the remaining three in the fifth column, as $H = 3$.

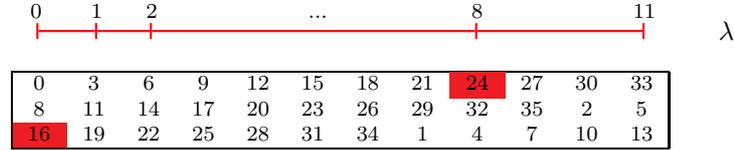


Figure 3: Matrix $M_{36}(3, 8)$; highlighted the elements $m_{R,k} = m_{3,1}$ and $m_{1,h} = m_{1,9}$, showing that $\lambda = 8$.

Consider an arbitrary element $m_{R,h}$ of $M_n(a, b)$, and its b -adjacent element $m_{1,k}$. Then, for an arbitrary integer t , $m_{R,(h+t) \bmod C}$ and $m_{R,(k+t) \bmod C}$ are b -adjacent, too. This shows that the quantity $(k - h) \bmod C$ is a constant.

Definition 3.2. [23] *The column-jump of $M_n(a, b)$ is $\lambda = (k - h) \bmod C$, where k and h are such that $m_{R,h}$ and $m_{1,k}$ are b -adjacent.*

In [23] it is also shown that $\lambda = \min\{z \in \mathbb{Z}^+ : \gcd(n, a)b \equiv za \pmod{n}\}$.

Example: Consider $M_{36}(3, 8)$. Then $\lambda = 8$, and in fact the b -cycle whose vertices are $v_0, v_8, v_{16}, v_{24}, v_{32}, v_4, v_{12}, v_{20}, v_{28}$, is read in column 1, then in column $1 + \lambda = 9$, finally in column $(9 + \lambda) \bmod C = 5$ as $C = 12$. After that, the b -cycle is back to column $(5 + \lambda) \bmod C = 1$ (see Fig. 3).

Notice that $0 \leq \lambda \leq C - 1$, by definition, and that $\lambda = 0$ if and only if the b -cycles are in one-to-one correspondence with the columns of $M_n(a, b)$ (this is to say, if and only if $n = \gcd(n, a) \gcd(n, b)$, i.e. $H = 1$).

3.3. Pseudo-matrix $S_n(1, b)$

This section is devoted to describe the pseudo-matrix $S_n(1, b)$ which is a convenient representation for the description of the colouring algorithms on well-defined $C_n(1, b)$'s.

$S_n(1, b) = [s_{i,j}]$ is an n -element irregularly shaped submatrix of $M_n^*(1, b)$, whose elements are in one-to-one correspondence with the vertices of $C_n(1, b)$ (see any closed area in Fig. 4). It is defined on b consecutive columns and $r = \lceil \frac{n}{b} \rceil \geq 2$ consecutive rows of $M_n^*(a, b)$ (as $b \leq \frac{n}{2}$), the last of which possibly not complete: if $n \bmod b = 0$ the last row is complete, and contains $p = b$ elements, otherwise it is not complete, and contains $p = n - (r - 1)b < b$ elements. We shall call *head* of S the submatrix defined on the first p columns (of r elements each), and *tail* of S the submatrix defined on the remaining $b - p$ columns (of $r - 1$ elements each), if any. W.l.o.g. we assume that vertex v_0 matches to element $s_{1,1}$ (thus element $s_{i,j}$ corresponds to vertex v_y where $y = ((i - 1)b + (j - 1)) \bmod n$, for $i = 1, \dots, r - 1$ and $j = 1, \dots, b$, or $i = r$ and $j = 1, \dots, p$). Then, from left to right, in the first row of S we find vertices v_0, \dots, v_{b-1} ; in the second row, vertices v_b, \dots, v_{2b-1} ; and so on up to row r , where we find vertices $v_{(r-1)b}, \dots, v_{n-1}$ (arithmetic is done modulo n).

13	14	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	0
4	5	6	7	8	9	10	11	12	13	14	0	1	2	3	4	5	6
10	11	12	13	14	0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	13	14	0	1	2	3
7	8	9	10	11	12	13	14	0	1	2	3	4	5	6	7	8	9
13	14	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	0
4	5	6	7	8	9	10	11	12	13	14	0	1	2	3	4	5	6
10	11	12	13	14	0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	13	14	0	1	2	3
7	8	9	10	11	12	13	14	0	1	2	3	4	5	6	7	8	9
13	14	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	0

Figure 4: Part of the infinite matrix $M_{15}^*(1, 6)$. Each area represents $S_{15}(1, 6)$, where $r = 3$, $b = 6$, and $p = 3$.

4. Colouring theorem

In the present section we recall the colouring theorem, and propose colouring algorithms for the trivial cases. Results limited to some subclasses of $C_n(a, b)$'s are due to [13, 28], while the whole theorem is proved in [15].

Theorem 4.1. [13, 15, 28] *Let $C_n(a, b)$ be a well-defined graph. Then*

$$\chi(C_n(a, b)) = \begin{cases} 2, & \text{if } n \text{ even, } a, b \text{ odd} \\ 5, & \text{if } n = 5 \\ 4, & \text{if } n = 13, a = 1, b = 5 \\ 4, & \text{if } n \neq 5, n \bmod 3 \neq 0, a = 1, b \in \{2, \frac{n-1}{2}\} \\ 3, & \text{otherwise.} \end{cases}$$

From now on we shall denote by \mathcal{B} , \mathcal{W} , \mathcal{R} , \mathcal{G} the colours Black, White, Red, and Green, respectively.

When n is even, and a and b are both odd, the graph $C_n(a, b)$ is bipartite: thus simply assign colour \mathcal{B} to the odd-indexed vertices and \mathcal{W} to the even-indexed ones.

The condition $n = 5$ identifies the unique well-defined graph on $n = 5$ vertices, namely $C_5(1, 2)$, which is isomorphic to K_5 , the complete graph on 5 vertices. The unique feasible and optimal colouring of its vertices consists of assigning a different colour to each vertex.

By requiring $n = 13$, $a = 1$, $b = 5$ we identify the well-defined graph $C_{13}(1, 5)$. It can be optimally 4-coloured, for example, by repeatedly assigning colours \mathcal{B} , \mathcal{W} , \mathcal{R} to vertices v_0, \dots, v_{11} , and colour \mathcal{G} to vertex v_{12} .

When $n \neq 5$ and $a = 1$, $b \in \{2, \frac{n-1}{2}\}$, the theorem states that any well-defined $C_n(a, b)$ is optimally 4-colourable if $n \bmod 3 \in \{1, 2\}$, and it is optimally 3-colourable if $n \bmod 3 = 0$. Simple optimal colouring algorithms are discussed in Section 5.1.

When the n, a, b at hand do not fit any of the above cases, we are left with 3-colourable graphs. Efficient algorithms to optimally colour these graphs are described in Sections 5.2 and 5.3.

5. Colouring algorithms

In this section we discuss optimal colouring algorithms for all the well-defined graphs $C_n(a, b)$, except the few ones dealt with in the previous section.

From Definition 2.7 it follows that the class of the well-defined graphs can be partitioned into two subclasses: one containing the well-defined graphs with $\min\{\gcd(n, a), \gcd(n, b)\} = 1$, the other one containing the well-defined graphs with $\min\{\gcd(n, a), \gcd(n, b)\} \geq 2$. Exact colouring algorithms for the former are found in Section 5.1 if $b \in \{2, \frac{n-1}{2}\}$, and in Section 5.2 if $2 < b \leq \frac{n}{2}$, $b \neq \frac{n-1}{2}$. Exact colouring algorithms for the latter are described in Section 5.3. In all cases, $M_n(a, b)$ or $S_n(1, b)$ will help in describing the algorithms and/or in verifying their correctness.

5.1. Colouring $C_n(1, 2)$ and $C_n(1, \frac{n-1}{2})$

In the present section we describe optimal colouring algorithms for all the well-defined $C_n(1, 2)$ and $C_n(1, \frac{n-1}{2})$, except $C_5(1, 2)$ which is dealt with in Section 4. W.l.o.g. we shall limit ourselves to describe a colouring algorithm for the $C_n(1, 2)$'s with arbitrary n : in fact, any $C_n(1, \frac{n-1}{2})$, which is defined for n odd only, admits an isomorphic $C_n(1, 2)$ (as stated by Corollary 2.4).

