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A. Galluccio, C. Gentile, P. Ventura

ON THE STABLE SET POLYTOPE OF CLAW-FREE GRAPHS

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Anna Galluccio – Istituto di Analisi dei Sistemi ed Informatica "Antonio Ruberti" del CNR, Viale Manzoni 30 - 00185 Roma, Italy. Email: galluccio@iasi.cnr.it.

Claudio Gentile – Istituto di Analisi dei Sistemi ed Informatica "Antonio Ruberti" del CNR, Viale Manzoni 30 - 00185 Roma, Italy. Email: gentile@iasi.cnr.it.

Paolo Ventura – Istituto di Analisi dei Sistemi ed Informatica "Antonio Ruberti" del CNR, Viale Manzoni 30 - 00185 Roma, Italy. Email: ventura@iasi.cnr.it.

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Collana dei Rapporti dell'Istituto di Analisi dei Sistemi ed Informatica "Antonio Ruberti", CNR

viale Manzoni 30, 00185 ROMA, Italy

tel. ++39-06-77161
fax ++39-06-7716461
email: iasi@iasi.cnr.it

Abstract

We define the class of *geared (fuzzy) line graphs* as the family of graphs obtained by repeated applications of the gear composition to a (fuzzy) line graph H. Using the decomposition theorem for claw-free graphs of Chudnovsky and Seymour [2], we show that this class represents a large subclass of claw-free graphs having stability number greater than 3.

We provide a complete linear description of the stable set polytope of geared (fuzzy) line graphs. This result gives the first positive answer to the longstanding open question of finding a defining linear system for the stable set polytope of claw-free graphs which are not quasi-line [10]. Furthermore, it opens the possibility of designing a polyhedral algorithm for solving the stable set problem in claw-free graphs, possibly computationally more effective than the existing ones of complexity $O(n^6)$.

Key words: stable set polytope, graph composition, claw-free graphs.

1. Introduction

Let $P \subseteq \mathbb{R}^n$ be a polyhedron; a linear system $Ax \leq b$ is said to be *defining* for P if $P = \{x \in \mathbb{R}^V_+ : Ax \leq b\}$. The *facet defining inequalities* (*facets*, for short) for P are those inequalities that constitute the unique (up to positive multiplications) nonredundant defining linear system of P. Given $c \in \mathbb{R}^n$, the *optimization problem* over P consists of finding the maximum value of $c^T x$ for $x \in P$. So, finding the defining linear system for P is equivalent to transform the original optimization problem into the linear program $\max\{c^T x : Ax \leq b\}$. Given $x^* \in \mathbb{R}^n$, the *separation problem* over P is the problem of finding an inequality valid for all points of P and violated by x^*, or proving that $x^* \in P$. A well-known result of Grötschel, Lovász and Schrijver [10] states that the existence of a polynomial time algorithm to optimize over P, for any $c \in \mathbb{R}^n$, is equivalent to the existence of a polynomial time separation algorithm over P, for any $x^* \in \mathbb{R}^n$. In practice, a consequence of this result is that a defining linear system for P may be dynamically determined by solving the optimization problem with respect to different objective functions. Therefore, a large accepted conjecture in the Combinatorial Optimization community is that if there exists a polynomial time algorithm to optimize over a polyhedron P, then an explicit description of the defining linear system of P can also be found. Only for very few known problems [3] this conjecture is still open and one of them is the stable set problem for claw-free graphs.

Given a graph $G = (V, E)$ and a vector $w \in \mathbb{Q}^V_+$ of node weights, the *stable set problem* is the problem of finding a set of pairwise nonadjacent nodes *(stable set)* of maximum weight. Let $\alpha(G, w)$ denote the maximum weight of a stable set of G; we refer to $\alpha(G) = \alpha(G, \mathbb{1})$ ($\mathbb{1}$ being the vector of all ones) as the *stability number* of G. The *stable set polytope*, denoted by $STAB(G)$, is the convex hull of the incidence vectors of the stable sets of G. Since the stable set problem is NP-hard, it is unlikely to find a defining linear system of $STAB(G)$ for general graphs. Nevertheless there are classes of graphs for which such systems are known, as bipartite graphs, line graphs [4], series-parallel graphs [12], perfect graphs. For all these classes of graphs, the weighted stable set problem is polynomial time solvable and an explicit linear description of $STAB(G)$ is known.

Claw-free graphs are those graphs such that the neighborhood of each node has no stable set of size three. There exist polynomial time algorithms for solving the maximum weight stable set problem on a claw-free graph [13, 14] but, despite many research efforts [8, 9, 11, 16] and many disproved conjectures [9, 6], a linear description of the stable set polytope of claw-free graphs is still unknown. Finding such a defining linear system is the first step towards the design of a new algorithm for claw-free graphs, possibly computationally more effective than the existing ones (the algorithm in [13] is $O(|V|^7)$ and the one in [14] is $O(|V|^6)$).

