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**CAPACITATED SURVIVABLE NETWORKS
AND POLYHEDRA**

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Abstract

In this paper we analyze a basic polyhedron arising in most models describing survivable networks. We are able to characterize completely the polyhedron via its extreme points and we describe some important classes of its facets.

Key words: Survivable Networks; Polyhedral Theory.

1. Introduction

Any major economic sector, from telecommunications and energy to financial service and transportation, relies on some type of underlying network and this magnifies the consequences of a failure and amplifies the vital importance of ensuring network survivability in many forms. It is no surprise, hence, that the term *survivability* has become of so widespread use among researchers and practitioners in the field of optimal network design and management since most practical applications need to include survivability constraints. This has been long practice in electrical networks, operated according to the so called “N minus 1 criterion”: “the system must be operated in such a way to remain secure upon failure of the most important component (generator or transmission line)”. More recently (in relative terms), “survivability” has become an issue also in telecommunications networks due to two main factors: customer requirement for higher and higher service level (up to zero tolerance for some critical service) and the switch to the new fiber optic technology. Fiber optic cables have, in fact, much higher capacity per line than copper cables and this translate in an almost tree-like network topology compared with the highly connected topology of copper-based networks.

In its more general sense, *survivability* refers to maintaining a certain level of *operation* upon *failure* of some network *component*. According to the precise definition of *operative*, *failure* and *component* we can have the most diverse setting and models. The least requirement of survivability is to require the network to be operational under a single failed component. This is the most widely imposed requirement in applications since usually “failures” are rare events, mostly modelled through a Poisson process in which the probability of two “arrivals” (using the classical terminology of stochastic processes) at the same time is zero. Moreover, it is usually assumed that the average time to recover a failure is much smaller than the average time between two failures, i.e., the average inter-arrival time. From this, it follows that the probability of having two failures at the same time is negligible. In this paper we will refer to this survivable model as the *single-fault* model. Under this hypothesis the topological requirement for the graph underlying the network is to be 2-connected, i.e., after the removal of any component (eventually from a pre-specified set) the resulting graph must be still connected. The generalization to k -connected graph to handle multiple, simultaneous failure is straightforward. It is not surprise, hence, that the first studies on survivable networks concern connectivity properties. See [31], [30], [18], [5], [19], [35], [7], [9], [27], [20], [4], [13]. More recent work on network topological structure can be found in [15], [14]. More or less, all these papers deal with the study of $0-1$ polyhedra based on cut-set inequalities, partition inequalities and variations of them.

In network design applications, though, there are several other issues other than topology to be kept in consideration: capacities, actual flow routing, flow-paths length, etc. In this framework the survivability requirement takes basically two forms: it can be required that each flow path does not carry more than a given percentage of end point demand (diversification models) or that, for any failure situation, a prescribed percentage of each end-point demand be fulfilled anyway (reservation models). Minoux, [28], is credited to be the first to investigate survivable multi-commodity models with continuous capacities. Dahl and Stoer, [10], [11], investigated the integral capacity version by using binary design variables, (indicating incremental capacity installation from a finite set) and metric inequalities based on the “Japanese theorem”, [21], [33]. In particular, in [11], a detailed $0-1$ polyhedral study, partly extending the work in [19] and [20], is presented. Different survivability models, combining capacity, demands and routing, together with a cutting plane algorithm, are presented in [1]. Further reference to survivable networks can be found in [2], [8], [12], [3], [34], [24], [26], [23].

4.

A typical polyhedral inequality used in cutting planes algorithms solving network design problems is the following:

$$\sum_{e \in C} x_e \geq \lceil D(C) \rceil$$

where x_e is an integer variables indicating the capacity value on the edge e , C is any cutset in the graph and $D(C)$ is the demand separated by the cut. This inequality simply states that the installed integer capacity on the elements of a cutset C has to be greater or equal to the ceiling of the demand $D(C)$ separated by the cut. Building on this, in [6] the authors introduce, for network models with survivability requirement *single fault*, the “survivable cut”, described by the following polyhedron $P_n(\lceil L \rceil, \lceil D \rceil)$:

$$\begin{aligned} \sum_{j=1; j \neq i}^n x_j &\geq \lceil L \rceil & \forall i = \{1 \dots n\} \\ \sum_{j=1}^n x_j &\geq \lceil D \rceil \\ x_i &\in Z_+^n & \forall i = \{1 \dots n\} \end{aligned}$$

Here, D is the demand separated by the cut and L is the prescribed percentage of the demand D that needs to survive in the presence of a failed component. A complete description of this polyhedron via facets and extreme points has been given in [6]. In the same paper the authors analyze, for single and multi-commodity scenario, the “survivable cutset” in the presence of both flow variables, supposed continuous, and capacity variables, supposed integer. The polyhedron $F_n(L)$ (for the single commodity case) is defined by:

$$\begin{aligned} \sum_{j \neq i} f_j &\geq L & \forall i = \{1 \dots n\} \\ x_i &\geq f_i & \forall i = \{1 \dots n\} \\ x_i \in Z_+ & f_i \geq 0 & \forall i = \{1 \dots n\} \end{aligned}$$

where f_i is the flow on the i^{th} element of a cutset with n elements and x_i is its corresponding capacity. Beside the cutset framework, this particular geometrical structure arises in energy markets models with security dispatch, where x_i represents capacity to be reserved at plant i and f_i represents the actual energy flow. In the next sections we will give a complete description of $F_n(L)$ via its extreme points. In theory, this is equivalent to have a complete description of its facets. We explicitly give some classes of important facets which usually close the integrality gap. From these, more facets can be generated by sequential application of the lifting procedure.

2. The Polyhedron $F_n(L)$: extreme points

If we consider the polyhedron obtained from $F_n(L)$ by relaxing the integrality constraint we obtain a very simple geometrical structure whose extreme points are:

$$\begin{aligned} x_i = f_i &= \frac{L}{|S| - 1} & \forall i \in S \\ x_i = f_i &= 0 & \forall i \notin S \end{aligned}$$

for any subset S of indices $\{1 \dots n\}$ with $|S| \geq 2$.

