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RECURSIVE FILTERING FOR LOG-RICE SIGNALS

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Abstract

This paper proposes a recursive filtering algorithm for log-Rice signals generated by the envelope of noisy narrow band Gaussian signals through noisy logarithmic amplifiers. A class of filters is considered, and within this class the filter providing the optimal estimate (minimum error variance) is computed. Some simulations results are presented to evaluate the filter performances.

Key words: Nonlinear filtering, Log-Rice signals.

1. Introduction

In many communication systems there are devices which provide some form of logarithmic compression of signal parameters (see e.g. [7, 8, 9]). For instance, in the field of avionics communications, the need for a wide dynamic range is a crucial issue that is commonly solved by means of a logarithmic compression of the envelope of the received signal [8]. It is well known that when noisy signals undergo a nonlinear transformation the problem of optimal filtering becomes not tractable, because the minimum error variance estimate requires the computation of the conditional probability density, a difficult infinite-dimensional problem in the general case [1, 2, 3]. For this reason suboptimal filters, often based on heuristics, are proposed in the literature.

In this paper we consider communication systems where the envelope of a signal affected by narrow band gaussian noise is processed by a noisy logarithmic amplifier. It is known that the envelope of a noisy narrow band signal follows a Rice distribution, so that the output of an ideal logaritmic amplifier would be a log-Rice signal. The problem of filtering a log-Rice signal has been investigated and solved in this paper for a signal generated by a linear Gaussian system that models the source and the channel; the samples of the envelope of the modulated signal are acquired after a logarithmic compression (log-Rice signal), in presence of a further additive Gaussian noise.

The estimate of a stationary linear Gaussian signal of a given spectral density, corrupted by an additive Gaussian noise is obtained according to the well known Wiener filter theory, providing the optimal stationary linear filter. In case of a time-varying framework and of a nonlinear noisy corruption, like Rice or Log-Rice distributions, such a methodology cannot be applied. In these cases the optimal filter, i.e. the one providing the conditional expectation, is prohibitive, due to its complexity, especially for real-time applications. When real time algorithms are needed, an approach is to adopt a preliminary linearization of the problem good for developing suboptimal algorithms. In this work according to a simple nonlinear treatment of the measured data a recursive suboptimal algorithm is proposed, optimal w.r.t. a suitably defined class of pre-processed outputs. More in details, a bilinear system is achieved, whose state vector is given by the second order powers of the original state so that, endowed with the pre-processed data, it comes out a linear drift and multiplicative noise. The optimal linear filter for bilinear systems is finally applied, according to the filtering polynomial approach succesfully used in the last decade in many non-Gaussian, nonlinear frameworks (see [3, 4] and the more recent [5, 6]).

2. Log-Rice signals

Consider the canonic form of a generically modulated pass-band signal $s(t)$ including narrow-band noise:

$$\begin{aligned} s(t) = z_c(t)\cos(2\pi f_0 t) + z_s(t)\sin(2\pi f_0 t) \\ + n_c(t)\cos(2\pi f_0 t) + n_s(t)\sin(\pi f_0 t), \end{aligned} \tag{2.1}$$

where f_0 represents the carrier frequency; $z_c(t)$, $z_s(t)$ represent the in phase and quadrature components of the modulated signal which will be assumed to be a Gaussian process; $n_c(t)$, $n_s(t)$ represent the in phase and quadrature components of the narrow-band Gaussian noise, with zero mean and known variance σ^2. In this paper we are interested in considering communication systems based on non-coherent envelope detection. Therefore, the final aim of the demodulator is to estimate the envelope of the signal $z(t)$, $E_z(t) = \sqrt{z_c^2(t) + z_s^2(t)}$, from the envelope of the received signal $s(t)$:

$$E_s(t) = \sqrt{\big(z_c(t) + n_c(t)\big)^2 + \big(z_s(t) + n_s(t)\big)^2}. \tag{2.2}$$

It is known that for any given $z_c(t)$ and $z_s(t)$, $E_s(t)$ follows a Rice distribution [9]. In some significant communication systems (e.g. for avionics communications) the need for a wide dynamic range requires a logarithmic compression of the samples of the envelope of the received signal E_s:

$$\tilde{y}(kT) = \widetilde{K} \log\left(\frac{E_s(t)}{V_0}\right)\bigg|_{t=kT} = a + K \ln\big(E_s(kT)\big), \tag{2.3}$$

with the gain $K = \widetilde{K}\log(e)$, V_0 a reference voltage and $a = -\widetilde{K}\log V_0$. In the sequel, it will be assumed $T = 1$ without loss of generality. Moreover, an additive Gaussian noise affects the acquired data, so that (2.3) becomes:

$$\tilde{y}(k) = a + K \ln\sqrt{\big(z_c(k) + n_c(k)\big)^2 + \big(z_s(t) + n_s(k)\big)^2} + gN''(k), \tag{2.4}$$

where $N''(k) \in \mathcal{G}(0,1)$, with $\mathcal{G}(\eta, \sigma^2)$ denoting the set of scalar Gaussian random variables with mean η and variance σ^2.