ALGORITHM FOR THE WELL-DEFINED $C_n(1, 2)$ 'S

If $n \bmod 3 = 0$
 then repeatedly assign colours \mathcal{B}, \mathcal{W} , and \mathcal{R} to v_0, \dots, v_{n-1} ;

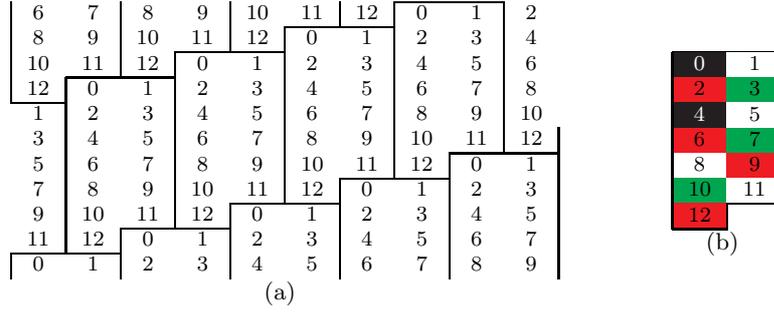
If $n \bmod 3 \in \{1, 2\}$ and $n \bmod 4 \in \{0, 3\}$
 then repeatedly assign colours $\mathcal{B}, \mathcal{W}, \mathcal{R}, \mathcal{G}$ to v_0, \dots, v_{n-1} ;

If $n \bmod 3 \in \{1, 2\}$ and $n \bmod 4 = 1$
 then repeatedly assign colours $\mathcal{B}, \mathcal{W}, \mathcal{R}, \mathcal{G}$ to v_0, \dots, v_{n-6} ,
 assign colours $\mathcal{W}, \mathcal{R}, \mathcal{G}, \mathcal{W}, \mathcal{R}$ to v_{n-5}, \dots, v_{n-1} ;

If $n \bmod 3 \in \{1, 2\}$ and $n \bmod 4 = 2$,
 repeatedly assign $\mathcal{B}, \mathcal{W}, \mathcal{R}, \mathcal{G}$ to v_0, \dots, v_{n-7} ,
 then repeatedly assign colours $\mathcal{W}, \mathcal{R}, \mathcal{G}$ to v_{n-6}, \dots, v_{n-1} .

It is easy to verify that the colouring output by the algorithm is feasible: it suffices to transfer it onto (the corresponding pseudo-matrix $S_n(1, 2)$ of) $M_n^*(1, 2)$, checking the adjacencies (an example is drawn in Fig. 5).

The colouring output by the algorithm is optimal, in fact $\chi(C_n(1, 2)) = 3$ if $n \bmod 3 = 0$ and $\chi(C_n(1, 2)) = 4$ in the other cases, by Theorem 4.1. Finally, we remark that the sequence of colours assigned to the last 5, 6 vertices in the last two cases is not unique, since there are other (few) feasible ways to colour them.

Figure 5: (a) Part of $M_{13}^*(1, 2)$; (b) An optimal 4-colouring for $C_{13}(1, 2)$.

5.2. Colouring $C_n(1, b)$ with $2 < b \leq \frac{n}{2}$ and $b \neq \frac{n-1}{2}$

This section is devoted to describe optimal colouring algorithms for the well-defined graphs $C_n(1, b)$ with $2 < b \leq \frac{n}{2}$, except $C_{13}(1, 5)$, the bipartite ones, and all the $C_n(1, \frac{n-1}{2})$'s, dealt with in previous sections. For the sake of shortness we shall denote by T the subset of graphs which we shall focus on, in this section. All the graphs in T have chromatic number equal to 3, as stated by Theorem 4.1, and are defined on $n \geq 2b \geq 6$ vertices. Optimal colourings will be constructed on $S_n(1, b)$ (recall that $r = \lceil \frac{n}{b} \rceil$ and $p = n - (r-1)b$): we shall denote by $(s_{i,j}, s_{h,k})$ the edge connecting the vertices corresponding to $s_{i,j}$ and $s_{h,k}$.

The approach we use has 2-phases. In the first phase we start by suitably colouring $S_n(1, b)$ with only two colours, \mathcal{B} and \mathcal{W} . Since bipartite graphs are excluded from this section, any 2-colouring has to be infeasible. In order to remove infeasibilities we proceed with the second phase, where we suitably modify into \mathcal{R} the colour of one element of each infeasible edge.

The remainder of this section is organized as follows: we start by defining the (2-colouring) *Chessboard Colouring* (C^2 , for short), then we list the infeasibilities it induces, introducing the definition of *steps* and *staircases*, and describe the *Zig-Zag modification* (Z^2 , for short); after that, we identify the cases where the 3-colouring output by C^2 followed by Z^2 would be infeasible, and describe the *S-block Chessboard Colouring* (SbC^2 , for short) and the *Corner Complemented Chessboard Colouring* (C^4 , for short), which, followed by Z^2 , give rise to feasible 3-colourings.

Throughout the whole section, by “colouring some elements according to a \mathcal{BW} -schema” we mean assigning colour \mathcal{B} to element $s_{i,j}$ iff $i+j$ is even, and colour \mathcal{W} iff $i+j$ is odd (thus $s_{1,1}$ is \mathcal{B}), while by “colouring some elements according to a \mathcal{WB} -schema” we mean assigning colour \mathcal{W} to element $s_{i,j}$ iff $i+j$ is even, and colour \mathcal{B} iff $i+j$ is odd (thus $s_{1,1}$ is \mathcal{W}).

To begin with, we describe the *Chessboard Colouring* (C^2 , for short) applied to $S_n(1, b)$:

C^2 - CHESSBOARD COLOURING

Colour all the elements of $S_n(1, b)$ according to a \mathcal{BW} -schema.

An edge $(s_{i,j}, s_{h,k})$ whose endpoints are assigned the same colour is *infeasible*. No infeasible edges are found “within” $S_n(1, b)$ w.r.t. C^2 , by definition. Thus, only edges “across” the geometric boundary of $S_n(1, b)$, the *boundary edges*, might be infeasible w.r.t. C^2 . As an example: $(s_{1,b}, s_{2,1})$ is a boundary edge, and it is infeasible iff b is even.

It is convenient to classify the boundary edges into “homogeneous” subsets: in Lemma 5.1 we prove, in fact, that a boundary edge in a subset is feasible (infeasible, resp.) w.r.t. C^2 iff the

whole subset is feasible (infeasible, resp.) w.r.t. C^2 . These homogeneous subsets are H_1 , H_2 , V_1 , and V_2 .

Subsets H_1 and H_2 are a partition of the set of boundary b -edges, that is of the set of edges crossing the horizontal portions of the geometric boundary of $S_n(1, b)$. Precisely, $H_1 = \{(s_{1,1}, s_{r-1,p+1}), (s_{1,2}, s_{r-1,p+2}), \dots, (s_{1,b-p}, s_{r-1,b})\}$, which is empty when the tail of $S_n(1, b)$ is empty, and $H_2 = \{(s_{1,b-p+1}, s_{r,1}), (s_{1,b-p+2}, s_{r,2}), \dots, (s_{1,b}, s_{r,p})\}$ (see Fig. 6).

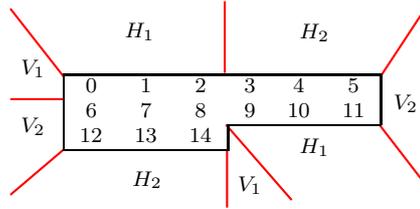


Figure 6: The boundary of $S_{15}(1, 6)$.

Subsets V_1 and V_2 are a partition of the set of boundary a -edges, that is of the set of edges crossing the vertical portions of the geometric boundary of $S_n(1, b)$. Precisely, $V_1 = \{(s_{1,1}, s_{r,p})\}$, and $V_2 = \{(s_{2,1}, s_{1,b}), (s_{3,1}, s_{2,b}), \dots, (s_{r,1}, s_{r-1,b})\}$ (see Fig. 6).

Lemma 5.1. *Consider a boundary edge $k \in K$, for $K \in \{H_1, H_2, V_1, V_2\}$. Then k is infeasible w.r.t. C^2 if and only if each boundary edge $h \in K$ is infeasible w.r.t. C^2 .*

Proof. Consider $K = V_2$ and $k = (s_{i,1}, s_{i-1,b}) \in K$ for an arbitrary $i \in \{2, \dots, r\}$. Since k is infeasible, $s_{i,1}$ and $s_{i-1,b}$ have the same colour, that is to say $i + 1$ and $i - 1 + b$ are either both even or both odd. Thus, also $h + 1$ and $h - 1 + b$, for $h = 2, \dots, r$, do always have the same parity. In the same way one can prove the result for the other three sets, and the claimed thesis follows. ■

Each subset among H_1, H_2, V_1, V_2 is feasible or infeasible w.r.t. C^2 of $S_n(1, b)$, depending on some conditions on b and $r + p$, as stated in the following lemmas.