The recent work of Chudnovsky and Seymour [2] on the structure of claw-free graphs settled new directions to investigate the problem of finding a defining linear system for $STAB(G)$ when G is claw-free. In this paper we consider the set of claw-free graphs with stability number greater 3, denoted by $\mathcal{C}$. A graph is quasi-line if the neighborhood of each node can be partitioned into two cliques (this implies that a quasi-line graph is claw-free); we denote by $\mathcal{Q}$ the set of *quasi-line* graphs with stability number greater than 3. Clearly, $\mathcal{Q} \subseteq \mathcal{C}$. In [1] Chudnovsky and Seymour proved that the set $\mathcal{Q}$ is partitioned into two sets: $\mathcal{Q}^\ell$ (*(fuzzy) line graphs*) and $\mathcal{Q}^c$ (*(fuzzy) circular interval graphs*). Then they showed that any graph in $\mathcal{C}$ either belongs to $\mathcal{Q}^c$ or it can be obtained by composing three types of graphs, called *(fuzzy) strips*.

A defining linear system for $STAB(G)$ was given by Chudnovsky and Seymour [1] when $G \in \mathcal{Q}^\ell$ and by Eisenbrand et al. [5] when $G \in \mathcal{Q}^c$. It remains open the problem of finding a linear description for $STAB(G)$ when $G \in \mathcal{C} \setminus \mathcal{Q}^c$. Here we consider the class $\mathcal{XX}$ of graphs obtained by composing only two types of strips: (fuzzy) linear interval strips and XX-strips. From [2], $\mathcal{XX} \subseteq \mathcal{C} \setminus \mathcal{Q}^c$ and the situation is illustrated in Fig. 1.

In this paper we provide a linear description of $STAB(G)$ when G

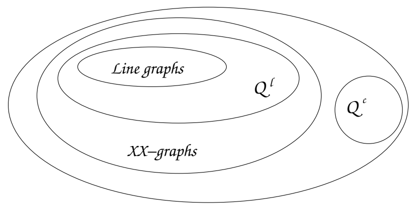

Figure 1: The class $\mathcal{C}$

starting from a given graph H and substituting an edge of H with the fixed graph (*gear*) shown in Fig. 2.

The gear composition produces new facets for the stable set polytope [6], called *geared inequalities*, which play an important role in solving the problem of finding a linear description for $STAB(G)$ when $G \in \mathcal{XX}$. Here, we prove that any graph in $G \in \mathcal{XX}$ can be built from a graph in $\mathcal{Q}^\ell$ via the gear composition and then we show that a defining linear system for $STAB(G)$ consists of: *rank inequalities, (lifted) 5-wheel inequalities and (lifted) geared inequalities.*

We now introduce some notations and basic definitions. We denote by G any graph with node set V_G and edge set E_G. Given a vector $\beta \in \mathbb{R}^m$ and a subset $S \subseteq \{1, \ldots, m\}$, define $\beta_S \in \mathbb{R}^{|S|}$ as the subvector of β restricted on the indices of S and $\beta(S) = \sum_{i \in S} \beta_i$. Given a subset $S \subseteq \{1, \ldots, m\}$, we denote by $x^S \in \mathbb{R}^m$ the incidence vector of S.

A linear inequality $\sum_{j \in V_G} \pi_j x_j \leq \pi_0$ is *valid* for $STAB(G)$ if it holds for all $x \in STAB(G)$. For short, we also denote a linear inequality $\pi^T x \leq \pi_0$ as (π, π_0). A *rank inequality* has $\pi \in \{0,1\}^{|V_G|}$.

We denote by $\delta(v)$ the set of edges of G having v as endnode and by $N(v)$ the set of nodes of V_G adjacent to v. We also denote by $G \setminus A$ the subgraph of G induced by $V_G \setminus A$ where $A \subseteq V_G$ and by $G - e$ $(G + e)$ the subgraph of G obtained by removing (adding) the edge e.

A k-*hole* $C_k = (v_1, v_2, \ldots, v_k)$ is a chordless cycle of length k. A k-*antiwheel* $W = (h : \overline{C}_k)$ is a graph consisting of a k-*antihole* $\overline{C}_k$ and a node h (*hub* of W) adjacent to every node of $\overline{C}_k$. If $k = 5$, then $\overline{C}_5$ is isomorphic to C_5 and we refer to W as a *5-wheel*. The inequality $\sum_{i=1}^5 x_{v_i} + 2x_h \leq 2$ is facet defining for $STAB(W)$ and it is called *5-wheel inequality*. A *claw* is 3-antiwheel denoted by $(y : w_1, w_2, w_3)$, where y is the hub of the claw. A *gear* B is a graph of eight nodes $\{a, b_1, b_2, c, d_1, d_2, h_1, h_2\}$ such that $W_1 = (h_1 : a, d_1, b_1, c, h_2)$ and $W_2 = (h_2 : a, d_2, b_2, c, h_1)$ are 5-wheels (see Fig. 2); moreover, the edges of these wheels are the only edges of B.

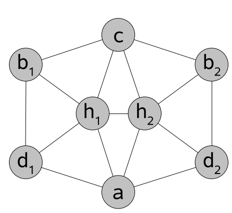

Figure 2: The gear with nodes $a, b_1, b_2, c, d_1, d_2, h_1, h_2$.

In Section 2, we recall the definition of *gear composition* and some of its polyhedral properties. In Section 3, we show that the graphs in $\mathcal{XX}$ can be built by iteratively applying the gear composition to

2. Gear composition

An edge v_1v_2 of a graph H is said to be *simplicial* if $K_1 = N(v_1) \setminus \{v_2\}$ and $K_2 = N(v_2) \setminus \{v_1\}$ are cliques of H and both $K_1 \setminus K_2$ and $K_2 \setminus K_1$ are nonempty. Notice that K_1 and K_2 might have nonempty intersection.