According to their Gaussian feature, the samples of z_c and z_s may be modeled as the output of the following linear Gaussian system:

$$x(k+1) = Ax(k) + FN'(k), \tag{2.5a}$$

$$z_c(k) = C_c x(k), \tag{2.5b}$$

$$z_s(k) = C_s x(k), \tag{2.5c}$$

where $x(k) \in \mathbb{R}^n$ is the state of the system and $N'(k)$ is a zero-mean Gaussian noise with unitary covariance matrix. The initial state $x_0 = x(0)$ is also a Gaussian random vector in $\mathbb{R}^n$. It is assumed that $\{n_c(k)\}$, $\{n_s(k)\}$, $\{N'(k)\}$, $\{N''(k)\}$ are independent sequences of independent random vectors.

It is well known that in case of high signal/noise ratio (i.e. $E_z \gg \sigma$), both the envelope $E_s(k)$ (Rice signal) and the data $\tilde{y}(k)$ (log-Rice signal) can be approximated by Gaussian variables:

$$E_s \simeq \mathcal{G}\big(E_z, \sigma^2\big), \quad \tilde{y} \simeq \mathcal{G}\left(K \ln\left(\frac{E_z}{V_0}\right), \frac{K^2\sigma^2}{E_z^2}\right). \tag{2.6}$$

On the other hand, such an approximation may well become too rough to be applied in many practical situations, where the noise component is not so lower than the signal amplitude. Therefore, in this paper no approximations are considered.

3. The bilinear systems

As a preliminary step, the measurements are pre-processed. The following samples, denoted by $y(k)$, straightforwardly acquired from the original data $\tilde{y}(k)$, will be considered in order to estimate $E_z(k)$:

$$y(k) = \exp\left\{\frac{2(\tilde{y}(k)-a)}{K}\right\} = \Big(\big(C_c x(k)+n_c(k)\big)^2 + \big(C_s x(k)+n_s(k)\big)^2\Big)e^{\frac{2g}{K}N''(k)}. \tag{3.1}$$

The squares in (3.1) are exploited according to the Kronecker algebra formalism (for a quick survey on the Kronecker product and its main properties, see [4]). By denoting in square brackets the Kronecker product of a vector, eq.(3.1) becomes:

$$y(k) = \big(C_c^{[2]}x^{[2]}(k)+2C_c x(k)n_c(k)+n_c^2(k)+C_s^{[2]}x^{[2]}(k)+2C_s x(k)n_s(k)+n_s^2(k)\big)e^{\frac{2g}{K}N''(k)}; \tag{3.2}$$

note that the signal $e^{\frac{2g}{K}N''(k)}$ is not zero-mean.

Lemma 3.1. *The output equation* (3.2) *is a quadratic transformation of the state of the system $x(k)$, driven by a multiplicative noise, according to the following equation:*

$$y(k) = C_2 x^{[2]}(k) + v + \eta(k), \tag{3.3}$$

with $C_2 = e^{\frac{2g^2}{K^2}}\big(C_c^{[2]}+C_s^{[2]}\big)$; $v = 2\sigma^2 e^{\frac{2g^2}{K^2}}$ is a known deterministic term; $\eta(k)$ is a sequence of zero-mean uncorrelated random variables (noise), given by:

$$\eta(k) = \big(C_c^{[2]}+C_s^{[2]}\big)x^{[2]}(k)\eta_2(k) + 2C_c x(k)\eta_{1,c}(k) + 2C_s x(k)\eta_{1,s}(k) + \eta_0(k), \tag{3.4}$$

where:

$$\begin{aligned}
\eta_0(k) &= \big(n_c^2(k)+n_s^2(k)\big)\, e^{\frac{2g}{K}N''(k)} - 2\sigma^2 e^{\frac{2g^2}{K^2}},\\
\eta_{1,c}(k) &= n_c(k)e^{\frac{2g}{K}N''(k)},\\
\eta_{1,s}(k) &= n_s(k)e^{\frac{2g}{K}N''(k)},\\
\eta_2(k) &= e^{\frac{2g}{K}N''(k)} - e^{\frac{2g^2}{K^2}}.
\end{aligned} \tag{3.5}$$