Lemma 5.2. *H_1 and V_1 are infeasible w.r.t. C^2 of $S_n(1, b)$ if and only if $r + p$ is even.*

Proof. Consider an arbitrary boundary edge in V_1 , say $(s_{1,1}, s_{r,p})$. It is infeasible if and only if $1 + 1$ and $r + p$ have the same parity, that is to say if and only if $r + p$ is even. Consider now an arbitrary boundary edge in H_1 , say $(s_{1,j}, s_{r-1,p+j})$ ($1 \leq j \leq b - p$, by construction). It is infeasible if and only if $1 + j$ and $r - 1 + p + j$ have the same parity, as claimed. ■

Lemma 5.3. *H_2 is infeasible w.r.t. C^2 of $S_n(1, b)$ if and only if $r + p$ is odd and b is even, or viceversa.*

Proof. Consider an arbitrary boundary edge in H_2 , say $(s_{1,j}, s_{r,j-(b-p)})$ ($b - p + 1 \leq j \leq b$, by construction). It is infeasible if and only if $1 + j$ and $r + j - (b - p)$ have the same parity, that is to say if and only if $r - (b - p)$ is odd, as claimed. ■

Lemma 5.4. *V_2 is infeasible w.r.t. C^2 of $S_n(1, b)$ if and only if b is even.*

Proof. Consider an arbitrary boundary edge in V_2 , say $(s_{i,1}, s_{i-1,b})$ ($2 \leq i \leq r$, by construction). It is infeasible if and only if $i + 1$ and $i - 1 + b$ have the same parity, as claimed. ■

it is easy to prove that

Lemma 5.5. *No subset among H_1, H_2, V_1, V_2 is infeasible w.r.t. C^2 of $S_n(1, b)$ if and only if $r + p$ and b are both odd.*

From the above lemmas it follows that four cases arise, depending on the parity of b and $r + p$. Notice that the last lemma identifies the bipartite graphs, which are excluded from this section. The other three cases identify three types of *steps*, summarized in the following (see Fig. 7, where solid (dashed, resp.) lines represent infeasible (feasible, resp.) portions of the boundary):

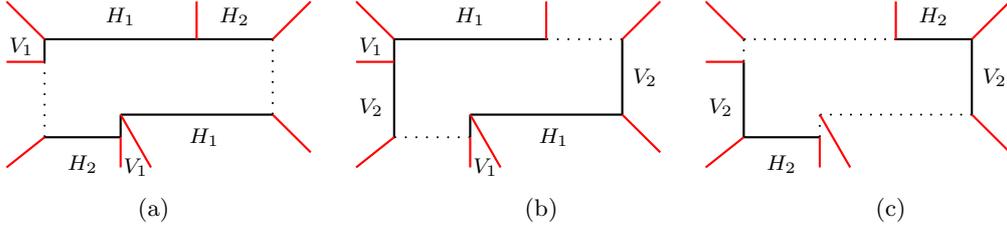


Figure 7: The three types of step on S : Step S1 (a), Step S2 (b), Step S3 (c), where solid (dotted, resp.) lines represent infeasible (feasible, resp.) portions of the boundary.

Definition 5.6. *Consider C^2 of $S_n(1, b)$, then*

- STEP S1: H_1, H_2, V_1 are infeasible w.r.t. C^2 of $S_n(1, b)$, and V_2 is not, if and only if $r + p$ is even and b is odd (see Fig. 7 (a));
- STEP S2: H_1, V_1, V_2 are infeasible w.r.t. C^2 of $S_n(1, b)$, and H_2 is not, if and only if $r + p$ and b are both even (see Fig. 7 (b));
- STEP S3: H_2, V_2 are infeasible w.r.t. C^2 of $S_n(1, b)$, and H_1, V_1 are not if and only if $r + p$ is odd and b is even (see Fig. 7 (c));

Consider $M_n^*(1, b)$, and colour all the infinitely many copies of $S_n(1, b)$ according to C^2 : the infeasible portions of the boundary of a copy of $S_n(1, b)$ end where the infeasible portions of some adjacent copy of $S_n(1, b)$ begin, whatever the step type is. Thus, the infeasibilities on the infinitely many (coloured) copies of $S_n(1, b)$ give rise to infinitely many parallel *staircases* of infinitely many steps each (see Fig. 8 (c)). We say that C^2 is a *staircase-generating* 2-colouring for $S_n(1, b)$. C^2 is not the only staircase-generating 2-colouring for $S_n(1, b)$: in the sequel we will define other two staircase-generating 2-colourings, precisely SbC^2 and C^4 .

Staircase-generating 2-colourings are important because we can obtain a feasible and optimal 3-colouring for $S_n(1, b)$ by suitably modifying the colour of the elements along any staircase. To this extent it is convenient to define a measure of how “close” two consecutive staircases are.

Definition 5.7. *Let K be an arbitrary staircase-generating 2-colouring for a graph $C_n(1, b) \in T$. The horizontal distance $HD(K)$ (vertical distance $VD(K)$, resp.) is the minimum number of elements in a row (column, resp.) of $M_n^*(1, b)$ separating two consecutive staircases generated by K on $S_n(1, b)$.*

Distances, clearly, depend on the chosen staircase-generating 2-colouring. If C^2 is chosen, then horizontal and vertical distances can be evaluated on the basis of the step type, resulting in the following:

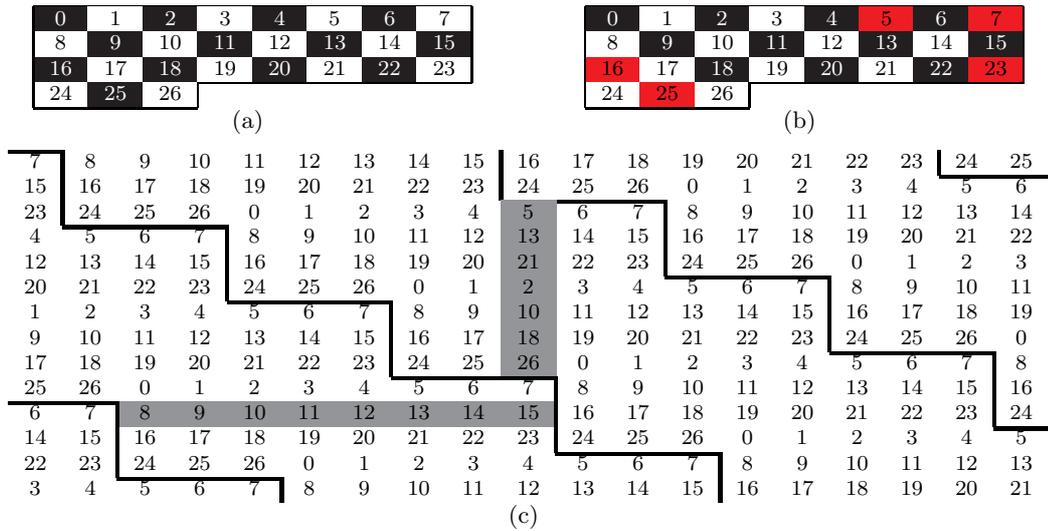


Figure 8: (a) C^2 for $C_{27}(1,8)$; (c) the staircases generated by C^2 on $M_{27}^*(1,8)$ (Z^2 changes into \mathcal{R} the colour of $v_5, v_{25}, v_7, v_{16}, v_{23}$) - the horizontal and vertical distances are highlighted in grey; (b) $C^2 + Z^2$ for $C_{27}(1,8)$.

Remark 5.8. Consider C^2 of $S_n(1,b)$, then

- in STEP S1: $HD(C^2) > b$ and $VD(C^2) = r - 1$;
- in STEP S2: $HD(C^2) = p$, $VD(C^2) = r - 1$ if $p < \frac{b}{2}$, and $VD(C^2) = r$ if $p \geq \frac{b}{2}$;
- in STEP S3: $HD(C^2) = b$ and $VD(C^2) > r$;

As an example, by applying C^2 to $C_{27}(1,8)$, one gets Step S3 and $HD(C^2) = b = 8$ with $VD(C^2) = 7 > r = 3$ (see Fig. 8 (c)).