Given an extreme point $W = (f_1, \dots, f_n, x_1, \dots, x_n)$ of $F_n(L)$ and a coordinate $i = \{1, \dots, n\}$ we will call f_i the continuous part of coordinate i and x_i the corresponding integral part. In order to determine the extreme point of $F_n(L)$ we note that any of them will have some coordinate, both the continuous and the integer part, equal to zero, (i.e) $x_i = f_i = 0$, some coordinate such that the integer and continuous part coincide, (i.e) $x_i = f_i > 0$ and some coordinate in which the continuous and the integer part are different and positive, (i.e) $x_i > f_i > 0$. Hence, given an extreme point, we can always partition the set of its coordinates into 3 subsets: the first, S , in which the continuous and integer part are different (and positive), the second, \tilde{S} , in which, instead, are equal and positive (and hence integer), and the third, T , in which both are equal to zero.

In the next Lemmas we will prove that any extreme point will have all the coordinates in S equal, (i.e) $f_i = \xi$ and $x_i = \lceil \xi \rceil \forall i \in S$, and all coordinates in \tilde{S} equal except for, at most, one coordinate (i.e.) $x_i = f_i = k \forall i \in \tilde{S} - \{j\}$ and $x_j = f_j = \mu$ for some $j \in \tilde{S}$. Note that $F_2(L)$ has a unique extreme point, namely $f_1 = f_2 = L, x_1 = x_2 = \lceil L \rceil$ and hence we will consider only extreme points with at least three positive components. Lemmas (2.1), (2.2), (2.3) hypothesis differ as to the cardinality of S . Accordingly, a relationship between ξ, k and μ will be established. In the following Lemmas we will refer to the partition (S, \tilde{S}, T) of $N = \{1, \dots, n\}$ previously described.

Lemma 2.1. *Let $W = (x, f)$ be an extreme point of $F_n(L)$ (with at least 3 positive coordinates) such that $|S| = m \geq 2$ (\tilde{S}, T possibly empty). Then $f_i = \xi, x_i = \lceil \xi \rceil \forall i \in S; x_i = f_i = k \forall i \in \tilde{S} - \{l\} x_l = f_l = \mu$ for some $l \in \tilde{S}$ where $\xi \in \mathfrak{R}, k, \mu$ integers, $\xi > k \geq \mu \geq 0$. If $|\tilde{S}| \geq 2$ then $k + 1 > \xi > k \geq \mu$.*

Proof. Let be $W = (x, f)$ an extreme point of $F_n(L)$ and let (S, \tilde{S}, T) be a partition of $N = \{1 \dots n\}$ as previously described. By contradiction suppose that $\exists i, j, k \in S$ such that $f_i > f_j \geq f_k > 0$. Then the points $A = (x, g)$ e $B = (x, h)$, with

$$\begin{aligned} g_j &= f_j + \epsilon & h_j &= f_j - \epsilon \\ g_k &= f_k - \epsilon & h_k &= f_k + \epsilon \\ g_l &= f_l & h_l &= f_l & \text{otherwise} \end{aligned}$$

are feasible for $\epsilon > 0$ small enough and $W = \frac{1}{2}A + \frac{1}{2}B$. By a similar argument one can show that $\nexists i, j, k \in \tilde{S}$ with $f_i > f_j \geq f_k > 0$.

Hence it must be that $f_t = \eta, x_t = \lceil \eta \rceil$ for some $t \in S, f_i = \xi, x_i = \lceil \xi \rceil$ for all $i \in S - \{t\}; f_l = x_l = \mu$ for some $l \in \tilde{S}, f_j = x_j = k$ for all $j \in S - \{l\}$ and $\xi \geq \eta, k \geq \mu$. Also, since $|S| \geq 2$, it can not be $k > \xi$. Otherwise, one can, as above, construct points A e B such that $W = \frac{1}{2}A + \frac{1}{2}B$. Similarly, if $|\tilde{S}| \geq 2$ it has to be $k + 1 > \xi \geq k \geq \mu$.

Now let us prove that $\xi = \eta$. By contradiction suppose not. Then we have:

$$\begin{aligned} (m - 2)\xi + \eta + (n - m - 1)k + \mu &= L \\ (m - 1)\xi + (n - m - 1)k + \mu &> L \\ (m - 1)\xi + \eta + (n - m - 2)k + \mu &> L \\ (m - 1)\xi + \eta + (n - m - 1)k &> L \end{aligned}$$

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and it exists $\epsilon > 0$ such that the points $A = (a, f^a)$ and $B = (b, f^b)$

$$\begin{aligned} a_h &= x_h & \forall h \\ f_i^a &= \xi + \frac{\epsilon}{m-2} & \forall i \in S-t \\ f_t^a &= \eta - \epsilon \\ f_l^a &= f_l & \text{otherwise} \end{aligned}$$

$$\begin{aligned} b_h &= x_h & \forall h \\ f_i^b &= \xi - \frac{\epsilon}{m-2} & \forall i \in S-t \\ f_t^b &= \eta + \epsilon \\ f_l^b &= f_l & \text{otherwise} \end{aligned}$$

are feasible. Let (\bar{c}, c) be a cost vector such that W is the unique solution to the optimization problem:

$$\min \sum_i \bar{c}_i x_i + \sum_i c_i f_i$$

s.t.

$$(x, f) \in F_n(L)$$

Then

$$\sum_i \bar{c}_i a_i + \sum_i c_i f_i^a > \sum_i \bar{c}_i x_i + \sum_i c_i f_i$$

and

$$\sum_i \bar{c}_i b_i + \sum_i c_i f_i^b > \sum_i \bar{c}_i x_i + \sum_i c_i f_i$$

Hence

$$\frac{\epsilon}{m-2} \sum_{i \in S-t} c_i - \epsilon c_t > 0$$

and

$$-\frac{\epsilon}{m-2} \sum_{i \in S-t} c_i + \epsilon c_t > 0$$

Contradiction.

Lemma 2.2. *Let $W = (x, f)$ be an extreme point of $F_n(L)$ (with at least 3 positive coordinates) such that $|S| = 1$. Then $f_i = \xi, x_i = \lceil \xi \rceil$ for $i \in S, x_i = f_i = k \forall i \in \tilde{S} - \{l\}, x_l = f_l = \mu$ for some $l \in \tilde{S}$ with $\xi \in \mathfrak{R}, k, \mu$ integers and $k > \xi > k - 1 \geq \mu$ or $k = \mu > \xi \geq 1$ or $k \geq \mu \geq 1 > \xi$.*

Proof.