Moreover, $\eta(k)$ is a uncorrelated with $x(k)$, and its variance $\Sigma_o(k) = \mathrm{Cov}\big(\eta(k)\big)$ is given by:

$$\begin{aligned}
\Sigma_o(k) = \big(C_c^{[2]}+C_s^{[2]}\big)^{[2]} Z_4(k) e^{\frac{4g^2}{K^2}}\Big(e^{\frac{4g^2}{K^2}}-1\Big) + 4\big(C_c^{[2]}+C_s^{[2]}\big)Z_2(k)\sigma^2 e^{\frac{4g^2}{K^2}}\Big(2e^{\frac{4g^2}{K^2}}-1\Big)&\\
+ 4\sigma^4 e^{\frac{4g^2}{K^2}}\Big(2e^{\frac{4g^2}{K^2}}-1\Big).&
\end{aligned} \tag{3.6}$$

where $Z_i = I\!E\big[x^{[i]}(k)\big]$, $i = 2,4$.

Proof. As a preliminary step, the mean value of the following random variable is computed:

$$\mu_\alpha(n) = e^{\alpha n}, \qquad n = \mathcal{G}(0,1). \tag{3.7}$$

According to the Gaussian feature of n:

$$\begin{aligned}
I\!E\big[\mu_\alpha(n)\big] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\alpha x}e^{-\frac{x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{x^2-2\alpha x}{2}}\,dx\\
&= \frac{e^{\frac{\alpha^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{(x-\alpha)^2}{2}}\,dx = e^{\frac{\alpha^2}{2}}.
\end{aligned} \tag{3.8}$$

Then, it comes that the moments of $e^{\frac{2g}{K}N''(k)}$ are:

$$\mathbb{E}\left[\left(e^{\frac{2g}{K}N''(k)}\right)^h\right] = \mathbb{E}\left[\mu_{\alpha=\frac{2gh}{K}}(n)\right] = e^{\frac{2g^2h^2}{K^2}}. \tag{3.9}$$

According to (3.9), taking into account the mean value of $e^{\frac{2g}{K}N''(k)}$, the output equation (3.2) may be rewritten as in (3.3), (3.4) and (3.5).

$\{\eta(k)\}$ is a sequences of zero-mean uncorrelated random variables because $x(k)$ is uncorrelated with each component of $\eta(k)$ (3.5), which are sequences of zero-mean uncorrelated random variables, according to the auto and mutual independence of the noisy sequences affecting the original system (see [4] for more details). In order to compute the covariance of $\eta(k)$, the following property of the Kronecker product will be here (and further) used, which holds for any set of suitably dimensioned matrices [4]:

$$(A \cdot B) \otimes (C \cdot D) = (A \otimes C) \cdot (B \otimes D). \tag{3.10}$$

Then, according to (3.4) and (3.5):

$$\begin{aligned}
\mathbb{E}\big[\eta^2(k)\big] &= \big(C_c^{[2]} + C_s^{[2]}\big)^{[2]}\mathbb{E}\big[x^{[4]}(k)\big]\mathbb{E}\big[\eta_2^2(k)\big] + 4C_c^{[2]}\mathbb{E}\big[x^{[2]}(k)\big]\mathbb{E}\big[\eta_{1,c}^2(k)\big] \\
&+ 4C_s^{[2]}\mathbb{E}\big[x^{[2]}(k)\big]\mathbb{E}\big[\eta_{1,s}^2(k)\big] + \mathbb{E}\big[\eta_0^2(k)\big] + 4\big(C_c^{[3]} + C_s^{[2]} \otimes C_c\big)\mathbb{E}\big[x^{[3]}(k)\big]\mathbb{E}\big[\eta_2(k)\eta_{1,c}(k)\big] \\
&+ 4\big(C_c^{[2]} \otimes C_s + C_s^{[3]}\big)\mathbb{E}\big[x^{[3]}(k)\big]\mathbb{E}\big[\eta_2(k)\eta_{1,s}(k)\big] + 2\big(C_c^{[2]} + C_s^{[2]}\big)\mathbb{E}\big[x^{[2]}(k)\big]\mathbb{E}\big[\eta_2(k)\eta_0(k)\big] \\
&+ 4\big(C_c \otimes C_s\big)\mathbb{E}\big[x^{[2]}(k)\big]\mathbb{E}\big[\eta_{1,c}(k)\eta_{1,s}(k)\big] + 4C_c\mathbb{E}\big[x(k)\big]\mathbb{E}\big[\eta_{1,c}(k)\eta_0(k)\big] \\
&+ 4C_s\mathbb{E}\big[x(k)\big]\mathbb{E}\big[\eta_{1,s}(k)\eta_0(k)\big],
\end{aligned} \tag{3.11}$$