The colour of the elements along a staircase are modified according to the so-called *Zig-Zag modification* (Z^2 , for short) here described.

Z^2 - ZIG-ZAG MODIFICATION

Consider an arbitrary staircase, and go downstairs.

Modify into \mathcal{R} the colour of the \mathcal{B} elements on the left,

and of the \mathcal{W} elements on the right.

Z^2 modifies into \mathcal{R} the colour of one endpoint of each infeasible edge. Moreover, whenever the endpoints of two infeasible edges are pairwise adjacent (that is, the four of them form a 2×2 matrix of adjacent rows and columns of $M_n^*(a,b)$), it modifies into \mathcal{R} the colour of two non-adjacent vertices. For these reasons, we can state the following theorem, where by $K + Z^2$ we denote the colouring algorithm obtained by running K followed by Z^2 .

Theorem 5.9. Let K be an arbitrary staircase-generating 2-colouring for a graph $C_n(1,b) \in \mathcal{T}$. If $HD(K) \geq 3$ and $VD(K) \geq 3$, then the 3-colouring output by $K + Z^2$ is feasible and optimal.

Proof. Z^2 removes the existing infeasibilities generated by K : in fact, it modifies the colour of one endpoint of each infeasible pair along a staircase. In addition, by definition of Z^2 , the

modified elements are not adjacent. Z^2 does not introduce any new infeasibility because the conditions $HD(K) \geq 3$ and $VD(K) \geq 3$ ensure that between any two elements along consecutive staircases, there exists at least one element whose colour is not affected by any modification. ■

Theorem 5.9 can be strengthened for C^2 , as follows:

Theorem 5.10. *Consider C^2 for a graph $C_n(1, b) \in T$. If $HD(C^2) \geq 2$ and $VD(C^2) \geq 3$, then the 3-colouring output by $C^2 + Z^2$ is feasible and optimal.*

Proof. We focus on the case $HD(C^2) = 2$, only, all the other cases following by Theorem 5.9 applied to C^2 . By definition of C^2 , $HD(C^2)$ is either p or $\geq b$. Since $b > 2$ as $C_n(1, 2) \notin T$, it is the case that $p = 2$. Such distance $HD(C^2) = p = 2$ is attained on row r of $S_n(1, b)$, which therefore has $p = 2$ elements, precisely $s_{r,1}$ and $s_{r,2}$. We can achieve this distance in the Step S2, only, which implies that r is even (see Definition 5.6). Thus, by definition of C^2 , $s_{r,1}$ is a \mathcal{W} element on the left, while $s_{r,2}$ is a \mathcal{B} element on the right, going downstairs along a staircase. For this reason Z^2 does not change the colour of $s_{r,1}$ and $s_{r,2}$, thus it does not induce new infeasibilities. Since Z^2 removes all the infeasibilities without introducing new ones, the claimed thesis follows. ■

In Fig. 8 (b) the optimal 3-colouring resulting from $C^2 + Z^2$ for the graph $C_{27}(1, 8)$ is depicted. It is worth observing that the 3-colouring output by $C^2 + Z^2$ is feasible also when $HD(C^2) = 2$ thanks to the mutual definition of C^2 and Z^2 . We could exchange the role of \mathcal{B} and \mathcal{W} in C^2 and/or in Z^2 , or the role of *left* and *right*, or the role of *downstairs* and *upstairs* in Z^2 : if we make an even number of such modifications, the colouring output by $C^2 + Z^2$ keeps being feasible, while if we make an odd number of them, it becomes infeasible.

The conditions of Theorem 5.10 show that there are cases where $C^2 + Z^2$ does not produce a feasible 3-colouring, because Z^2 happens to introduce new infeasibilities (assigning colour \mathcal{R} to adjacent endpoints), instead of just removing those generated by C^2 . For these cases C^2 is not the suitable staircase-generating 2-colouring in the first phase of our approach. These cases can be identified *a-priori*, that is without applying C^2 , because they depend on some conditions on r, b, p . For them, staircase-generating 2-colourings other than C^2 are proposed in the first phase, namely SbC^2 and C^4 , which ensure that $SbC^2 + Z^2$ or $C^4 + Z^2$ output feasible 3-colourings.

The three cases where $C^2 + Z^2$ does not work are those excluded by Theorem 5.10, that is, the cases where the horizontal and vertical distances of C^2 verify $VD(C^2) = 2$, $HD(C^2) = 1$, or $VD(C^2) = 1$. This last case never happens because the distance $VD(C^2) = 1$ can be attained on columns of the tail of $S_n(1, b)$, only. These columns are shorter than the columns in the head of $S_n(1, n)$, and have $r - 1$ elements each. The contradiction is found in the fact that, for all the well-defined $C_n(1, b)$'s, $r = 2$ implies $n = 2b$, and the tail of $S_n(1, b)$ is empty. As for the other two cases, we now start by characterizing the conditions on r, b, p under which $VD(C^2) = 2$ or $HD(C^2) = 1$, then we go on describing the staircase-generating 2-colourings suitable for them. Consider $VD(C^2) = 2$, first. The following lemma holds:

Lemma 5.11. *Consider C^2 for a graph $C_n(1, b) \in T$. $VD(C^2) = 2$ if and only if $r = 3$, and either b is even and $2 \leq p < \frac{b}{2}$ and odd, or b is odd and $2 \leq p < b - 1$ and odd.*

Proof. The distance $VD(C^2) = 2$ can be attained on the tail of $S_n(1, b)$, only: in Step S1 for any p , and in Step S2 for $p < \frac{b}{2}$. By Remark 5.8 the thesis follows. ■

The reason why $C^2 + Z^2$ in this case outputs an infeasible colouring is the following. Consider two (consecutive) columns where $VD(C^2) = 2$ is attained, say $j, j + 1$ (it is easy to prove that

$VD(C^2) = 2$ can not be attained in only one column). Either one among $s_{1,j}$ and $s_{1,j+1}$ is a \mathcal{B} element on the left going downstairs along a staircase, and the element below it in the second row of $S_n(1, b)$ is a \mathcal{W} element on the right going downstairs along a different staircase. For this reason the colour of both of them is changed into \mathcal{R} by Z^2 , generating an infeasibility. As an example, see the staircases generated by C^2 for the graph $C_{17}(1, 7)$, in Fig. 9: if Z^2 is applied, the adjacent $s_{1,5} = 4$ and $s_{2,5} = 11$, for example, would both receive colour \mathcal{R} .

15	16	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
5	6	7	8	9	10	11	12	13	14	15	16	0	1	2	3	4	5	6
12	13	14	15	16	0	1	2	3	4	5	6	7	8	9	10	11	12	13
2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	0	1	2	3
9	10	11	12	13	14	15	16	0	1	2	3	4	5	6	7	8	9	10
16	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	0
6	7	8	9	10	11	12	13	14	15	16	0	1	2	3	4	5	6	7
13	14	15	16	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
3	4	5	6	7	8	9	10	11	12	13	14	15	16	0	1	2	3	4
10	11	12	13	14	15	16	0	1	2	3	4	5	6	7	8	9	10	11
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	0	1
7	8	9	10	11	12	13	14	15	16	0	1	2	3	4	5	6	7	8
14	15	16	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Figure 9: The staircases generated by C^2 on $M_{17}^*(1, 7)$.

However, the problem of optimally 3-colouring the graphs $C_n(1, b) \in T$ on which C^2 yields $VD(C^2) = 2$, can be solved thanks to the definition of the *S-block Chessboard Colouring* (SbC^2 , for short), a staircase-generating 2-colouring whose horizontal and vertical distances $HD(SbC^2)$ and $VD(SbC^2)$ are at least 3. Theorem 5.9 ensures that $SbC^2 + Z^2$ will output a feasible (and optimal) 3-colouring for this particular case.

SbC^2 consists of partitioning $S_n(1, b)$ into column-submatrices of suitable size, the *S-blocks*, and colouring each *S-block* according to a *BW-schema* or a *WB-schema*. *S-block* sizes and colouring schema depend on some conditions on r, b, p .