As in the previous proof one can show that $W = (x, f)$ is of the form:

$$\begin{aligned} x_i &= \lceil f_i \rceil & i \in S \\ f_i &= \xi & i \in S \\ f_l &= \mu & l \in \tilde{S} \\ f_j &= k & j \in \tilde{S} - l \end{aligned}$$

and $k \geq \mu$. It can not be $\xi > k$ In this case in fact the points $A = (a, f^a)$ and $B = (b, f^b)$

$$\begin{aligned} a_h &= x_h & \forall h \\ f_i^a &= f_i - \epsilon & i \in S \\ f_l^a &= f_l & \text{otherwise} \end{aligned}$$

$$\begin{aligned} b_h &= x_h & \forall h \\ f_i^b &= f_i + \epsilon & i \in S \\ f_l^b &= f_l & \text{otherwise} \end{aligned}$$

are feasible for $\epsilon > 0$ small enough and $W = \frac{1}{2}A + \frac{1}{2}B$.

If $k > \xi > \mu$ then it must be the case that $k > \xi > k - 1$. Otherwise the points $A = (a, f^a)$ and $B = (b, f^b)$

$$\begin{aligned} a_h &= \lceil f_h^a \rceil & \forall h \\ f_i^a &= f_i - 1 & i \in S \\ f_l^a &= f_l + 1 & l \in \tilde{S} \\ f_j^a &= f_j & \text{otherwise} \end{aligned}$$

$$\begin{aligned} b_h &= \lceil f_h^b \rceil & \forall h \\ f_i^b &= f_i + 1 & i \in S \\ f_l^b &= f_l - 1 & l \in \tilde{S} \\ f_j^b &= f_j & \text{otherwise} \end{aligned}$$

are feasible and $W = \frac{1}{2}A + \frac{1}{2}B$.

A similar argument will show that if $k \geq \mu \geq \xi \geq 1$ then $k = \mu$.

Lemma 2.3. *Let $W = (x, f)$ be an extreme point of $F_n(L)$ (with at least 3 positive coordinates) such that $|S| = \emptyset$. Then $x_j = f_j = \mu$ for some $j \in \tilde{S}$, $x_i = f_i = k \forall i \in \tilde{S} - \{j\}$ with k, μ integers and $k \geq \mu$.*

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Proof. See the first part of the proof of Lemma (2.1).

Now, let us see which are the possible values of ξ, k, μ . Note that any extreme point of $F_m(L)$ with exactly t positive coordinates naturally correspond to extreme points of $F_{m+k}(L)$ with exactly t positive coordinates (i.e) $F_n(L) \cap \{x_{i_1} = 0\} \cap \dots \cap \{x_{i_t} = 0\} \simeq F_{n-t}(L)$ up to renaming variables. Hence we need to characterize only the extreme points with exactly n positive components (i.e) $T = \emptyset$.

Theorem 2.1. *Let $t = L - \lfloor L \rfloor, v = \lfloor L \rfloor - (n-1)\lfloor \frac{\lfloor L \rfloor}{n-1} \rfloor, w = \lfloor L \rfloor - (n-2)\lfloor \frac{\lfloor L \rfloor}{n-2} \rfloor$ and define*

$$\begin{aligned} k_1(N) &= \lceil \frac{\lfloor L \rfloor - N}{n-1} \rceil \\ k_2(N) &= \lfloor \frac{\lfloor L \rfloor - N}{n-2} \rfloor \end{aligned}$$

If $W = (x, f)$ is an extreme point of $F_n(L)$ with n strictly positive components such that $2 \leq |S| = m \leq n-1$ then

1. $\lfloor L \rfloor \geq n-2$

2.

$$\begin{aligned} x_h &= \lceil f_h \rceil && \forall h \\ f_i &= k + \frac{t + N^1}{m-1} && \forall i \in S \\ f_j &= \lfloor L \rfloor - N^1 - (n-2)k && \text{for some } j \in \tilde{S} \\ f_l &= k = \lceil \frac{\lfloor L \rfloor - N^1}{n-1} \rceil && \forall k \in \tilde{S} - \{j\} \end{aligned}$$

for some $N^1 \in \{a^1, b^1, c^1, d^1\}$ (whenever a^1, b^1, c^1 , or d^1 exist) where

$$a^1 = \min\{p : 0 \leq p \leq \min(v-1; m-2) \text{ and } k_1(p) \leq k_2(p)\}$$

$$b^1 = \max\{p : 0 \leq p \leq \min(v-1; m-2) \text{ and } k_1(p) \leq k_2(p)\}$$

$$c^1 = \min\{p : v \leq p \leq m-2 \text{ and } k_1(p) \leq k_2(p)\}$$

$$d^1 = \max\{p : v \leq p \leq m-2 \text{ and } k_1(p) \leq k_2(p)\}$$

or

$$\begin{aligned} x_h &= \lceil f_h \rceil && \forall h \\ f_i &= k + \frac{t + N^2}{m-1} && \forall i \in S \\ f_j &= \lfloor L \rfloor - N^2 - (n-2)k && \text{for some } j \in \tilde{S} \\ f_l &= k = \lfloor \frac{\lfloor L \rfloor - N^2}{n-2} \rfloor && \forall k \in \tilde{S} - \{j\} \end{aligned}$$

for some $N^2 \in \{a^2, b^2, c^2, d^2\}$, (whenever a^2, b^2, c^2 or d^2 exist), where

$$a^2 = \min\{p : 0 \leq p \leq \min(w; m-2) \text{ and } k_1(p) \leq k_2(p)\}$$

$$b^2 = \max\{p : 0 \leq p \leq \min(w; m-2) \text{ and } k_1(p) \leq k_2(p)\}$$

$$c^2 = \min\{p : w+1 \leq p \leq m-2 \text{ and } k_1(p) \leq k_2(p)\}$$

$$d^2 = \max\{p : w+1 \leq p \leq m-2 \text{ and } k_1(p) \leq k_2(p)\}$$

Proof.

By Lemma (2.1) we can write:

$$\begin{aligned} x_i &= \lceil f_i \rceil & \forall i \\ f_i &= k + q & \forall i \in S \\ f_j &= \mu & \text{for some } j \in \tilde{S} \\ f_l &= k & \forall k \in \tilde{S} - \{j\} \end{aligned}$$

where $0 < q < 1$ and $k \geq \mu$.