Definition 2.1. *Let $H = (V_H, E_H)$ be a graph with a simplicial edge v_1v_2 and let $B = (V_B, E_B)$ be a gear. The* gear composition *of H and B produces a new graph $G = (H, B, v_1v_2)$, called* geared graph, *such that:*

$$V_G = V_H \setminus \{v_1, v_2\} \cup V_B,$$
$$E_G = E_H \setminus (\delta(v_1) \cup \delta(v_2)) \cup E_B \cup F_1 \cup F_2, \text{ where } F_i = \{d_i u, b_i u | u \in K_i\} \text{ for } i = 1, 2.$$

A sketch of how the gear composition works is shown in Fig. 3.

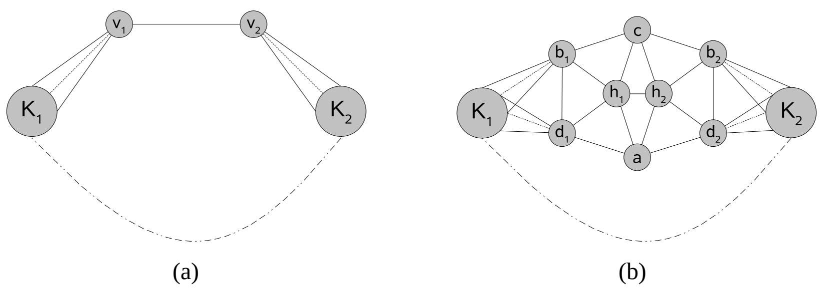

Figure 3: (a) A graph H with a simplicial edge v_1v_2; (b) The geared graph $G = (H, B, v_1v_2)$.

Definition 2.2. *Let $H = (V_H, E_H)$ be a graph containing the simplicial edge v_1v_2 and let (π, π_0) be a valid inequality for $STAB(H)$ such that $\pi_{v_1} = \pi_{v_2} = \lambda > 0$. Let $B = (V_B, E_B)$ be a gear and $G = (H, B, v_1v_2)$ a geared graph. Then the inequalities*

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B \setminus \{h_1, h_2\}} x_i + 2\lambda(x_{h_1} + x_{h_2}) \le \pi_0 + 2\lambda \tag{1}$$

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B \setminus A} x_i \le \pi_0 + \lambda \tag{2}$$
$$where \quad A \in \{\{b_1, c\}, \{b_2, c\}, \{d_1, a\}, \{d_2, a\}, \{a, c\}\}$$

are called geared inequalities *associated with (π, π_0). The unique geared inequality that is full support on V_B is (1) and it will be called* proper geared inequality.

Definition 2.3. *Let H^e be a graph obtained from $H = (V_H, E_H)$ by subdividing the simplicial edge $e = v_1v_2$ with a node t. An inequality (π, π_0) which is valid for $STAB(H^e)$ is said to be* g-liftable *(with respect to v_1v_2) if $\pi_{v_1} = \pi_{v_2} = \pi_t = \lambda > 0$.*

Definition 2.4. *Let $G = (H, B, v_1v_2)$ be a geared graph and (π, π_0) a g-liftable inequality. Then the*

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B} x_i \leq \pi_0 + \lambda, \tag{3}$$

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B \setminus A} x_i \leq \pi_0 \tag{4}$$

$$where \quad A \in \{\{b_1, c, b_2, h_1, h_2\}, \{d_1, a, d_2, h_1, h_2\}\}$$

are called g-lifted inequalities *associated with* (π, π_0). *The unique g-lifted inequality that is full support on* V_B *is (3) and it will be called* proper g-lifted inequality.

In [7], we showed how the gear composition affects the stable set polytope of a geared graph $G = (H, B, e)$. More precisely, we proved that the linear description of $STAB(G)$ is completely determined by the linear description of $STAB(H)$ and $STAB(H^e)$. In fact, we have that:

Theorem 2.5. *Let* $G = (H, B, e)$ *be a geared graph. Then the stable set polytope* $STAB(G)$ *is described by the following linear inequalities:*

- *clique-inequalities,*
- *(lifted)* 5*-wheel inequalities,*
- *geared inequalities associated with facet defining inequalities of* $STAB(H)$ *having nonzero coefficient on the endnodes of* e*,*
- *g-lifted inequalities associated with facet defining inequalities of* $STAB(H^e)$ *having nonzero coefficient on the endnodes of* e*,*
- *facet defining inequalities of* $STAB(H)$ *having zero coefficient on the endnodes of* e*.*

This result implies that, if H and H^e "well behave" with respect to the stable set problem (meaning that there exists a defining linear system for their stable set polytopes) then the geared graphs obtained from H do the same. In the following we will extend some of the polyhedral properties of the gear composition in order to prove that a large subclass of claw-free graphs well behave with respect to the stable set problem.