SbC^2 - S-BLOCK CHESSBOARD COLOURING

If $r = 3$, and $2 \leq p < \frac{b}{2}$ and odd,
then colour the *S-block* of the first $p + (b \bmod p)$ columns of $S_n(1, b)$ according to a *BW-schema*;
partition the remaining columns into *S-blocks* of consecutive p columns;
colour each *S-block* according to a *BW-schema* if $b \bmod p$ is even, or according to a *WB-schema* if $b \bmod p$ is odd;

If $r = 3$, b odd, and $\frac{b}{2} < p < b$ and odd,
then colour the *S-block* of the first $p - 1$ columns of $S_n(1, b)$ according to a *BW-schema*;
colour the remaining columns according to a *WB-schema*.

An example of SbC^2 is depicted in Fig. 10 (a). It is easy to verify that SbC^2 is staircase-generating (see Fig. 10 (c)). Its distances are summarized in the following:

Remark 5.12. Consider a graph $C_n(1, b) \in T$ such that $r = 3$, and either b is even and $2 \leq p < \frac{b}{2}$ and odd, or b is odd and $2 \leq p < b - 1$ and odd, and consider SbC^2 of the corresponding $S_n(1, b)$.

i) If $2 \leq p < \frac{b}{2}$ odd, then $HD(SbC^2) = p$ and $VD(SbC^2) \geq r$.

ii) If $\frac{b}{2} < p < b$ odd, b odd, then $HD(SbC^2) \geq b - p + 1$ and $VD(SbC^2) \geq r$.

As an example, on the graph $C_{17}(1, 7)$ one gets $HD(SbC^2) = p = 3$ and $VD(SbC^2) = 17 > r = 3$ (see Fig. 10 (c)).

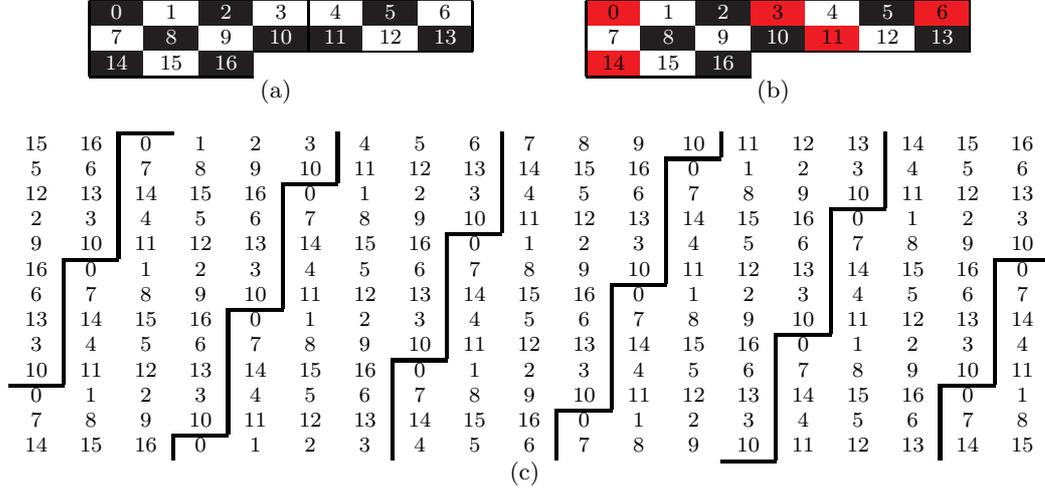


Figure 10: (a) SbC^2 for $C_{17}(1, 7)$; (c) the staircases generated by SbC^2 on $M_{17}^*(1, 7)$ (Z^2 changes into \mathcal{R} the colour of $v_0, v_6, v_{14}, v_3, v_{11}$); (b) $SbC^2 + Z^2$ for $C_{17}(1, 7)$.

As in both cases $i)$ and $ii)$ of the above remark, one has $HD(SbC^2) \geq 3$ and $VD(SbC^2) \geq 3$, Theorem 5.9 applies, resulting in the following:

Corollary 5.13. *Let $C_n(1, b) \in T$ such that $r = 3$, b even, and $2 \leq p < \frac{b}{2}$ and odd, or $r = 3$, b is odd, and $2 \leq p < b - 1$ and odd. Then the 3-colouring output by $SbC^2 + Z^2$ is feasible and optimal.*

In Fig. 10 (b) the 3-colouring output by $SbC^2 + Z^2$ for the graph $C_{17}(1, 7)$ is depicted. Consider now $HD(C^2) = 1$. The following lemma holds:

Lemma 5.14. *Consider C^2 for a graph $C_n(1, b) \in T$. $HD(C^2) = 1$ if and only if $r \geq 5$ and odd, b is even, and $p = 1$.*

Proof. Recalling that $HD(C^2)$ is either p or $\geq b$, and that $b \geq 3$, the distance $HD(C^2) = p = 1$ can be attained in Step S2, only, and in row r of $S_n(1, b)$, only. This row must have $p = 1$ element, namely $s_{r,1}$. Since a Step S2 arises iff $r + p$ and b are both even, $p = 1$ implies r odd, and the thesis follows. ■

The reason why $C^2 + Z^2$ in this case produces an infeasible colouring is now explained. By definition of C^2 , $s_{r,1}$ is a \mathcal{B} element on the left of a staircase and, at the same time, on the right of the consecutive one, going downstairs. In being a \mathcal{B} element on the left of a staircase Z^2 would change its colour into \mathcal{R} , while in being a \mathcal{B} element on the right of the consecutive staircase Z^2 would not have to change its colour, yielding a contradiction. As an example, see the staircases generated by C^2 for the graph $C_{17}(1, 4)$ in Fig. 11, where $s_{r,1} = 16$.

However, the problem of optimally 3-colouring the graphs $C_n(1, b) \in T$ on which a Chessboard Colouring yields $HD(C^2) = 1$ can be solved thanks to the definition of the *Corner Complemented*

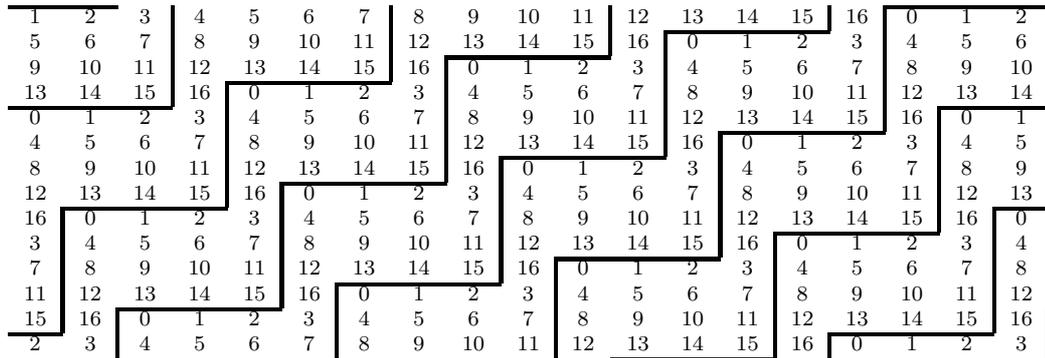


Figure 11: The staircases generated by C^2 on $M_{17}^*(1,4)$.

Chessboard Colouring (C^4 , for short), a staircase-generating 2-colouring whose horizontal and vertical distances $HD(C^4)$ and $VD(C^4)$ are at least 3. Theorem 5.9 ensures that $C^4 + Z^2$ will output a feasible (and optimal) 3-colouring for this particular case.

C^4 - CORNER COMPLEMENTED CHESSBOARD COLOURING

Colour elements $s_{1,1}, s_{1,2}, s_{2,1}$ according to a WB -schema;
 colour the remaining elements according to a BW -schema.