The survivability constraint becomes:

$$\begin{aligned} (n-2)k + \mu + mq &= L + u \\ (n-1)k + mq &\geq L + u \\ (n-2)k + \mu + (m-1)q &= L \end{aligned}$$

with $u > 0$ implying $u = q$.

Hence

$$\begin{aligned} (n-2)k + \mu - \lfloor L \rfloor &= t - (m-1)q \\ (n-1)k - \lfloor L \rfloor &\geq t - (m-1)q \end{aligned}$$

Then $t - (m-1)q = -N$ where $0 \leq N \leq m-2$ and N integer.

For fixed N , from the previous inequalities we get:

$$k \geq \lceil \frac{\lfloor L \rfloor - N}{n-1} \rceil$$

and

$$\mu = \lfloor L \rfloor - N - (n-2)k$$

Since μ has to be non negative, it must be that $\mu = \lfloor L \rfloor - N - (n-2)k \geq 0$ i.e

$$k \leq \lfloor \frac{\lfloor L \rfloor - N}{n-2} \rfloor$$

Note that from $k \geq \lceil \frac{\lfloor L \rfloor - N}{n-1} \rceil$ we also get $\mu \leq k$.

Altogether hence, for fixed N it has to be:

$$0 < k_1(N) = \lceil \frac{\lfloor L \rfloor - N}{n-1} \rceil \leq k \leq \lfloor \frac{\lfloor L \rfloor - N}{n-2} \rfloor = k_2(N)$$

A convexity argument can be used to prove that $k = k_1(N)$ or $k = k_2(N)$.

Also, it has to be $\lfloor L \rfloor \geq n-2$. Otherwise $k_1(N) > k_2(N)$ or $k_2(N) \leq 0$ for all $0 \leq N \leq m-2$.

Since $m \leq n-1$ then $m-2 < n-2$ and $\lfloor L \rfloor - N > 0$. Hence $k_1(N) > 0$ for all $0 \leq N \leq m-2$.

Note that

$$\begin{aligned} \lceil \frac{\lfloor L \rfloor - N}{n-1} \rceil &= \lceil \frac{\lfloor L \rfloor}{n-1} \rceil & 0 \leq N \leq v-1 \\ \lceil \frac{\lfloor L \rfloor - N}{n-1} \rceil &= \lfloor \frac{\lfloor L \rfloor}{n-1} \rfloor & v \leq N \leq m-2 < n-1 \end{aligned}$$

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This is because

$$\lceil \frac{\lfloor L \rfloor - N}{n-1} \rceil = \lfloor \frac{\lfloor L \rfloor}{n-1} \rfloor + \lceil \frac{v-N}{n-1} \rceil$$

and

$$\lceil \frac{\lfloor v \rfloor - N}{n-1} \rceil = \begin{cases} 0 & \text{if } v - N \leq 0 \\ 1 & \text{if } v - N > 0 \end{cases}$$

Similarly

$$\begin{aligned} \lfloor \frac{\lfloor L \rfloor - N}{n-2} \rfloor &= \lfloor \frac{\lfloor L \rfloor}{n-2} \rfloor & 0 \leq N \leq w \\ \lfloor \frac{\lfloor L \rfloor - N}{n-2} \rfloor &= \lfloor \frac{\lfloor L \rfloor}{n-2} \rfloor - 1 & w+1 \leq N \leq m-2 < n-1 \end{aligned}$$

Define a, b, c, d and a^1, b^1, c^1, d^1 as in the statement of the theorem. A convexity argument can be used to show that the point corresponding to N where for example $a < N < b$ can not be an extreme point.

Theorem 2.2. *If $W = (x, f)$ is an extreme point of $F_n(L)$ with $|S| = n$ then $f_i = \frac{L}{n-1}$, $x_i = \lceil \frac{L}{n-1} \rceil$ for all $i \in \{1 \dots n\}$.*

Proof. Standard.

Theorem 2.3. *If $W = (x, f)$ is an extreme point of $F_n(L)$ with n positive components such that $|S| = 1$ then L is not integer and W is one of the following points:*

$$\begin{aligned} x_l &= \lceil f_l \rceil & \forall l \\ f_i &= \xi & \text{for some } i \\ f_j &= \mu & \text{for some } j \\ f_g &= k & \forall g \neq i, j \end{aligned}$$

where

$$\begin{aligned} \xi &= k - (\lceil L \rceil - L) \\ \mu &= \lceil L \rceil - (n-2)k \\ k &= \begin{cases} k_1 = \lceil \frac{\lceil L \rceil + (\lceil L \rceil - L)}{n-1} \rceil \\ k_2 = \lfloor \frac{\lceil L \rceil}{n-2} \rfloor \end{cases} \\ & \text{if } k_1 \leq k_2 \end{aligned}$$

or

$$\begin{aligned} \xi &= L - (n-2)k \\ \mu &= k \\ k &= \begin{cases} k_1 = \lceil \frac{\lceil L \rceil}{n-1} \rceil \\ k_2 = \lfloor \frac{L-1}{n-2} \rfloor \end{cases} \end{aligned}$$

if $k_1 \leq k_2$

or

$$\begin{aligned} \xi &= L - \lfloor L \rfloor \\ \mu &= \lfloor L \rfloor - (n-3)k \\ k &= \begin{cases} k_1 = \lceil \frac{\lfloor L \rfloor}{n-2} \rceil \\ k_2 = \lfloor \frac{\lfloor L \rfloor - 1}{n-3} \rfloor \end{cases} \end{aligned}$$

if $k_1 \leq k_2$

Proof. Similar to the proof in the theorem (2.1).

Theorem 2.4. *If $W = (x, f)$ is an extreme point of $F_n(L)$ with n positive components such that $S = \emptyset$ then W is one of the points:*

$$\begin{aligned} x_i &= k & \forall i \in \{1 \dots n\} - \{j\} \\ x_j &= \lceil L \rceil - (n-2)k & \text{for some } j \\ k &= \begin{cases} k_1 = \lceil \frac{\lceil L \rceil}{n-1} \rceil \\ k_2 = \lfloor \frac{\lceil L \rceil}{n-2} \rfloor \end{cases} \end{aligned}$$

Proof. See [6].