3. Geared (fuzzy) line graphs

In 2004, Chudnovsky and Seymour proved that claw-free graphs with stability number greater than 3 that are not quasi-line are obtained by composing only three kinds of graphs, called *(fuzzy) strips*. One kind of strips, the *(fuzzy) linear interval strips*, is used to generate quasi-line graphs while the other two kinds of strips, the *(fuzzy)* XX*-strips* and the *(fuzzy) antihat-strips*, contain 5-wheels and so, they are used to generate claw-free graphs that are not quasi-line.

This structure suggests the idea that claw-free graphs which are not quasi-line and have stability number greater than 3 might not be so distant from line graphs in terms of polyhedral description of their stable set polytope. In the following sections we give an evidence of this fact by showing that a defining linear system for the stable set polytope of *almost all* claw-free graphs in $\mathcal{C} \setminus \mathcal{Q}^c$ is built starting from the defining linear system of the stable set polytope of line graphs.

Definition 3.1. *A* strip (G, a, b) *consists of a claw-free graph G together with two designated simplicial vertices a, b called the* ends *of the strip. Two strips can be composed as follows: let A and B be the nodes of $G \setminus \{a, b\}$ adjacent in G to a and b respectively, and define A' and B' similarly. Take the disjoint union of $G \setminus \{a, b\}$ and $G' \setminus \{a', b'\}$; and let H be the graph obtained from this by adding all possible edges between A and A' and between B and B'.*

Definition 3.2. *A* homogeneous pair of cliques *in G is a pair (A, B) such that:*

- *A and B are cliques in G and $A \cap B = \emptyset$,*
- *$|A| \geq 2$ or $|B| \geq 2$,*
- *no vertex of $G \setminus (A \cup B)$ has both a neighbour and a non-neighbour in A, and the same in B.*

Definition 3.3. *Let T be a graph with vertex set $\{u_1, \ldots, u_{13}\}$ and with adjacency as follows. $(u_1, \ldots, u_6)$ is a hole of G of length 6. Next, u_7 is adjacent to u_1, u_2; u_8 is adjacent to u_4, u_5; u_9 is adjacent to u_6, u_1, u_2, u_3; u_{10} is adjacent to u_3, u_4, u_5, u_6, u_9; u_{11} is adjacent to $u_3, u_4, u_6, u_1, u_9, u_{10}$; u_{12} is adjacent to $u_2, u_3, u_5, u_6, u_9, u_{10}$; u_{13} is adjacent to $u_1, u_2, u_4, u_5, u_7, u_8$. Let $X \subseteq \{u_{11}, u_{12}, u_{13}\}$; then the strip $(T \setminus X, u_7, u_8)$ is called an XX*-strip.

We can now state the decomposition theorem of Chudnovsky and Seymour; the decomposition involves the antihat strips, but we omit their definition since they will never be used in the following.

Theorem 3.4. *[2] For every claw-free graph G with $\alpha(G) \geq 4$, if G does not admit a 1-join and there is no homogeneous pair of cliques in G, then either G is a circular interval graph, or G is a composition of linear interval strips, XX-strips, and antihat strips.*

Since graphs containing homogeneous pair cannot be represented with the above strips, Chudnovsky and Seymour were forced to introduce the concept of *fuzzyness* [2] and to give a "fuzzy" version of the above theorem were all the strips are fuzzy strips. Since the fuzzyness is a very technical concept we do not go into the detail of its definition (we refer the interested reader to [2]). To our purpose it suffices to observe that (fuzzy) linear interval strips are quasi-line graphs and to refer to a quasi-line graph that is a composition of (fuzzy) linear interval strips as a *(fuzzy) line graphs*. Recall that the class of (fuzzy) line graphs is denoted by $\mathcal{Q}^\ell$ and the class of (fuzzy) circular interval graphs is denoted by $\mathcal{Q}^c$.

It is worth noticing that the fuzzyness does not have much relevance from the polyhedral point of view. This was already noticed by Chudnovsky and Seymour who proved that:

Theorem 3.5. *[1] If G is a (fuzzy) line graph, then $STAB(G)$ is described by the Edmonds' inequalities.*

And it was further confirmed by the work of Eisenbrand et al. on (fuzzy) circular inteval graphs.

Lemma 3.6. *[5] Let F be a facet of $STAB(G)$ where G is a fuzzy circular interval graph. Then F is also a facet of $STAB(G')$, where G' is a circular interval graph obtained from G by removing some edges.*

By Theorem 3.4 and its "fuzzy" version, finding a linear description of $STAB(G)$ for claw-free graphs with $\alpha(G) \geq 4$ is equivalent to finding a linear description of $STAB(G)$ for (fuzzy) circular interval graphs, namely the graphs in $\mathcal{Q}^c$, and for graphs that are composition of fuzzy linear interval strips, XX-strips and antihat strips. Since the first case has been solved in [5], we focus our attention on the graphs that are a

We call these graphs XX-*graphs* and their family will be denoted as $\mathcal{XX}$.

In the following we show that any XX-graph can be obtained by iteratively applying the gear composition defined in Section 2 to a (fuzzy) line graph, i.e., a graph in $\mathcal{Q}^\ell$. We start by showing that the gear is a subgraph of an XX-strip.