with

$$\begin{aligned}
\mathbb{E}\big[\eta_2(k)\eta_{1,c}(k)\big] &= \mathbb{E}\big[\eta_2(k)\eta_{1,s}(k)\big] = \mathbb{E}\big[\eta_{1,c}(k)\eta_0(k)\big] \\
&= \mathbb{E}\big[\eta_{1,s}(k)\eta_0(k)\big] = \mathbb{E}\big[\eta_{1,c}(k)\eta_{1,s}(k)\big] = 0,
\end{aligned} \tag{3.12}$$

and:

$$\begin{aligned}
&\mathbb{E}\big[\eta_2^2(k)\big] = e^{\frac{8g^2}{K^2}} - e^{\frac{4g^2}{K^2}} = e^{\frac{4g^2}{K^2}}\left(e^{\frac{4g^2}{K^2}} - 1\right), \\
&\mathbb{E}\big[\eta_{1,c}^2(k)\big] = \mathbb{E}\big[\eta_{1,s}^2(k)\big] = \sigma^2 e^{\frac{8g^2}{K^2}}, \\
&\mathbb{E}\big[\eta_0^2(k)\big] = 8\sigma^4 e^{\frac{8g^2}{K^2}} + 4\sigma^4 e^{\frac{4g^2}{K^2}} - 4\sigma^2 e^{\frac{2g^2}{K^2}} \cdot 2\sigma^2 e^{\frac{2g^2}{K^2}} = 4\sigma^4 e^{\frac{4g^2}{K^2}}\left(2e^{\frac{4g^2}{K^2}} - 1\right), \\
&\mathbb{E}\big[\eta_2(k)\eta_0(k)\big] = 2\sigma^2\mathbb{E}\big[e^{\frac{4g}{K}N''(k)} - e^{\frac{2g^2}{K^2}}e^{\frac{2g}{K}N''(k)}\big] = 2\sigma^2\left(e^{\frac{8g^2}{K^2}} - e^{\frac{4g^2}{K^2}}\right) = 2\sigma^2 e^{\frac{4g^2}{K^2}}\left(e^{\frac{4g^2}{K^2}} - 1\right),
\end{aligned} \tag{3.13}$$

so that, eq.(3.6) is obtained. ∎

In order to achieve a bilinear generation model for the output (3.3), the second order Kronecker power of the original state $x(k)$ is considered. This issue is achieved according to the formula concerning the expansion of a Kronecker power of order ν of a binomial:

$$(a+b)^{[\nu]} = \sum_{i=0}^{\nu} M_i^{\nu}(n)\big(a^{[i]} \otimes b^{[\nu-i]}\big), \qquad a, b \in \mathbb{R}^n, \tag{3.14}$$

with $M_i^{\nu}(n)$ suitably defined matricial coefficients in $\mathbb{R}^{n\times n}$ (see [4] for more details). Before stating the main Theorem of the session some formalism concerning the *stack* operator are recalled. The stack of a matrix $A \in \mathbb{R}^{r\times c}$ is the vector in $\mathbb{R}^{r\cdot c}$ that piles up all the columns of matrix A, and is denoted by $\mathrm{st}(A)$. The inverse operation is denoted by $\mathrm{st}^{-1}_{r,c}(\cdot)$, and transforms a vector of size $r\cdot c$ in an $r\times c$ matrix. When written without any subscript, the inverse stack operator should be intended to generate a square matrix, so that if A is a square matrix then $\mathrm{st}^{-1}\big(\mathrm{st}(A)\big) = A$.

In the sequel, the following notation will be adopted for the (known) moments of N':

$$\mathbb{E}\big[N'^{[i]}\big] = \xi_i, \qquad i = 1,\dots,4, \tag{3.15}$$

with $\xi_1 = 0$, $\xi_3 = 0$ (recall that $N'(k)$ is a zero-mean Gaussian vector with unitary covariance matrix).