So far, we have proved that if W is an extreme point of $F_n(L)$ with n positive components it must be one of those previously described. As previously pointed this is enough to describe all the extreme points of $F_n(L)$.

3. Classes of Facets

In the previous section we have characterized a set containing all the extreme points of our basic polyhedron. We could use this information in order to find a characterization of all its facets. Here we will give some important classes of them and more can be generated via a simple application of the lifting procedure.

The polyhedron $P_n(\lceil D \rceil, \lceil L \rceil)$, defined by

$$\begin{aligned} \sum_{i=1}^n x_i &\geq \lceil D \rceil \\ \sum_{i \neq j} x_i &\geq \lceil L \rceil \quad \forall j = 1 \dots n \\ x &\in Z_+^n \end{aligned}$$

was completely characterized in [6]. When $D = L$, i.e., 100% survivability is required, we have that the first equation is redundant (is implied by the other constraints) and hence all the facets of this polyhedron are also facets for $F_n(L)$.

Theorem 3.1. *All the facets of $P_n(\lceil L \rceil, \lceil L \rceil)$ are facets of $F_n(L)$.*

Proof. The inequalities describing $P_n(\lceil L \rceil, \lceil L \rceil)$ are implied by those describing $F_n(L)$, hence all facets of $P_n(\lceil L \rceil, \lceil L \rceil)$ are valid inequalities for $F_n(L)$. By Theorem (2.4) they are also facets for $F_n(L)$.

Next, we will give, for completeness, the proof of a Theorem, stated in [6] for the multi-commodity case. In order to grasp some intuition, let us give an example. Suppose that we are given the following optimization problem over $F_5(40.6)$:

$$\begin{aligned} \text{Min } & \sum_{i \neq j} f_i \\ & \text{s.t.} \\ & (f, x) \in F_5(40.6) \end{aligned}$$

If we consider its LP relaxation we get an optimal objective value of 40.6 and an optimal solution $x_i = f_i = \frac{40.6}{4} = 10.15$ for all $i = \{1 \dots 5\}$. If we fix some variable j to its floor we ask ourself by how much the objective function will increase (i.e) what is the optimal value and optimal solution of the problem with the additional constraint $f_j = 10$. We need to redistribute the amount 0.15 of flow among the other variables in a way that minimize the objective function. It is easy to see that the resulting point will be $f_j = 10$, $f_i = 10.15 + \frac{0.15}{3} = 10.2$ for all $i \neq j$ $x_i = \lceil f_i \rceil$ for all $i = \{1, \dots, 5\}$. The total increase of the flow is hence $4 * \frac{0.15}{3} = 0.2$ If we now fix $f_j = 9$ we get that the flow must increase by $4 * 1/3$. We can, hence, consider the inequality:

$$0.2x_j + \sum_{i \neq j} f_i \geq 0.2 * 10 + 0.2 * 1 + 40.6$$

which is valid for $x_j \leq 10$ by the same type of reasoning and it is obviously valid for $x_j \geq 11$.

In general we can prove the following:

Theorem 3.2. *Let $n \geq 3$. Let $s = \frac{L}{n-1} - \lfloor \frac{L}{n-1} \rfloor$ and $q = \frac{s(n-1)}{n-2}$. If $s \neq 0$ and exists a point such that $\tilde{x}_j = \lceil \frac{L}{n-1} \rceil$, $\sum_{i \neq j} \tilde{f}_i = L$ and $f_l \neq f_i$ for some $l \neq i, j$ then*

$$qx_j + \sum_{i \neq j} f_i \geq q \lceil \frac{L}{n-1} \rceil + L \quad (1)$$

is a facet of $F_n(L)$ for any $j \in \{1, \dots, n\}$

Proof.

The constraints defining $F_n(L)$ imply that $\sum_{i \neq j} f_i \geq \frac{(n-1)L}{n-2} - \frac{(n-1)f_j}{n-2}$. Hence for any unit decrease in f_j the total flow on the remaining variables must increase by $\frac{n-1}{n-2}$. All the points (x^*, f^*) with $x_j^* \leq \lfloor \frac{L}{n-1} \rfloor$ must also have $f_j^* \leq \lfloor \frac{L}{n-1} \rfloor$. Hence, for $p \in N^+$ we get:

$$\begin{aligned} qx_j^* + \sum_{i \neq j} f_i^* & \geq q(\lfloor \frac{L}{n-1} \rfloor - p) + \frac{(n-1)p}{n-2} + L + (n-1)\frac{s}{n-2} = \\ & s\frac{n-1}{n-2} \lceil \frac{L}{n-1} \rceil + \frac{n-1}{n-2}p(1-s) + L > q \lceil \frac{L}{n-1} \rceil + L \end{aligned}$$

It is easy to see that all the points (x^*, f^*) with $x_j^* \geq \lceil \frac{L}{n-1} \rceil$ are feasible for (1). This proof validity. The points A and B:

$$\begin{aligned} f_j^a &= \lfloor \frac{L}{n-1} \rfloor \\ f_i^a &= \frac{L}{n-1} + \frac{s}{n-2} \quad \forall i \neq j \\ x_i^a &= \lceil f_i^a \rceil \quad \forall i \end{aligned}$$

$$\begin{aligned} f_i^b &= \frac{L}{n-1} \quad \forall i \\ x_i^b &= \lceil f_i^b \rceil \quad \forall i \end{aligned}$$

are feasible and tight for (1).

To prove that (1) is indeed a facet let us consider another valid inequality $\alpha x + \beta f \geq \pi$ which is tight for all tight points of (1). This inequality must have $\alpha_i = 0 \quad \forall i \neq j$ and $\beta_j = 0$ since for any M the points

$$\begin{aligned} x_i &= \lceil \frac{L}{n-1} \rceil + M \quad i \neq j \\ x_j &= \lceil \frac{L}{n-1} \rceil \\ f_i &= \frac{L}{n-1} \quad \forall i \end{aligned}$$

are tight for (1).

Also, we can increase f_j by ϵ without changing x_j because by the hypothesis $s \neq 0$. From the hypothesis of existence of (\tilde{x}, \tilde{f}) (many points described in the previous section will do it) it also follows that $\beta_i = \beta = 1$ for all $i \neq j$. *c.v.d.*

Now we will present classes of valid inequalities, that are facet defining under some parametric conditions.