Lemma 3.7. *The graph obtained by composing a strip* (G, v_1, v_2) *with the* XX*-strip* $(T\setminus\{u_{11}, u_{12}, u_{13}\}, u_7, u_8)$ *is a geared graph.*

Proof. Rename the nodes $a, b_1, b_2, c, d_1, d_2, h_1, h_2$ of a gear B as $u_6, u_2, u_4, u_3, u_1, u_5, u_9, u_{10}$, respectively. Thus, the strip composition of (G, v_1, v_2) and the XX-strip $(T \setminus \{u_{11}, u_{12}, u_{13}\}, u_7, u_8\})$, as defined in Definition 3.1, corresponds to the gear composition of $G' = (V_G, E_G \cup \{v_1v_2\})$ and the gear B. In fact, being the nodes v_1 and v_2 simplicial, we have that the edge v_1v_2 of G' is simplicial. The graph obtained by applying the above strip composition is precisely the geared graph (G', B, v_1v_2). ∎

As a consequence of the above lemma and Definition 3.3 we have that each XX-strip composition produces a geared graph $G = (H, B, e)$ plus an extra set Y of nodes which are properly adjacent to B. This, together with Theorem 3.4, implies that a large number of claw-free graphs can be seen as geared graphs. We now prove that we can restrict ourselves to consider only XX-strips not containing node u_{13} since this node can be added using an appropriate linear interval strip.

Lemma 3.8. *The class of* XX*-graphs coincides with the subclass of claw-free graphs obtained by composition of* XX*-strips of type* $(T \setminus \{u_{13}\}, u_7, u_8)$ *and (fuzzy) linear interval strips.*

Proof. Let G be an XX-graph obtained by composing a strip (L, v_1, v_2) and an XX-strip $(T \setminus X, v_7, v_8)$. Suppose that $u_{13} \notin X$. Define L' as the graph obtained from L by adding two new nodes v'_1 and v'_2 whose neighborhoods are $N(v'_i) = N(v_i) \cup \{v_i\}$, for $i = 1, 2$. Clearly, (L', v'_1, v'_2) is a strip. Moreover, let L'' be obtained by composing (L', v'_1, v'_2) with the linear interval strip (P, a_0, b_0) where P is the simple path (a_0, u_{13}, b_0). Then (L'', v_1, v_2) is also a strip. Finally, it is easy to check that G may also be obtained by composing (L'', v_1, v_2) with the XX-strip $(T \setminus (X \cup \{u_{13}\}), u_7, u_8\})$. ∎

In order to show that the gear composition can be used to build XX-graphs, we need to show that the removal of a simplicial edge preserves the property of the graph of being (fuzzy) line.

Proposition 3.9. *H is a (fuzzy) line graph with a simplicial edge $e = v_1v_2$ if and only if $(H - e, v_1, v_2)$ is a strip with $H - e$ (fuzzy) line graph.*

Proof. First we prove the "if" direction. It suffices to observe that H is obtained by composing the strip $(H - e, v_1, v_2)$ with a strip (P, a_0, b_0) consisting of a path $P = (a_0, u_1, u_2, b_0)$ and then rename the nodes u_i as v_i, $i = 1, 2$. To prove the other direction observe that a (fuzzy) line graph is actually a quasi-line graph. Hence to prove that $H - e$ is a (fuzzy) line graph we must first show that it is quasi-line, i.e., it contains neither a claw nor an odd-antiwheel. Suppose by contradiction that $H - e$ contains a claw C. Since the only edge which was removed from H is $e = v_1v_2$, we have that C contains both v_1 and v_2. So, $C = (y : v_1, v_2, w)$ with $y \in K_1 \cap K_2$ and $w \in V_H \setminus (K_1 \cup K_2 \cup \{v_1, v_2\})$. Since e is simplicial, there exists a node $z_1 \in K_1 \setminus K_2$ and a node $z_2 \in K_2 \setminus K_1$. Thus, $wz_i \in E_H$, $i = 1, 2$, since otherwise $(y : v_1, z_2, w)$ or $(y : v_2, z_1, w)$ would be claws in H, contradicting the hypothesis that H is quasi-line. Hence, the edge e belongs to the 5-wheel $(y : v_1, v_2, z_2, w, z_1)$ contained in H, contradicting the hypothesis that H is quasi-line. Similarly, it can be proved that $H - e$ does not contain odd-antiwheels and so, it is a quasi-line graph. Thus, by the decomposition of quasi-line graphs in [1], $H - e$ is either a (fuzzy) line graph or a (fuzzy) circular interval graph. Since it is possible to prove that if $H - e$ is a (fuzzy) circular interval graph, then H is also a (fuzzy) circular interval graph, we have that $H - e$ is a (fuzzy) line graph as claimed. ∎

Definition 3.10. *Let $B = (V_B, E_B)$ be a gear. An extended gear B_Y is a graph whose vertex set is $V_B \cup Y$ where $Y \subseteq \{u_{11}, u_{12}\}$ and whose edge set is $E_B \cup \{u_{11}d_1, u_{11}a, u_{11}h_1, u_{11}h_2, u_{11}c, u_{11}b_2\} \cup \{u_{12}d_2, u_{12}a, u_{12}h_1, u_{12}h_2, u_{12}c, u_{12}b_1\}$ (see Fig. 4).*

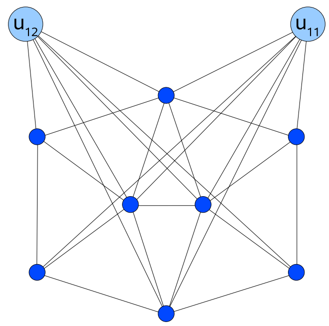

Figure 4: The extended gear B_Y with $Y = \{u_{11}, u_{12}\}$.