Theorem 3.2. *Define the vector $X(k) = x^{[2]}(k) \in \mathbb{R}^{n^2}$, with $x(k)$ the state vector of system (2.5). Then, $X(k)$ obeys the following difference equation:*

$$X(k+1) = \mathcal{A}X(k) + \mathcal{U} + \mathcal{N}(k), \tag{3.16}$$

where:

$$\mathcal{A} = A^{[2]}, \qquad \mathcal{U} = F^{[2]}\xi_2, \tag{3.17}$$

and $\mathcal{N}(k)$ is a sequence of zero-mean random vectors (noise), given by:

$$\mathcal{N}(k) = \Gamma_x\big(x(k)\otimes N'(k)\big) + \Gamma_2\big(N'^{[2]}(k) - \xi_2\big), \tag{3.18}$$

with:

$$\Gamma_x = M_1^2(n)\big(A\otimes F\big), \qquad \Gamma_2 = F^{[2]}. \tag{3.19}$$

Moreover, $\mathcal{N}(k)$ is a uncorrelated with $x(k)$, with covariance matrix $\Sigma_s(k) = \mathrm{Cov}\big(\mathcal{N}(k)\big)$ given by:

$$\Sigma_s(k) = \Gamma_x\Big(\mathrm{st}^{-1}\big(Z_2(k)\big)\otimes I_b\Big)\Gamma_x^T + \Gamma_2\mathrm{st}^{-1}(\xi_4 - \xi_2^{[2]})\Gamma_2^T. \tag{3.20}$$

Proof. Exploit the second order Kronecker power of $x(k+1)$ and rearrange the matricial coefficients according to (3.14) and (3.10). Then:

$$X(k+1) = A^{[2]}x^{[2]}(k) + F^{[2]}N'^{[2]}(k) + M_1^2(n)\big(A\otimes F\big)\big(x(k)\otimes N'(k)\big), \tag{3.21}$$

from which, eq.(3.16) readily comes, according to (3.17), (3.18) and (3.19).

$\{\mathcal{N}(k)\}$ is a sequences of zero-mean uncorrelated random vectors because $x(k)$ is uncorrelated with $N'(k)$, which is a sequence of zero-mean uncorrelated random vectors. As far as the computation of the covariance matrix $\Sigma_s(k)$ observe that, given any pair of vectors u, v:

$$\mathrm{st}(u\cdot v^T) = v\otimes u. \tag{3.22}$$

Then:

$$\begin{aligned}\Sigma_s(k) &= \Gamma_x\mathrm{Cov}\big(x(k)\otimes N'(k)\big)\Gamma_x^T + \Gamma_2\mathrm{Cov}\big(N'^{[2]}(k) - \xi_2\big)\Gamma_2^T\\ &= \Gamma_x\Big(\mathrm{st}^{-1}\big(\mathbb{E}\big[x^{[2]}(k)\big]\big)\otimes\mathrm{Cov}\big(N'(k)\big)\Big)\Gamma_x^T + \Gamma_2\mathrm{st}^{-1}\Big(\mathbb{E}\Big[\big(N'^{[2]}(k) - \xi_2\big)^{[2]}\Big]\Big)\Gamma_2^T,\end{aligned} \tag{3.23}$$

from which (3.20) readily comes. ∎

Remark 3.3. Note that the computation of the covariance matrices of the extended state and pre-processed output noises, $\mathcal{N}(k)$ and $\eta(k)$ respectively, require the knowledge of the expectation value of the Kronecker powers of the original state of 2-nd and 4-th order. •

Remark 3.4. According to Theorem 3.2, the output equation (3.3) can be written as:

$$y(k) = C_2X(k) + v + \eta(k). \tag{3.24}$$

•

4. The filtering algorithm

The square of the envelope of z is estimated by using the extended state estimate $\widehat{X}(k)$, that is:

$$\widehat{E_z^2}(k) = \left(C_c^{[2]} + C_s^{[2]}\right)\widehat{X}(k). \tag{4.1}$$

As is well known, the optimal choice for $\widehat{X}(k)$ would be the conditional expectation w.r.t. all the Borel transformations of the measurements, whose computation in general can not be obtained through algorithms of finite dimension. Nevertheless, from an applicative point of view, it is useful to look for finite-dimensional approximations of the optimal filter. In the present paper an implementable recursive filter is proposed, providing the optimal estimate w.r.t. the Hilbert space of all the linear transformations of the output y, defined as:

$$\widehat{X}(k) = I\!\!E\left[X(k)|L(Y_k)\right], \quad L(Y_k) = \mathrm{span}\{y(0), \dots, y(k)\}, \tag{4.2}$$

and performed as the projection of $X(k)$ onto the space $L(Y_k)$. As a consequence of the linear relationship between E_z^2 and X (2.5), it comes that the estimate $\widehat{E_z^2}$ is the optimal linear estimate of the square of the envelope w.r.t. the output y.