Theorem 3.3. *Let $t = L - \lfloor L \rfloor \neq 0$, $v = \lfloor L \rfloor - (n-1)\lfloor \frac{\lfloor L \rfloor}{n-1} \rfloor$. If $\lceil \frac{\lfloor L \rfloor - v}{n-1} \rceil \leq \lfloor \frac{\lfloor L \rfloor - v}{n-2} \rfloor$ then for any \tilde{m} such that $v+2 \leq \tilde{m} \leq n-2$ the following inequality*

$$\sum_{i \neq j} x_i + (2\tilde{m} - 1 - v)x_j + \frac{\tilde{m}(\tilde{m} - 1)}{v+t} \sum_{i \neq j} f_i \geq \lceil L \rceil + (2\tilde{m} - 1 - v)\lceil \frac{\lfloor L \rfloor}{n-1} \rceil + \frac{\tilde{m}(\tilde{m} - 1)}{v+t} L \quad (2)$$

is a facet of $F_n(L)$.

Before we prove it let us prove the following Lemma:

Lemma 3.1.

$$\frac{m}{v+1} + \frac{v}{m-1} \geq 2 \quad (3)$$

for all $m \geq 2$, $v \geq 0$, m and v integers.

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Proof.

Fix v . Then (3) is a function in m that is not increasing for $m \leq v + 1$ and not decreasing for $m \geq v + 1$ i.e. it has a global minimum at $m = v + 1$. In fact

$$\frac{m+1}{v+1} + \frac{v}{m} \geq \frac{m}{v+1} + \frac{v}{m-1}$$

is equivalent to

$$\frac{1}{v+1} \geq \frac{v}{m(m-1)}$$

is equivalent to

$$m(m-1) \geq v(v+1)$$

Hence $\frac{m}{v+1} + \frac{v}{m-1} \geq \frac{v+1}{v+1} + \frac{v}{v} = 2$

We are ready to prove Theorem (3.3).

Proof.

Let us call $\gamma = (2\tilde{m} - 1 - v)$, $\beta = \frac{\tilde{m}(\tilde{m}-1)}{v+1}$ and $\beta' = \frac{\tilde{m}(\tilde{m}-1)}{v+1}$. We will show first that the inequality (2) is valid for $F_n(L)$ and then that is facet defining. In order to prove that this inequality is valid for $F_n(L)$ we will show that it is valid for all its extreme points, whose characterization was given in the previous section. Let (x, f) be an extreme point of $F_n(L)$. If $x_j \geq \lceil \frac{|L|}{n-1} \rceil$ the point is valid for (2). So suppose that $x_j < \lceil \frac{|L|}{n-1} \rceil$. From Lemmas (2.1), (2.2) and (2.3) we know that all extreme points have the form:

$$\begin{aligned} x_a &= \mu & a \in I \\ x_b &= k & \forall b \in I - \{a\} \\ x_c &= \lceil \xi \rceil & \forall c \in NI \\ f_a &= \mu & a \in I \\ f_b &= k & \forall b \in I - \{a\} \\ f_c &= \xi & \forall c \in NI \end{aligned}$$

where (I, NI) is a partition of indices $\{1 \dots n\}$ and $k \geq \mu$, k, μ integer. (One of the set I or NI can be possibly empty). Suppose that $\exists l \in \{1 \dots n\} - \{j\}$ such that $x_l = f_l = k > x_j$. Then the following inequality holds:

$$\sum_{i \neq j, l} x_i + x_l + \gamma x_j + \beta \sum_{i \neq j, l} f_i + \beta f_l \geq \sum_{i \neq j, l} x_i + x_j + \gamma x_l + \beta \sum_{i \neq j, l} f_i + \beta f_j$$

In fact this is true iff

$$\frac{\beta}{\gamma - 1} \geq \frac{x_l - x_j}{f_l - f_j}$$

Note that

$$\frac{x_l - x_j}{f_l - f_j} = \frac{k - x_j}{k - f_j} \leq 1$$

and hence it is enough to show that

$$\frac{\beta}{\gamma-1} \geq 1$$

Since $\frac{\beta}{\gamma-1} \geq \frac{\beta'}{\gamma-1}$ the result follows from Lemma (3.1). It is easy to see that $k \geq \lfloor \frac{\lfloor L \rfloor}{n-1} \rfloor$ with equality holding only if (x, f) is one of these points:

$$\begin{aligned} x_i = f_i = k &= \lceil \frac{\lfloor L \rfloor - N}{n-1} \rceil & i \in I - \{j\} & |I| = n - m - 1 \\ x_j = f_j = \mu &= \lfloor L \rfloor - N - (n-2)k \\ f_i &= \lceil \frac{\lfloor L \rfloor - N}{n-1} \rceil + \frac{t+N}{m-1} & i \in NI & |NI| = m \\ x_i &= \lceil \frac{\lfloor L \rfloor - N}{n-1} \rceil + 1 & i \in NI \end{aligned}$$

with $v+2 \leq m \leq n-1$ and $v \leq N \leq m-2$. In this case in fact $k = \lfloor \frac{\lfloor L \rfloor}{n-1} \rfloor$. Let us show that

$$\begin{aligned} (n-m-2)k + \lfloor L \rfloor - N - (n-2)k + m(k+1) + \gamma k + \beta((n-m-2)k + \\ \lfloor L \rfloor - N - (n-2)k + m(k + \frac{t+N}{m-1})) \geq \lceil L \rceil + \gamma \lceil \frac{\lfloor L \rfloor}{n-1} \rceil + \beta L \end{aligned}$$

This holds if and only if

$$\lfloor L \rfloor - N + m + \beta(\lfloor L \rfloor - N + m \frac{t+N}{m-1}) - \gamma \geq \lceil L \rceil + \beta L$$

which is equivalent to

$$m-1-N + \beta \frac{t+N}{m-1} - \gamma \geq 0$$

When $N = v$ we have:

$$m-1-v-2\tilde{m}+v+1 + \frac{\tilde{m}(\tilde{m}-1)}{m-1} \geq 0$$

which is equivalent to

$$\frac{m}{\tilde{m}} + \frac{\tilde{m}-1}{m-1} \geq 2$$

which holds true for any m and \tilde{m} as in the theorem.