Notice that C^4 can be obtained from C^2 by complementing the colour of the three elements $s_{1,1}, s_{1,2}, s_{2,1}$. It is easy to verify that C^4 is staircase-generating (see Fig. 12 (c)). Its distances are summarized in the following:

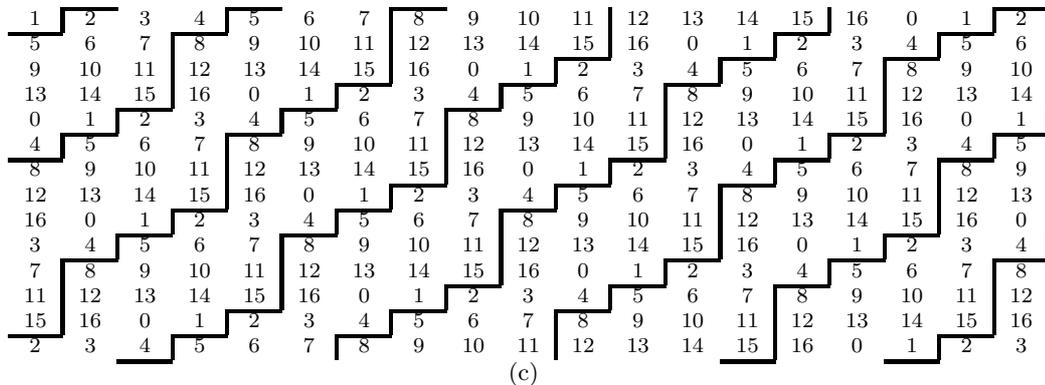
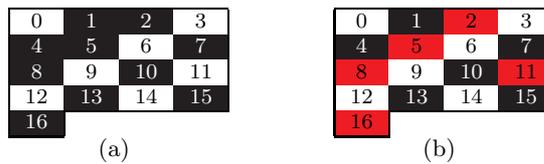


Figure 12: (a) C^4 for $C_{17}(1,4)$; (c) the staircases generated by C^4 on $M_{17}^*(1,4)$ (Z^2 changes into \mathcal{R} the colour of $v_2, v_5, v_8, v_{11}, v_{16}$); (b) $C^4 + Z^2$ for $C_{17}(1,4)$.

Remark 5.15. Consider a graph $C_n(1, b) \in T$ such that $r \geq 5$ and odd, b is even, and $p = 1$, and consider C^4 of the corresponding $S_n(1, b)$, then $HD(C^4) = 3$ and $VD(C^4) = r \geq 5$.

As an example, on the graph $C_{17}(1, 4)$ one gets $HD(C^4) = 3$ and $VD(C^4) = r \geq 5$ (see Fig. 12 (c)).

As horizontal and vertical distances of C^4 verify $HD(C^4) \geq 3$ and $VD(C^4) \geq 3$, Theorem 5.9 applies, resulting in the following:

Corollary 5.16. Let $C_n(1, b) \in T$ such that $r \geq 5$ and odd, b is even, and $p = 1$. Then the 3-colouring output by $C^4 + Z^2$ is feasible and optimal.

An example can be found in Fig. 12.

As a final result, the following theorem can be stated:

Theorem 5.17. Let $C_n(1, b) \in T$. Then there exists a staircase-generating 2-colouring $K \in \{C^2, SbC^2, C^4\}$ such that the 3-colouring output by $K + Z^2$ is feasible and optimal.

To summarize we have the following algorithm:

```

ALGORITHM FOR THE WELL-DEFINED  $C_n(1, b)$ 'S WITH  $2 < b \leq \frac{n}{2}$ 

Let  $r := \lceil \frac{n}{b} \rceil$ ;
Let  $p := n - (r - 1)b$ ;
If  $r \geq 5$  and odd,  $b$  even, and  $p = 1$ ,
    then apply  $C^4$ ;

If  $r = 3$  and  $((b$  even,  $p$  odd,  $2 \leq p < \frac{b}{2}$ ) or  $(b, p$  odd,  $2 \leq p < b - 1))$ ,
    then apply  $SbC^2$ ;

In all other cases apply  $C^2$ ;
Apply  $Z^2$ .

```

We just remark that it is an easy task to describe the 3-colouring resulting from the above algorithm in a closed form.

5.3. Colouring $C_n(a, b)$ with $\gcd(n, a), \gcd(n, b) \geq 2$

This section is devoted to describe optimal 3-colouring algorithms for the non-bipartite (well-defined) graphs $C_n(a, b)$ verifying $\gcd(n, a), \gcd(n, b) \geq 2$. Optimal colourings for these graphs will be constructed on the representative matrix $M_n(a, b)$. To this extent, recall that $R = \gcd(n, a)$, $C = \frac{n}{\gcd(n, a)} = H \gcd(n, b)$, and that $\lambda = \min\{z \in \mathbb{Z}^+ : \gcd(n, a)b \equiv za \pmod{n}\}$ (see also Section 3.2). Finally, since $C_n(a, b) \simeq C_n(b, a)$, throughout the rest of the present section, we assume that $\gcd(n, b) > \gcd(n, a)$. Recall that $\gcd(n, b) > \gcd(n, a) \geq 2$ and $\gcd(n, a, b) = 1$ imply both $\gcd(n, b) \geq 3$ and $\gcd(n, a) \geq 2$, that is to say, $C \geq 3$, $R \geq 2$, and $n \geq 6$.

In the whole section we shall make use of the *blocks*, which are defined as follows:

Definition 5.18. [23] Block β_k , for $k = 0, \dots, H - 1$, is the submatrix of $M_n(a, b)$ defined on $R = \gcd(n, a)$ rows and the $(k - 1)$ -th set of $\gcd(n, b)$ consecutive columns.

Notice that blocks are horizontally aligned in $M_n^*(a, b)$ (by definition). We remark that they are also vertically aligned in $M_n^*(a, b)$. In fact, the j -th column of each block represents a portion of a same b -cycle. As an example, the blocks of $M_{36}(3, 8)$ are 3, and each one is defined on 3 rows and 4 columns. Now consider one of its four b -cycles, say the one whose vertices are $v_0, v_8, v_{16}, v_{24}, v_{32}, v_4, v_{12}, v_{20}, v_{28}$: the vertices are found in the first column of β_0, β_2 , and β_1 , respectively.

Two different methods are proposed to optimally 3-colour the graphs of the present section. One method uses the same technique of the algorithms in the preceding section: it consists of a suitable staircase-generating 2-colouring K of $M_n(a, b)$, followed by a Z^2 modification ($K + Z^2$, for short), which assigns a third colour to one suitable endpoint for each infeasible edge. The other method, the *Tile Colouring* (TC , for short), consists of directly applying a same suitable 3-colouring to each block of $M_n(a, b)$.

First Method

Like already done in the previous section, the approach we use has two phases. In the first one we start with a suitable staircase-generating \mathcal{BW} -colouring of $M_n(a, b)$. There will be *infeasible* edges $(m_{i,j}, m_{h,k})$ (recall in fact that bipartite graphs are excluded from this section). In order to remove infeasibilities we proceed with the second phase, suitably modifying into \mathcal{R} the colour of one endpoint in each infeasible edge, by means of the Zig-Zag modification (see Section 5.2).

Theorem 5.9 can be extended to the graphs of the present section. In most cases C^2 is a staircase-generating 2-colouring whose horizontal and vertical distances are ≥ 3 . When this is not the case, BC^2 , the *Block Chessboard Colouring*, described later, is a staircase-generating 2-colouring whose horizontal and vertical distances are ≥ 3 . In order to characterize the cases where C^2 does not verify the conditions of Theorem 5.9, we need to preliminarily analyse the sets of infeasible edges. No infeasible edges are found “within” $M_n(a, b)$ w.r.t. C^2 , by definition, but only across its boundary. Reasoning as in Section 5.2, it is convenient to classify the boundary edges into the “homogeneous” subsets H_1, H_2, V , precisely: $H_1 = \{(m_{1,1}, m_{R,C-\lambda+1}), (m_{1,2}, m_{R,C-\lambda+2}), \dots, (m_{1,\lambda}, m_{R,C})\}$; $H_2 = \{(m_{1,\lambda+1}, m_{R,1}), (m_{1,\lambda+2}, m_{R,2}), \dots, (m_{1,C}, m_{R,C-\lambda})\}$; and $V = \{(m_{1,1}, m_{1,C}), (m_{2,1}, m_{2,C}), \dots, (m_{R,1}, m_{R,C})\}$ (see Fig. 13). Notice the dependence of H_1, H_2 from λ , and that H_1 is empty when $\lambda = 0$. In [23] it is proved that λ is a multiple of $\gcd(n, b)$, thus $|H_1|$ and $|H_2|$ also are.

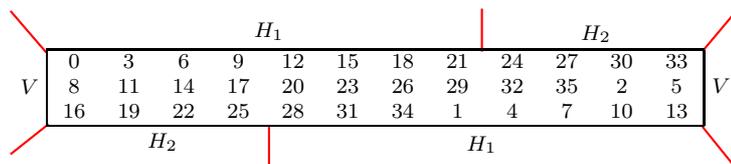


Figure 13: The boundary of $M_{36}(3, 8)$.