Theorem 4.1. *Denote $\zeta_i = I\!\!E\left[x_0^{[j]}\right]$, $j = 2, 4$, the moments of the initial state and define $Z(k) = \left[Z_2^T(k) \quad Z_4^T(k)\right]^T$. The optimal linear filter of $E_z^2(k)$ w.r.t. the output $y(k)$ is achieved according to the following equations:*

$$\begin{aligned}
&\widehat{X}(0|-1) = \zeta_2,\\
&\widehat{X}(k) = \widehat{X}(k|k-1) + K(k)\Big(y(k) - v - C_2\widehat{X}(k|k-1)\Big),\\
&\widehat{X}(k+1|k) = \mathcal{A}\widehat{X}(k) + \mathcal{U},\\
&\widehat{E_z^2}(k) = \left(C_c^{[2]} + C_s^{[2]}\right)\widehat{X}(k),
\end{aligned} \tag{4.3}$$

where the gain matrix $K(k)$ is computed by using the recursive equations:

$$\begin{aligned}
&Z(0) = \left[\zeta_2^T \quad \zeta_4^T\right]^T, \qquad P_P(0) = \mathrm{st}^{-1}\left(\zeta_4 - \zeta_2^{[2]}\right),\\
&\Sigma_o(k) \;\text{ as in }\; (3.6) \;\text{ by using }\; Z(k),\\
&K(k) = \frac{1}{C_2 P_P(k) C_2^T + \Sigma_o(k)} P_P(k) C_2^T,\\
&P(k) = \left[I_{n^2} - K(k)C_2\right]P_P(k),\\
&\Sigma_s(k) \;\text{ as in }\; (3.20) \;\text{ by using }\; Z(k),\\
&P_P(k+1) = \mathcal{A}P(k)\mathcal{A}^T + \Sigma_s(k),\\
&Z(k+1) = \overline{A}Z(k) + \overline{U},
\end{aligned} \tag{4.4}$$

The explicit computation of matrix $\overline{A}$ and vector $\overline{U}$ can be found in Appendix.

Proof. Taking into account the state equation (3.16) endowed with the output equation (3.24), the filter equations are computed according to the Kalman filter algorithm, providing the optimal linear estimate among all the linear transformations of the output. In this case, according to the multiplicative feature of the noise sequences affecting both the state and output equations, the covariance matrices are not stationary, in that they depend on the expectation value of the Kronecker powers of the original state. ∎

Remark 4.2. Note that the optimal initialization requires the moments of the initial state of degree 2 and 4. •

Remark 4.3. An estimate of $E_z(k)$ is readily obtained from Theorem 4.1 as:

$$\widehat{E}_z(k) = \sqrt{\widehat{E_z^2}(k)}. \tag{4.5}$$

•

5. Simulation results

Some simulation results are here reported in order to show the effectiveness of the proposed algorithm. The modulated signal is assumed in a realistic noise environment. Consider the matrices:

$$\begin{aligned} A &= \begin{bmatrix} 0.8 & 0.1 & 0 \\ 0.1 & 0 & -0.1 \\ 0 & 0.2 & -0.6 \end{bmatrix}, & B &= \begin{bmatrix} 1 \\ 0.6 \\ -1.3 \end{bmatrix}, \\ F &= 2 \cdot 10^{-6} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0.5 & 1 & -1 \end{bmatrix}, & C_c &= \begin{bmatrix} 1 & 1.2 & -0.8 \end{bmatrix}, \\ & & C_s &= \begin{bmatrix} 0 & -0.4 & 1.3 \end{bmatrix}. \end{aligned} \tag{5.1}$$

The initial state x_0 is a zero-mean Gaussian vector, with covariance matrix $\mathrm{Cov}(x_0) = 10^{-14} \cdot I_3$. The in phase and the quadrature Gaussian noises n_c, n_s have standard deviation $\sigma = 3.54 \cdot 10^{-6}$. Data are acquired according to a logarithmic compression as in (2.3) with $\widetilde{K} = 0.5$, $V_0 = 3.16 \cdot 10^{-6}$, $g = 0.01$. Note that the signal/noise ratio is not so great to allow Gaussian approximations (2.6): as it is shown in fig. 5.1, such a ratio is rarely greater than 3 (a good Gaussian approximation is obtained for S/N ratios greater than 5).