Now we will show that, if $N = m-2$, we have that

$$1 + \beta \frac{t+m-2}{m-1} - \gamma \geq 0$$

This is true if and only if

$$1 + \beta' \left(1 + \frac{1-t}{v+t}\right) \left(\frac{t+m-2}{m-1}\right) - \gamma \geq 0$$

which is equivalent to

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$$1 + \beta' \left(1 + \frac{1-t}{v+t}\right) \left(\frac{t}{m-1} + \frac{m-1}{m-1} - \frac{1}{m-1}\right) - \gamma \geq 0$$

which is equivalent to

$$1 + \beta' \left(1 + \frac{1-t}{v+t}\right) \left(1 + \frac{t-1}{m-1}\right) - \gamma \geq 0$$

which is equivalent to

$$1 + \beta' \left(1 + \frac{t-1}{m-1} + \frac{1-t}{v+t} \frac{t-1}{m-1} + \frac{1-t}{v+t}\right) - \gamma \geq 0$$

which holds if

$$\frac{-1}{m-1} + \frac{-(1-t)}{(v+t)(m-1)} + \frac{1}{v+t} \geq 0$$

which is equivalent to

$$-v - t - 1 + t + m - 1 \geq 0$$

which is equivalent to

$$m - 2 \geq v$$

which is true.

From Theorems (2.1), (2.2), (2.3) and (2.4) we see that the only points for which $\exists l \in \{1 \dots n\}$ such that $x_l = f_l > f_j$ are those with:

$$\begin{aligned} f_j = x_j = \mu &= \lfloor L \rfloor - N - (p-2)k \\ f_i = \xi &= k + \frac{t+N}{p-2} \quad \forall i \in I \\ x_i = \lceil \xi \rceil &= k+1 \quad \forall i \in I \end{aligned}$$

where $I \subset \{1 \dots n\} - \{j\}$ such that $|I| = p-1$; $0 \leq N \leq p-3$ for any $3 \leq p \leq n$ and k as in theorem (2.1).

We know that for these points is $x_i \geq \lceil \frac{\lfloor L \rfloor}{n-1} \rceil$ $i \neq j$.

Note that for fixed p and N and $\tilde{k} > k$

$$\begin{aligned} (p-1)(\tilde{k}+1) + \gamma(\lfloor L \rfloor - N - (p-2)\tilde{k}) + \beta(p-1)\left(\tilde{k} + \frac{t+N}{p-2}\right) &\geq \\ (p-1)(k+1) + \gamma(\lfloor L \rfloor - N - (p-2)k) + \beta(p-1)\left(k + \frac{t+N}{p-2}\right) & \end{aligned}$$

which is equivalent to

$$(p-1) - \gamma(p-1) + \beta(p-1) + \gamma \geq 0$$

which holds true by Lemma (3.1).

So we should be considering only those points with $k = \lceil \frac{\lfloor L \rfloor - N}{p-1} \rceil$. Let us consider a feasible point with p positive coordinates.

$$\begin{aligned}
x_j &= f_j = \mu \\
f_i &= k + q \quad 0 < q < 1 \\
x_i &= k + 1 \quad \forall i \neq j
\end{aligned}$$

Suppose that $q + \frac{1}{p-2} > 1$. Then the point

$$\begin{aligned}
y_j &= g_j = \mu - 1 \\
g_i &= k + q + \frac{1}{p-2} \quad 0 < q < 1 \\
y_i &= k + 2 \quad \forall i \neq j
\end{aligned}$$

is feasible and

$$\sum_{i \neq j} y_i + \gamma y_j + \beta \sum_{i \neq j} g_i \geq \sum_{i \neq j} x_i + \gamma x_j + \beta \sum_{i \neq j} f_i$$

In fact it is enough to show that:

$$(p-1) - \gamma + \beta \left(1 + \frac{1}{p-2}\right) \geq 0$$

which is true by Lemma (3.1).

Then we should consider only those points such that:

$$\begin{aligned}
f_j &= x_j = \mu = \lfloor L \rfloor - N - (p-2)k \\
f_l &= \xi = k + \frac{t+N}{p-2} \quad \forall l \in I \subset \{1 \dots n\} - \{j\} \\
x_l &= \lceil \xi \rceil = k + 1 \quad \forall l \in I \subset \{1 \dots n\} - \{j\}
\end{aligned}$$

such that $|I| = p-1$, $v_p \leq N \leq p-3$ for any $3 \leq p \leq n$ and $v_p = \lfloor L \rfloor - (p-1)\lfloor \frac{\lfloor L \rfloor}{p-1} \rfloor$. We know that in this case is $\lceil \frac{\lfloor L \rfloor - N}{p-1} \rceil = \lfloor \frac{\lfloor L \rfloor}{p-1} \rfloor$

Suppose $p = n$.

$$\begin{aligned}
&(n-1)(k+1) + \gamma(\lfloor L \rfloor - N - (n-2)k) + \beta((n-1)(k + \frac{t+N}{n-2})) \geq \\
&(v+1)(k+1) + (n-v-2)k + \gamma(k+1) + \beta((v+1)(k + \frac{v+t}{v+1}) + (n-v-2)k) = \\
&\lceil L \rceil + \gamma \lceil \frac{\lfloor L \rfloor}{n-1} \rceil + \beta L
\end{aligned}$$

which is equivalent to

$$n - v - 2 + \gamma(v - N - 1) + \beta((n-1)\frac{t+N}{n-2} - v - t) \geq 0$$

Suppose $N = n - 3$

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$$(n - v - 2)(1 - \gamma + \beta) + \beta(-1 + \frac{t + (n - 3)}{n - 2}) \geq 0$$

which is equivalent to

$$(n - v - 2)(1 - \gamma + \beta) + \beta(\frac{-n + 2 + t + n - 3}{n - 2}) \geq 0$$

which is equivalent to

$$(n - v - 2)(1 - \gamma + \beta) + \beta\frac{t - 1}{n - 2} \geq 0$$

since $v \leq n - 3$ if

$$(1 - \gamma + \beta) + \beta\frac{t - 1}{n - 2} \geq 0$$

$$1 - \gamma + \beta'(1 + \frac{1 - t}{v + t}) + \beta'(1 + \frac{1 - t}{v + t})\frac{t - 1}{n - 2} \geq 0$$

if

$$\beta'(\frac{1 - t}{v + t} + \frac{t - 1}{n - 2} + \frac{(1 - t)(t - 1)}{(v + t)(n - 2)}) \geq 0$$

which is equivalent to

$$\frac{1}{v + t} - \frac{1}{n - 2} - \frac{1 - t}{(v + t)(n - 2)} \geq 0$$

which is equivalent to

$$n - 3 - v \geq 0$$

When $N = v$ we have that

$$n - v - 2 + \gamma(v - N - 1) + \beta(t + N - v - t + \frac{t + N}{n - 2}) \geq 0$$

which is equivalent to

$$n - 1 - v - 1 - 2\tilde{m} + v + 1 + \frac{\tilde{m}(\tilde{m} - 1)}{n - 2} \geq 0$$

which is equivalent to

$$\frac{n - 1}{\tilde{m}} + \frac{\tilde{m} - 1}{n - 2} \geq 2$$

which holds true because for fixed \tilde{m} this function achieves its minimum at $n - 1 = \tilde{m}$.