Here too, a result similar to that of Lemma 5.1 holds, and each subset among H_1, H_2, V is feasible or infeasible w.r.t. C^2 of $M_n(a, b)$, depending on some conditions on C and $R + \lambda$. Since C^2 is a staircase-generating 2-colouring for $M_n(a, b)$ (see Fig. 14 (b)), we can extend Definition 5.7 to the following one, which also includes the distances (recall that no infeasibilities exist iff the graph is bipartite, that is iff $R + \lambda$ and C are both even).

Definition 5.19. Consider a C^2 of $M_n(a, b)$, then

- STEP M1: H_1, V are infeasible w.r.t. C^2 of $M_n(a, b)$, and H_2 is not, if and only if $R + \lambda$

is even and C is odd; in this case one has $HD(C^2) = C$ for any value of λ , $VD(C^2) > R$ if $\lambda \leq \frac{C}{2}$ and $VD(C^2) = R$ if $\lambda > \frac{C}{2}$;

- STEP M2: H_2, V are infeasible w.r.t. C^2 of $M_n(a, b)$, and H_1 is not, if and only if $R + \lambda$ and C are both odd; in this case one has $HD(C^2) = C$ for any value of λ , $VD(C^2) = R$ if $\lambda < \frac{C}{2}$ and $VD(C^2) > R$ if $\lambda \geq \frac{C}{2}$;
- STEP M3: H_1, H_2 are infeasible w.r.t. C^2 of $M_n(a, b)$, and V is not if and only if $R + \lambda$ is odd and C is even; in this case one has $HD(C^2) = \infty$ and $VD(C^2) = R$, for any value of λ .

0	8	16	24	32	40	48	56	4	12	20	28	36	44	52
5	13	21	29	37	45	53	1	9	17	25	33	41	49	57
10	18	26	34	42	50	58	6	14	22	30	38	46	54	2
15	23	31	39	47	55	3	11	19	27	35	43	51	59	7

(a)

19	27	35	43	51	59	7	15	23	31	39	47	55	3	11	19	27	35	43
24	32	40	48	56	4	12	20	28	36	44	52	0	8	16	24	32	40	48
29	37	45	43	51	59	17	25	33	41	49	57	5	13	21	29	37	45	53
34	42	50	58	6	14	22	30	38	46	54	2	10	18	26	34	42	50	58
39	47	55	3	11	19	27	35	43	51	59	7	15	23	31	39	47	55	3
44	52	0	8	16	24	32	40	48	56	4	12	20	28	36	44	52	0	8
49	57	5	13	21	29	37	45	53	1	9	17	25	33	41	49	57	5	13
54	2	10	18	26	34	42	50	58	6	14	22	30	38	46	54	2	10	18
59	7	15	23	31	39	47	55	3	11	19	27	35	43	51	59	7	15	23
4	12	20	28	36	44	52	0	8	16	24	32	40	48	56	4	12	20	28
9	17	25	33	41	49	57	5	13	21	29	37	45	53	1	9	17	25	33
14	22	30	38	46	54	2	10	18	26	34	42	50	58	6	14	22	30	38
19	27	35	43	51	59	7	15	23	31	39	47	55	3	11	19	27	35	43
24	32	40	48	56	4	12	20	28	36	44	52	0	8	16	24	32	40	48

(b)

0	8	16	24	32	40	48	56	4	12	20	28	36	44	52
5	13	21	29	37	45	53	1	9	17	25	33	41	49	57
10	18	26	34	42	50	58	6	14	22	30	38	46	54	2
15	23	31	39	47	55	3	11	19	27	35	43	51	59	7

(c)

Figure 14: (a) C^2 for $C_{60}(8, 5)$; (b) the staircases generated by C^2 on $M_{60}^*(8, 5)$ (Z^2 changes into \mathcal{R} the colour of $v_0, v_3, v_{16}, v_{19}, v_{32}, v_{35}, v_{48}, v_{51}, v_4, v_7, v_{10}, v_{57}$); (c) $C^2 + Z^2$ for $C_{60}(8, 5)$.

The three types of steps are depicted in Fig. 15, where solid (dashed, resp.) lines represent infeasible (feasible, resp.) portions of the boundary. As an example, consider C^2 of $C_{60}(8, 5)$ (where $\lambda = 10$). One gets Step M1, $VD(C^2) = 4$ and $HD(C^2) = C = 15$ (see Fig. 14 (b)).

We now proceed by applying Z^2 . As an extension of Theorem 5.9 we have

Theorem 5.20. *Let K be an arbitrary staircase-generating 2-colouring for a well-defined graph $C_n(a, b)$ with $\gcd(n, b) > \gcd(n, a) > 1$. If $HD(K) \geq 3$ and $VD(K) \geq 3$, then the 3-colouring output by $K + Z^2$ is feasible and optimal.*

The 3-colouring output by $C^2 + Z^2$ for the graph $C_{60}(8, 5)$ is depicted in Fig. 14 (c). Recalling that, in the present section, $C \geq 3$ and $R \geq 2$, one has $HD(C^2) \geq 3$ for all the step types, while there are cases where $VD(C^2) = 2$, to which the above theorem does not apply. The reason is that, under these conditions, it follows from the definition of H_1, H_2 that there are at least $\gcd(n, b)$ consecutive columns (the number of columns of one block) where $VD(C^2) = 2$

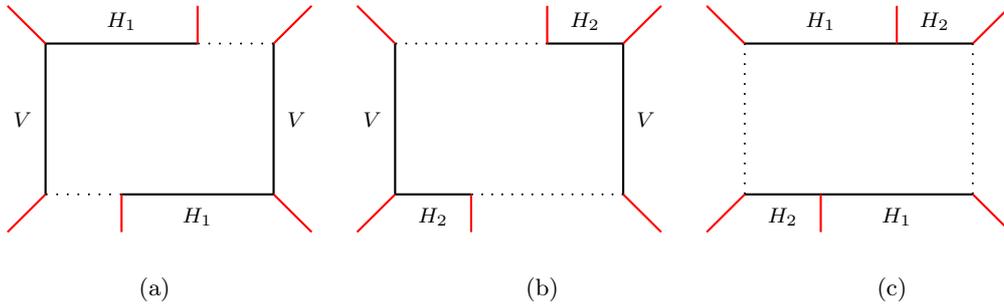


Figure 15: The three types of step on M : Step M1 (a), Step M2 (b), Step M3 (c). Solid (dotted, resp.) lines represent infeasible (feasible, resp.) portions of the boundary.

is attained. Let $j, j + 1$ be two (consecutive) of them. Either one among $m_{1,j}$ and $m_{1,j+1}$ is a \mathcal{B} element on the left going downstairs along a staircase, and the element below it in the second row of $M_n(a, b)$ is a \mathcal{W} element on the right going downstairs along a different staircase. For this reason the colour of both of them is changed into \mathcal{R} by Z^2 , generating an infeasibility. This shows that $C^2 + Z^2$ outputs an infeasible colouring of $M_n(a, b)$.

Precisely, the three cases where $VD(C^2) = 2$ are identified by the following conditions: when $R = 2$, C odd, $\lambda > \frac{C}{2}$, and λ even; when $R = 2$, C odd, $\lambda < \frac{C}{2}$, and λ odd; and when $R = 2$, C even, and λ odd. In these cases there exists a staircase-generating 2-colouring, namely the *Block Chessboard Colouring* (BC^2 , for short), whose horizontal and vertical distances are ≥ 3 . This method colours $M_n(a, b)$ by blocks, as now described.

BC^2 - BLOCK CHESSBOARD COLOURING

Colour each block $\beta_0, \dots, \beta_{H-1}$ of $M_n(a, b)$ like a chessboard, in such a way that the upper left corner of each block has always the same colour (\mathcal{B} or \mathcal{W} does not matter).

No infeasible edges are found within a block; they are found across the boundary of each block. In particular, the horizontal boundary separating two blocks on top of one another in $M_n^*(a, b)$ is always feasible: in fact the number $\gcd(n, a)$ of rows of a block is even, as we are assuming that $\gcd(n, a) = 2$. On the contrary, the vertical boundary separating two consecutive blocks is always infeasible: in fact the number $\gcd(n, b)$ of columns of a block is odd, as the assumed conditions $\gcd(n, a) = 2$, $\gcd(n, a, b) = 1$ and $\gcd(n, b) > \gcd(n, a)$ imply $\gcd(n, b) \geq 3$ and odd. This proves that $HD(BC^2) = \gcd(n, b) \geq 3$ and $VD(BC^2) \geq 3$ as $VD(BC^2) > \gcd(n, a) = 2$, and Theorem 5.20 applies.