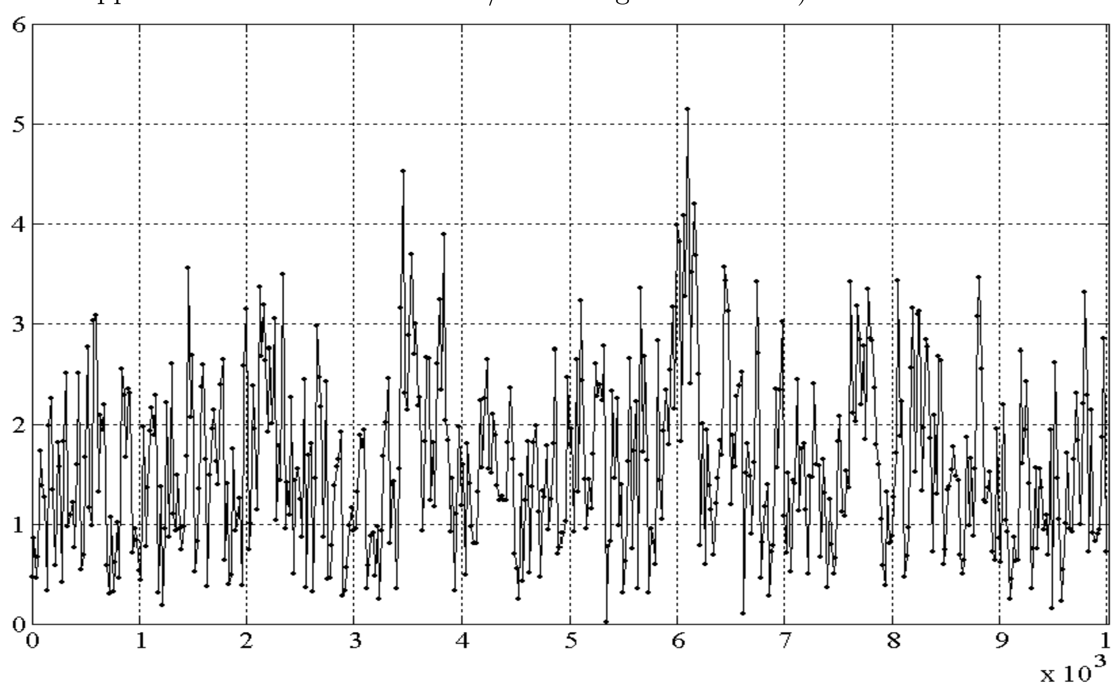

Fig. 5.1 - Signal/Noise ratio: $E_z(k)/\sigma$.

Fig. 5.2 shows the evolution of the estimated envelope $\widehat{E}_z$ w.r.t. the true value on a window of 100 time steps, to best appreciate the filter performance. A comparison is performed with the estimate obtained by simply assuming absence of noises:

$$\widetilde{E}_z(k) = e^{\frac{\tilde{y}(k)-a}{K}}. \tag{5.2}$$

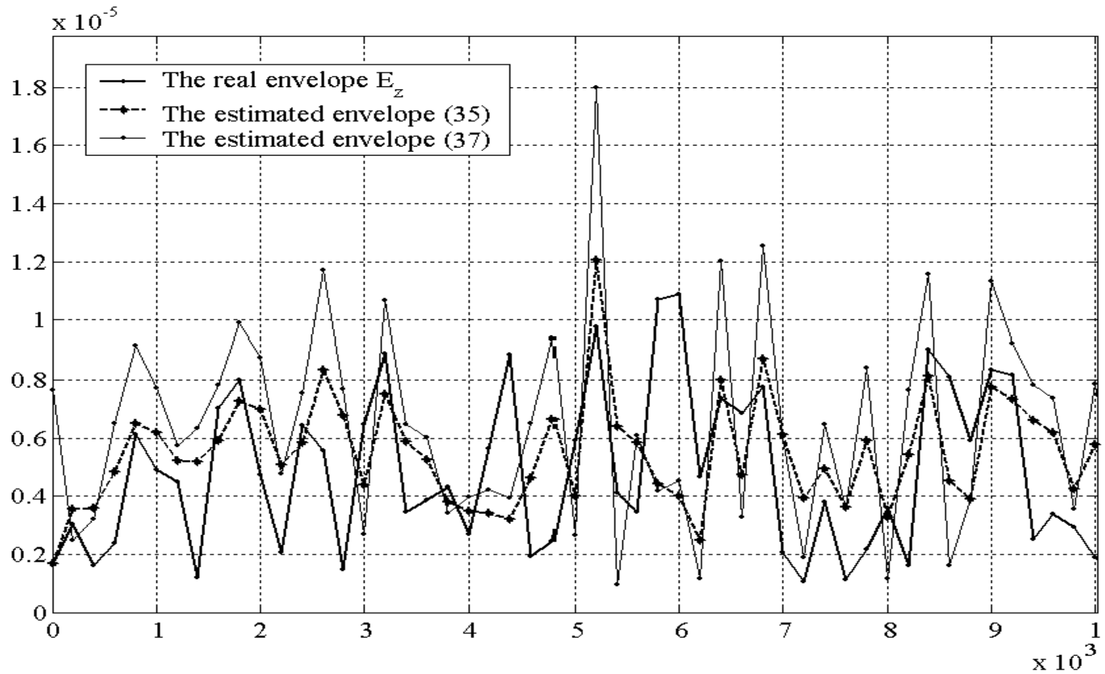

Fig. 5.2 - Estimates of $E_z(k)$.

There are apparent improvements by filtering the envelope according to Theorem 4.1, instead of trivially neglecting the noise contributes. These improvements may be seen also by considering the following ratios:

$$\widehat{R} = \frac{I\!E\big[E_z^2\big]}{I\!E\big[|E_z - \widehat{E}_z|^2\big]}, \qquad \widetilde{R} = \frac{I\!E\big[E_z^2\big]}{I\!E\big[|E_z - \widetilde{E}_z|^2\big]}. \tag{5.3}$$

Below are reported the ratios according to the samples of the simulation above described on a 1000 time steps:

$$\widehat{R} \simeq 6.7240, \qquad \widetilde{R} \simeq 3.1963. \tag{5.4}$$

5. Conclusions

A recursive filtering algorithm is proposed for log-Rice signals. The problem has been approached assuming the signal to be generated by Gaussian linear systems; however we are in a position to say that the whole theory can be extended also for the filtering of signals generated by linear systems driven by white non-Gaussian noises.

Appendix

In order to compute the matrix $\overline{A}$ and the vector $\overline{U}$ in Theorem 4.1, formula (3.14) will be used, expressing the binomial expansion of a generic Kronecker power.

Lemma A.1. *The matrix $\overline{A}$ and the vector $\overline{U}$ in Theorem* 4.1*, are computed according to the following equations:*

$$\overline{A} = \begin{bmatrix} A^{[2]} & O_{n^2 \times n^4} \\ \overline{A}_{42} & A^{[4]} \end{bmatrix}, \qquad \overline{U} = \begin{bmatrix} F^{[2]}\xi_2 \\ F^{[4]}\xi_4 \end{bmatrix}, \tag{A.1}$$

with:

$$\overline{A}_{42} = M_2^4\Big(A^{[2]} \otimes \big(F^{[2]}\xi_2\big)\Big). \tag{A.2}$$

Proof. According to (3.14), it is:

$$x^{[4]}(k+1) = \sum_{i=0}^{4} M_i^4(n)\Big(\big(Ax(k)\big)^{[i]} \otimes \big(FN'(k)\big)^{[4-i]}\Big); \tag{$A.3$}$$

taking into account the expectation value and (3.10):

$$\begin{aligned} Z_4(k+1) &= I\!E\Big[\big(Ax(k)\big)^{[4]}\Big] + F^{[4]}\xi_4 + M_2^4(n)\Big(I\!E\Big[\big(Ax(k)\big)^{[2]}\Big] \otimes \big(F^{[2]}\xi_2\big)\Big) \\ &= A^{[4]}Z_4(k) + F^{[4]}\xi_4 + M_2^4(n)\Big(A^{[2]} \otimes \big(F^{[2]}\xi_2\big)\Big)Z_2(k), \end{aligned} \tag{$A.4$}$$

because $N'(k)$ is a zero-mean Gaussian sequence. Then, (A.1), (A.2) are readily obtained. ∎

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