Suppose now that $p < n$.

Let us consider the case $N = p - 3$. Let $\tilde{k} = \lfloor \frac{[L]}{p-1} \rfloor$ and $k = \lfloor \frac{[L]}{n-1} \rfloor$. Let $v_p = [L] - (p - 1)\tilde{k}$.

We will show that

$$(p - 1)(\tilde{k} + 1) + \gamma([L] - (p - 3) - (p - 2)\tilde{k}) + \beta((p - 1)(\tilde{k} + \frac{t + (p - 3)}{p - 2})) \geq [L] + \gamma(k + 1) + \beta L$$

This is true if and only if

$$\lfloor L \rfloor - v_p + p - 1 + \gamma(\tilde{k} + v_p - (p - 3) - k - 1) + \beta(\lfloor L \rfloor - v_p + t + p - 3 + \frac{t + p - 3}{p - 2}) \geq \lfloor L \rfloor + \beta L$$

Note that $v_p \leq p - 3 < p - 2$. Then

$$(p - 3) - v_p - \gamma(-\tilde{k} + k + (p - 2) - v_p) + \beta((p - 3) - v_p + \frac{t + p - 3}{p - 2}) \geq 0$$

Since $p < n$ is $\tilde{k} - k \geq 1$. Hence

$$(p - 3) - v_p - \gamma(-\tilde{k} + k + (p - 2) - v_p) + \beta((p - 3) - v_p + \frac{t + p - 3}{p - 2}) \geq (p - 3) - v_p - \gamma((p - 3) - v_p) + \beta((p - 3) - v_p + \frac{t + p - 3}{p - 2}) \geq 0$$

where the last inequality is true by Lemma (3.1).

When $N = v_p$ $\mu = \lfloor L \rfloor - v_p - (p - 2)\tilde{k} = \tilde{k} \geq \lceil \frac{\lfloor L \rfloor}{n-1} \rceil$.

Now let us show that the inequality (2) is indeed a facet.

We note that the points:

A:

$$\begin{aligned} x_i^a &= k = \lfloor \frac{\lfloor L \rfloor}{n-1} \rfloor & \forall i \in I - \{j\} \\ x_j^a &= k \\ x_l^a &= k + 1 & \forall l \in NI \\ f_i^a &= k & \forall i \in I - \{j\} \\ f_j^a &= k \\ f_l^a &= k + \frac{t+v}{\tilde{m}-1} & \forall l \in NI \end{aligned}$$

with $|NI| = \tilde{m}$ and (I, NI) a partition of $\{1 \dots n\}$

B:

$$\begin{aligned} x_i^b &= k = \lfloor \frac{\lfloor L \rfloor}{n-1} \rfloor & \forall i \in I \\ x_j^b &= k + 1 \\ x_l^b &= k + 1 & \forall l \in NI - \{j\} \\ f_i^b &= k & \forall i \in I \\ f_j^b &= k + \frac{t+v}{v+1} \\ f_l^b &= k + \frac{t+v}{v+1} & \forall l \in NI - \{j\} \end{aligned}$$

with $|NI| = v + 2$ and (I, NI) a partition of $\{ \dots n\}$

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C :

$$\begin{aligned}
x_i^c &= k = \lfloor \frac{\lfloor L \rfloor}{n-1} \rfloor & \forall i \in I - \{j\} \\
x_j^c &= k \\
x_l^c &= k+1 & \forall l \in NI \\
f_i^c &= k & \forall i \in I - \{j\} \\
f_j^c &= k \\
f_l^c &= k + \frac{t+v}{\tilde{m}} & \forall l \in NI
\end{aligned}$$

with $|NI| = \tilde{m} + 1$ and (I, NI) a partition of $\{1 \dots n\}$ are tight for (2).
Let us consider an inequality

$$\sum_i \alpha_i x_i + \sum_i \beta_i f_i \geq \pi \tag{4}$$

which is tight for all points that are tight for (2).

The point D :

$$\begin{aligned}
x_i^d &= x_i^b & \forall i \neq j \\
f_j^d &= f_j^b + \epsilon \\
f_i^d &= f_i^b & \forall i \neq j
\end{aligned}$$

which is tight for (2) implies $\beta_j = 0$.

Since (I, NI) can be any partition we get from point A that:

$$\alpha_i k + \alpha_l (k+1) + \beta_i k + \beta_l (k + \frac{t+v}{\tilde{m}-1}) = \alpha_i (k+1) + \alpha_l k + \beta_i (k + \frac{t+v}{\tilde{m}-1}) + \beta_l k$$

$\forall i, l \neq j$ i.e.

$$\alpha_l - \alpha_i = -(\beta_l - \beta_i) \frac{t+v}{\tilde{m}-1} + \beta_l k$$

while from the point C we get:

$$\alpha_l - \alpha_i = -(\beta_l - \beta_i) \frac{t+v}{\tilde{m}}$$

$\forall i, l \neq j$.

These two inequalities imply that

$$\beta_i = \beta_l = \beta \quad \forall i, l \neq j$$

and hence

$$\alpha_i = \alpha_l = \alpha = 1 \quad \forall i, l \neq j$$

The inequality (4) now is:

$$\sum_{i \neq j} x_i + \gamma x_j + \beta \sum_{i \neq j} f_i \geq \pi$$

and since the points A, B, C are tight for (4) and are linearly independent we have that (4) is equivalent to (2).

4. Conclusion

We are working on more facets for our polyhedron and later work will include a computational study with a real network.

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