By what above, we have the following

Theorem 5.21. *Let $C_n(a, b)$ be a (well-defined) graph with $\gcd(n, b) > \gcd(n, a) \geq 2$. Then there exists a staircase-generating 2-colouring $K \in \{C^2, BC^2\}$ such that the 3-colouring output by $K + Z^2$ is feasible and optimal.*

An example can be found in Fig. 16.

To summarize we have the following algorithm:

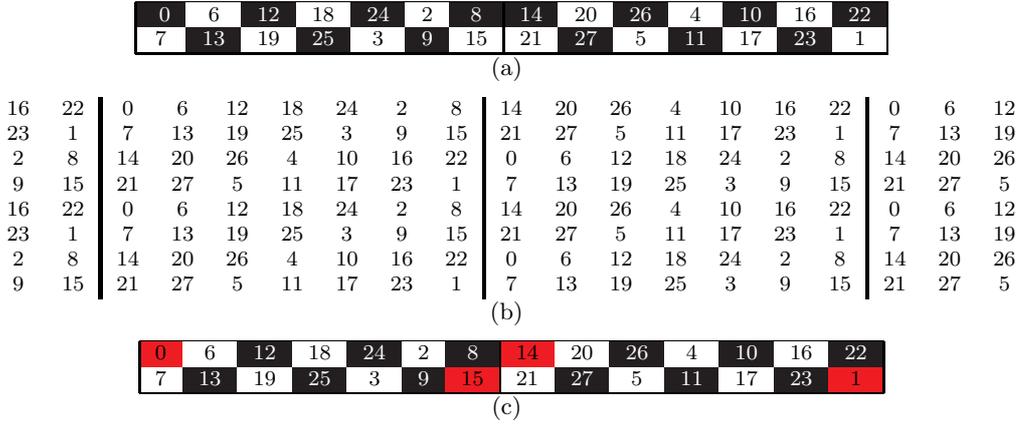


Figure 16: (a) BC^2 for $C_{28}(2, 7)$; (b) the staircases generated by BC^2 on $M_{28}^*(2, 7)$ (Z^2 changes into \mathcal{R} the colour of v_0, v_1, v_{14}, v_{15}); (c) $BC^2 + Z^2$ for $C_{28}(2, 7)$.

ALGORITHM FOR THE $C_n(a, b)$ 'S WITH $\gcd(n, b) > \gcd(n, a) \geq 2$

Let $R := \gcd(n, a)$;

Let $C := \frac{n}{\gcd(n, a)}$;

Let $\lambda := \min\{z \in \mathbb{Z}^+ : \gcd(n, a)b \equiv za \pmod{n}\}$;

If $(R = 2, C \text{ odd}, (\lambda > \frac{C}{2} \text{ and even}) \text{ or } (\lambda < \frac{C}{2} \text{ and odd}))$
 or $(R = 2, C \text{ even, and } \lambda \text{ odd})$,
 then apply BC^2 ;

Else apply C^2 ;

Apply Z^2 .

Second Method

We here describe the *Tile Colouring* (TC , for short). It consists of directly applying a *same* 3-colouring to each block of $M_n(a, b)$. Recalling that blocks are horizontally and vertically aligned, in order to ensure that adjacent blocks do not give rise to infeasibilities, the 3-colouring is chosen so as to satisfy the properties that the first and last element of each row and of each column, as well as adjacent elements within a block, must have different colours.

To this extent, recalling that the block is defined on $\gcd(n, a)$ rows and $\gcd(n, b)$ columns, we introduce the 3-coloured *tile* $T_{\gcd(n, a) \times \gcd(n, b)}$. It is obtained by adding or removing columns or rows of the *basic tile* $T_{3 \times 3}$ (depicted in Fig. 17 (a)), as we are going to explain.

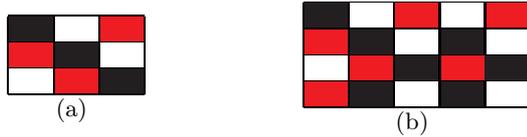


Figure 17: (a) The basic tile $T_{3 \times 3}$; (b) the (4×5) -tile $T_{4 \times 5}$.

Without loss of generality, consider the desired number $\gcd(n, b)$ of columns, first. Two cases arise: $\gcd(n, b) = 3$, or $\gcd(n, b) > 3$. If $\gcd(n, b) = 3$ consider the 3 columns of $T_{3 \times 3}$. If

$\gcd(n, b) > 3$ keep adding a copy of the second column and a copy of the third column of $T_{3 \times 3}$ until the desired number $\gcd(n, b)$ of columns is reached. The result is a tile $T_{3 \times \gcd(n, b)}$.

We now construct $T_{\gcd(n, a) \times \gcd(n, b)}$ from $T_{3 \times \gcd(n, b)}$. We have to modify the number of rows of $T_{3 \times \gcd(n, b)}$ so as to reach the desired number $\gcd(n, a)$ of rows. Three cases arise: $\gcd(n, a) = 2$, $\gcd(n, a) = 3$, or $\gcd(n, a) > 3$. If $\gcd(n, a) = 2$ consider the first and second rows of $T_{3 \times \gcd(n, b)}$, only. If $\gcd(n, a) = 3$ consider all the 3 rows of $T_{3 \times \gcd(n, b)}$. If $\gcd(n, a) > 3$ keep adding a copy of the second row and a copy of the third row of $T_{3 \times \gcd(n, b)}$ until the desired number $\gcd(n, a)$ of rows is obtained. The resulting tile is $T_{\gcd(n, a) \times \gcd(n, b)}$ (see $T_{4 \times 5}$ in Fig. 17 (b)). Since the basic tile has the required properties, and the operations of enlarging or reducing its sizes preserve them, $T_{\gcd(n, a) \times \gcd(n, b)}$ has the required properties. Thus the algorithm is:

ANOTHER ALGORITHM FOR THE $C_n(a, b)$ 'S WITH $\gcd(n, b) > \gcd(n, a) \geq 2$

Colour each block $\beta_0, \dots, \beta_{H-1}$ of $M_n(a, b)$ according to $T_{\gcd(n, a) \times \gcd(n, b)}$.

By the discussion above, the following theorem holds.

Theorem 5.22. *Let $C_n(a, b)$ be a non-bipartite (well-defined) graph with $\gcd(n, b) > \gcd(n, a) \geq 2$. Then the 3-colouring of $M_n(a, b)$ which results from TC is feasible and optimal.*

An example is depicted in Fig. 18.

0	8	16	24	32	40	48	56	4	12	20	28	36	44	52
5	13	21	29	37	45	53	1	9	17	25	33	41	49	57
10	18	26	34	42	50	58	6	14	22	30	38	46	54	2
15	23	31	39	47	55	3	11	19	27	35	43	51	59	7

Figure 18: TC for $C_{60}(8, 5)$. The three blocks of $M_{60}(8, 5)$ are separated in the picture and coloured according to $T_{4 \times 5}$.

6. Conclusions

In this paper we propose exact linear colouring algorithms for the vertex-colouring problem on arbitrary circulant graphs $C_n(a, b)$. In $O(\log^2 n)$ one can determine which is the correct algorithm for the case at hand. In fact, the most expensive task, from a computational point of view, is recognizing if the given graph is well-defined: for this reason, one has to compute $\gcd(n, a)$, $\gcd(n, b)$, and possibly solve the linear congruence of Corollary 2.5. The claimed complexity follows, since all these operations can be done $O(\log^2 n)$ by means of Euclid's Algorithm [10]. Once the suitable algorithm for the case at hand has been determined, one has to apply it: any algorithm requires $O(n)$ time, as most vertices are coloured only once, and a few of them twice.

The characteristics of our colouring algorithms, based on a matrix representation of the graph, allow for a possible extension to the circulant graphs $C_n(a_1, a_2, \dots, a_k)$. An immediate generalization of our method, for example, allows for 3-colouring the $C_n(a_1, a_2, \dots, a_k)$'s such that every a_i , for $i \geq 3$, is a weighted combination of a and b , and the sum of the weights is odd and not "too large" [24]. The simple combinatorial approach proposed in this paper also applies to the vertex colouring problem on similar graph classes, such as Toeplitz and integer distance graphs [25].

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