An extended gear composition *is a gear composition where the gear B is replaced by B_Y for some $Y \subseteq \{u_{11}, u_{12}\}$.*

Definition 3.11. *Let H be a (fuzzy) line graph which is not a clique and let E^*_H be the set of its simplicial edges. A* g-operation *on $e \in E^*_H$ is either an extended gear composition or an edge subdivision of e. A graph $G \in \mathcal{G}^*_H$ if and only if*

either $G = H$*,*

or $G = (L, B_Y, e)$*, where $L \in \mathcal{G}^*_H$, B_Y is an extended gear, and $e \in E^*_H \cap E_L$ (i.e., e is a simplicial edge of H on which no g-operation has been performed),*

or $G = L^e$*, where $L \in \mathcal{G}^*_H$ and $e \in E^*_H \cap E_L$.*

The graphs in $\mathcal{G}^*_H$ will be called geared (fuzzy) line graphs.

In the following we show that the geared (fuzzy) line graphs are exactly the XX-graphs, i.e., $\mathcal{G}^*_H = \mathcal{XX}$.

Lemma 3.12. *The geared (fuzzy) line graphs are the XX-graphs.*

Proof. By Proposition 3.9 and Lemmas 3.7 and 3.8, it is trivial to see that XX-graphs are geared (fuzzy) line graphs. To prove the opposite, suppose by contradiction that there exists a graph G in $\mathcal{G}^*_H$ which is not an XX-graph, with H being a (fuzzy) line graph. In particular, assume G be obtained by performing the smallest number of g-operations. If $G = H$ then, by definition, H is an XX-graph. Hence, either $G = (L, B_Y, e)$ or $G = L^e$, where $e = v_1v_2$ is a simplicial edge of H.

Suppose first that $G = L^e$. By the minimality of G we know that L is an XX-graph. Since e is simplicial it does not belong to any XX-strip of L. So, we can build a new graph $\tilde{L}$ from L by replacing each XX-strip $(T \setminus X, v_1, v_2)$ with the simple path (a_0, v_1, v_2, b_0). It follows that $\tilde{L}$ is a (fuzzy) line graph and, by Proposition 3.9, $\tilde{L} - e$ is also a (fuzzy) line graph. Now we reconstruct $L - e$ from $\tilde{L} - e$ by replacing the simple paths previously introduced with the corresponding XX-strips. Thus $L - e$ is obtained as a composition of XX-strips and (fuzzy) linear interval strips, and so, it is an XX-graph. Since G is obtained by composing the strip $(L - e, v_1, v_2)$ and (a, v'_1, t, v'_2, b) and renaming v'_i as v_i, $i = 1, 2$, we have that G is an XX-graph, as claimed.

Consider now the case $G = (L, B_Y, e)$. As above, we can prove that $L - e$ is an XX-graph. If we

From the above results it follows that the XX-graphs can be built in two different ways: either using the strip composition defined by Chudnovsky and Seymour in [2] or using the extended gear composition defined in [6]. This result allows us to exploit the polyhedral properties of the gear composition to find a linear description for the stable set polytope of XX-graphs. This will be discussed in the next section.

4. Stable set polytope of XX-graphs

In this section we consider the geared graph $G = (H, B_Y, e)$ obtained as an extended gear composition of the graph H and the extended gear B_Y, with $Y \subseteq \{u_{11}, u_{12}\}$.

Let (γ, γ_0) be a non trivial facet defining inequality of $STAB(G)$ which is not a clique or a (lifted) 5-wheel inequality. If $Y = \emptyset$ then, by Theorem 2.5, (γ, γ_0) is one of the following inequalities:

- an inequality of type (1) or (2) associated with a facet of $STAB(H)$;
- an inequality of type (3) or (4) associated with a facet of $STAB(H^e)$;
- a facet of $STAB(H)$.

We now show that the properties of the extended gear composition do not change substantially when $Y \neq \emptyset$. In particular we prove that the above inequalitities can be "lifted" to the higher dimensional space containing u_{11} and u_{12} using the *sequential lifting* procedure [15] described in the following.

Let $\sum_{j\in V_G\setminus\{v\}} \pi_j x_j \leq \pi_0$ be a facet defining inequality of $STAB(G\setminus\{v\})$. Then $\sum_{j\in V_G\setminus\{v\}} \pi_j x_j + \pi_v x_v \leq \pi_0$ with $\pi_v = \pi_0 - \max_{x\in STAB(G\setminus N(v))} \pi^T x$ is facet defining for $STAB(G)$. This procedure can be iterated to generate facet defining inequalities, called *lifted inequalities*, in a higher dimensional space.

We start finding the lifting coefficient of u_{11} and u_{12} for inequalities (1), (2), (3), and (4).

Lemma 4.1. *Let $G = (H, B_Y, e)$ with $Y = \{u_{11}\}$ and (β, β_0) be a geared inequality that is facet defining for $STAB(G')$, where $G' = (H, B, e)$. If (β, β_0) is of type (1), then the node u_{11} is lifted with coefficient $\beta_{u_{11}} = \lambda$. If (β, β_0) is of type (2), then the node u_{11} is lifted with coefficient $\beta_{u_{11}} = \lambda$ if $A = \{b_1, c\}$ or $A = \{d_2, a\}$, and $\beta_{u_{11}} = 0$ otherwise.*

Proof. Let (π_H, π_0) be the original facet defining inequality for $STAB(H)$, then $\beta_0 = \pi_0 + 2\lambda$. By definition of lifting

$$\beta_{u_{11}} = \pi_0 + 2\lambda - \max_{x^S: S\cap N(u_{11})=\emptyset} \beta x^S.$$

There exists a feasible tight solution S for (π_H, π_0) such that $S \cap (K_2 \cup \{v_2\}) = \emptyset$. Thus, $v_1 \in S$ and $S' = S \setminus \{v_1\} \cup \{b_1, d_2\}$ is a stable set such that $\beta(S') = \pi_H(S \setminus \{v_1\}) + 2\lambda = \pi_0 - \lambda + 2\lambda = \pi_0 + \lambda$. Therefore, $\beta_{u_{11}} \leq \lambda$. Suppose now that there exists a feasible solution x^S such that $\max_{x^S: S\cap N(u_{11})=\emptyset} \beta x^S > \pi_0 + \lambda$ and, consequently, $\beta_{u_{11}} < \lambda$. Then $b_1, d_2 \in S$, since otherwise $S \cup \{h_1\}$ or $S \cup \{h_2\}$ are stable sets violating (β, β_0). Therefore, $S' = S \setminus \{b_1\} \cup \{d_1, c\}$ is feasible and, consequently, $\beta(S') = \beta(S) + \lambda > \pi_0 + 2\lambda$, a contradiction. Hence, the thesis follows. Using similar arguments, the second part of the statement can proved. ∎

The next lemma has a similar proof that we omit.

Lemma 4.2. *Let $G = (H, B_Y, e)$ with $Y = \{u_{11}\}$ and (β, β_0) be a g-lifted inequality that is facet defining for $STAB(G')$, where $G' = (H, B, e)$. If (β, β_0) is of type (3), then the node u_{11} is lifted with coefficient $\beta_{u_{11}} = \lambda$. If (β, β_0) is of type (4) then the node u_{11} is lifted with coefficient $\beta_{u_{11}} = 0$.*

The next lemma shows the lifting coefficient $\beta_{u_{12}}$ for geared and g-lifted inequalities once the node u_{11}

Lemma 4.3. *Let $G = (H, B, e)$, $G' = (H, B_{\{u_{11}\}}, e)$, and $G'' = (H, B_{\{u_{11},u_{12}\}}, e)$. Moreover, let (β', β_0) be a facet defining inequality for $STAB(G')$, obtained by lifting the inequality (β, β_0) of type (1) ÷ (4) on node u_{11}. The lifting coefficient $\beta_{u_{12}}$ equals*

- *λ, if (β, β_0) is a proper geared inequality (1) or a proper g-lifted inequality (3),*
- *0, otherwise.*

By symmetry, the results of Lemmas 4.1÷4.3 hold if we interchange the role of u_{11} and u_{12}.

However the extension of the gear B with the nodes u_{11} and u_{12} does generate not only inequalities that are (sequential liftings of) inequalities defined in definitions 2.2 and 2.4. Indeed, new facet defining inequalities appear. More precisely,

Theorem 4.4. *Let $G = (H, B, e)$, $G' = (H, B_{\{u_{11}\}}, e)$, and $G'' = (H, B_{\{u_{11},u_{12}\}}, e)$. Let (π, π_0) be a g-liftable facet defining inequality for $STAB(H^e)$, then*

$$\sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda(x_{d_1} + x_{u_{11}} + x_{b_2}) \leq \pi_0 \tag{5}$$

is facet defining for both $STAB(G')$ and $STAB(G'')$; moreover,

$$\sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda(x_{b_1} + x_{u_{12}} + x_{d_2}) \leq \pi_0 \tag{6}$$

is facet defining for $STAB(G'')$.

Here we want to focus on a few remarks that will be useful to better understand the next results. First, observe that the hub v_1 of a 5-wheel can not be the endnode of a simplicial edge v_1v_2, since $N(v_1) \setminus \{v_2\}$ is not a clique. This implies that geared and g-lifted inequalities associated with non negativity, rank, and 5-wheel inequalities have only coefficients 1 or 2. In particular, the extreme nodes of a simplicial edge, have only coefficient 1. We now define recursively a new family of inequalities:

An inequality $(\gamma, \gamma_0) \in \mathcal{G}$ if and only if (γ, γ_0) is (the sequential lifting of)

either a rank inequality,

or a 5-wheel inequality,

or a geared or a g-lifted inequality associated with an inequality in $\mathcal{G}$.

Using definitions 2.2 and 2.4, and the fact that the gear composition does not produce new simplicial edges, it is not difficult to see that any inequality in $\mathcal{G}$ has only coefficients 1 or 2. For the sake of simplicity, from now on, we call geared inequalities the inequalities in $\mathcal{G}$ containing both coefficients 1 and 2 that are not (lifted) 5-wheel inequalities. Consider now the polyhedron

$$\mathcal{G}STAB(G) = \{x \in \mathbb{R}^V_+ | \ x \ satisfies \ \mathcal{G}\}. \tag{7}$$

In [6] it was proved that if G is a geared graph, then $STAB(G) \subseteq \mathcal{G}STAB(G)$; moreover, a graph G is said to be $\mathcal{G}$-perfect if the equality holds. Theorem 2.5 states that, if $G = (H, B, e)$ is a geared graph, then a defining linear system for $STAB(G)$ can be easily provided once we know a defining linear system for $STAB(H)$ and $STAB(H^e)$. In other words, Using the results of Lemmas 4.1÷4.3 and Theorem 4.4, we can extend the above result to geared (fuzzy) line graphs, i.e., graphs in $\mathcal{G}^*_H$, as follows:

Theorem 4.6. *Let H be a graph and E^*_H the set of its simplicial edges. Let H^F be the graph obtained from H by subdividing all the edges in $F \subseteq E^*_H$. If H and H^F are $\mathcal{G}$-perfect for any $F \subseteq E^*_H$, then every graph $G \in \mathcal{G}^*_H$ is $\mathcal{G}$-perfect.*

We now prove that the claw-free graphs in $\mathcal{XX}$ are $\mathcal{G}$-perfect. More precisely we prove that

Theorem 4.7. *If G is an XX-graph then $STAB(G)$ is defined by the (sequential lifting of the) following inequalities:*

- *rank inequalities,*
- *5-wheel inequalities,*
- *geared inequalities.*

Proof. By Lemma 3.12, the graph G is a geared (fuzzy) line graph, namely it belongs to the family $\mathcal{G}^*_H$ where H is a (fuzzy) line graph. By Proposition 3.9, the graph $H - e$ is also a (fuzzy) line graph. To obtain H^e one just needs to compose $(H - e, v_1, v_2)$ with the strip (P, a, b) where P is the path (a, u_1, t, u_2, b) and then rename u_i as v_i, $i = 1, 2$. Hence, H^e is a (fuzzy) line graph and the same holds for H^F for any subset $F \subseteq E^*_H$ of simplicial edges of H. By Theorem 3.5, we know that $STAB(H)$ and $STAB(H^F)$ are described only by rank-inequalities, i.e., the Edmonds' inequalities [4], and so, H and H^F are $\mathcal{G}$-perfect. Hence, by Theorem 4.6, every graph $G \in \mathcal{G}^*_H$ is $\mathcal{G}$-perfect, i.e., $STAB(G)$ is completely described by inequalities in $\mathcal{G}$ and the theorem follows. ∎

References

[1] M. Chudnovsky and P. Seymour, "Claw-free graphs VI: Quasi-line graphs," 2004. Manuscript.

[2] M. Chudnovsky and P. Seymour, "Claw-free graphs IV: Decomposition theorem," *J. Comb. Th. B*, 2007. To appear.

[3] M. Conforti, G. Rinaldi, and L. Wolsey, "On the cut polyhedron," *Discrete Mathematics*, vol. 277, pp. 279–285, 2004.

[4] J. Edmonds, "Maximum matching and a polyhedron with $0, 1$ vertices," *J. Res. of Nat. Bureau of Stand. B*, vol. 69B, pp. 125–130, 1963.

[5] F. Eisenbrand, G. Oriolo, G. Stauffer, and P. Ventura, "The stable set polytope of quasi-line graphs," *Combinatorica*. To appear.

[6] A. Galluccio, C. Gentile, and P. Ventura, "Gear composition and the stable set polytope," To appear in Operations Research Letters, 2008.

[7] A. Galluccio, C. Gentile, and P. Ventura, "Gear composition of stable set polytopes and $\mathcal{G}$-perfection," Submitted to Mathematics of Operations Research, 2006.

[8] A. Galluccio and A. Sassano, "The rank facets of the stable set polytope for claw-free graphs," *J. Comb. Th. B*, vol. 69, pp. 1–38, 1997.

[10] M. Grötschel, L. Lovász, and A. Schrijver, *Geometric algorithms and combinatorial optimization*. Springer Verlag, Berlin, 1988.

[11] T. Liebling, G. Oriolo, B. Spille, and G. Stauffer, “On non-rank facets of the stable set polytope of claw-free graphs and circulant graphs,” *Math. Methods of Oper. Research*, 2004.

[12] A. Mahjoub, “On the stable set polytope of a series-parallel graph,” *Mathematical Programming*, vol. 40, pp. 53–57, 1988.

[13] G. Minty, “On maximal independent sets of vertices in claw-free graphs,” *J. Comb. Th. B*, vol. 28, pp. 284–304, 1980.

[14] G. Oriolo, U. Pietropaoli, and G. Stauffer, “A new algorithm for the maximum weighted stable set problem in claw-free graphs,” *Proceedings of IPCO 2008*, 2008. To appear.

[15] M. W. Padberg, “On the facial structure of vertex packing polytope,” *Mathematical Programming*, vol. 5, pp. 199–215, 1973.

[16] W. Pulleyblank and B. Shepherd, “Formulations of the stable set polytope,” in *Proceedings Third IPCO Conference* (G. Rinaldi and L. Wolsey, eds.), pp. 267–279, 1993.