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STATE OBSERVATION FOR SYSTEMS WITH LINEAR DYNAMICS AND POLYNOMIAL OUTPUT

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Abstract

This paper deals with the problem of system identification and state estimation for nonlinear uncertain stochastic system, in the discrete-time framework. A polynomial filtering algorithm is here proposed and implemented by opportunely extending the state space with the inclusion of the unknown vector of parameters, so that the filtering and identification problems are simultaneously solved. Such an algorithm is achieved by applying the optimal polynomial filter of a chosen degree μ to the Carleman approximation of the same degree of the extended nonlinear system.

Key words: Polynomial filtering, Extended Kalman Filter, Carleman approximation, System identification, Uncertain systems.

1. Introduction

In this work it is investigated the problem of the simultaneous filtering and parameters identification for the following nonlinear uncertain stochastic system:

$$\begin{aligned} x(k+1) &= f\big(\theta, x(k)\big) + v(k), \quad x(0) = x_0, \quad k \geq 0 \\ y(k) &= h\big(\theta, x(k)\big) + w(k) \end{aligned} \tag{1.1}$$

where $x(k) \in I\!R^n$ is the state of the system, $y(k) \in I\!R^q$ is the measured output, $\theta \in I\!R^m$ is the unknown vector of parameters, $v(k)$ and $w(k)$ are sequences of zero-mean, auto and mutually independent random vectors, not necessarily Gaussian, x_0 is a random variable, independent of both the sequences $v(k)$ and $w(k)$. The most popular real time algorithm for simultaneous state and parameter estimation for this kind of systems is the Extended Kalman Filter (EKF) applied to the extended system, whose state is made of the original state and the parameter vector [8, 3]. The diffusion of the EKF is due to its simplicity and to the fact that in many applications it provides good estimates. The EKF is based on the linear approximation of a nonlinear system around the current estimate, and therefore it performs well in those cases in which the initial state estimate is good and the noises have low variance and approximately gaussian distribution. In such cases the state estimate remains close to the true state and the first-order Taylor expansion around such estimate remains a good approximation of the system dynamics. However, in the presence of high level non gaussian noises the state estimate deteriorates and the first-order approximation is no more a good model for the nonlinear system.

This paper deals with the problem of simultaneous filtering and parameter identification for system (1.1), in the case in which each component θ_i of the unknown parameter vector $\theta \in I\!R^m$ satisfies an internal constraint

$$\theta_i \in [\theta_{i,\min}, \theta_{i,\mathrm{Max}}], \quad i = 1, \dots, m. \tag{1.2}$$

The state and output transition maps $f : I\!R^m \times I\!R^n \mapsto I\!R^n$ and $h : I\!R^m \times I\!R^n \mapsto I\!R^q$ are nonlinear maps, smooth with respect to both the parameters and the state vectors. Like the EKF-based algorithm previously mentioned, also the one here proposed considers the unknown parameters as further components of an extended state, including the original state components: $X^T(k) = \big[x^T(k) \quad \theta^T(k)\big]^T$, with $\theta(k+1) = \theta(k)$. However, here the Polynomial Extended Kalman Filter (PEKF) is adopted, instead of the standard EKF. Such an algorithm [5] belongs to the group of nonlinear filtering methods where the nonlinear system is approximated by a the stochastic system for which known filtering procedures are available (the EKF, for instance, and all the existing modified versions [4, 10, 6]; an effective modification of the EKF is the Unscented Kalman Filter (UKF) [7], that uses the so-called *unscented transform* for the state and output prediction steps in the EKF equations.) The PEKF is obtained by the application of the optimal polynomial filter of [1, 2] to the Carleman approximation of a nonlinear system (see [9, 11]), whereas the standard EKF applies the classical Kalman filter to the linear approximation of nonlinear systems. The Carleman approximation of order μ of a nonlinear system is achieved by suitably defining an extended state made of the Kronecker powers of the original state up to a given order μ. The analogous definition of an extended output is also required for the construction of a polynomial filter. In the stochastic discrete-time framework the Carleman approximated system consists of a bilinear system (linear drift and multiplicative noise) with respect to the extended state. The extended output turns out to be a linear function of the extended state, corrupted by multiplicative noise. Once the approximation is obtained, the recursive equations of the optimal polynomial filter of order μ are available and can be applied with no further approximations (see [1, 2]). It is interesting to note that the implementation of the PEKF of a given degree μ does not require the complete knowledge of the noises distributions: only the moments up to order 2μ are needed. When $\mu = 1$ the PEKF reduces to the classical EKF. As in the case of the classical EKF, the Polynomial Extended Kalman Filter (PEKF) is a time-varying recursive algorithm whose performances depend on the specific application. A better behavior with respect to the classical EKF is expected for two reasons: i) a higher degree of approximation of the nonlinear system is adopted; ii) the optimal polynomial estimate is implemented for the approximate system, instead of the linear Kalman estimate of the EKF.

2. Carleman approximation of stochastic systems

In order to simultaneously estimate both the state of the system x and the unknown parameters vector θ, as already mentioned, the latter is treated as a further state component with no dynamical evolution so that, considering the additional state equation $\theta(k+1) = \theta(k)$, system (1.1) becomes:

$$\begin{aligned} X(k+1) &= f_e\big(X(k)\big) + v_e(k), \qquad X(0) = X_0 = \big(x_0^T \;\; \theta^T\big)^T, \quad k \geq 0 \\ y(k) &= h_e\big(X(k)\big) + w(k), \end{aligned} \tag{2.1}$$

where $f_e : I\!R^{n_e} \mapsto I\!R^{n_e}$, $h_e : I\!R^{n_e} \mapsto I\!R^q$, with $n_e = n + m$, are:

$$f_e(X) = \begin{bmatrix} f(X_2, X_1) \\ X_2 \end{bmatrix}, \qquad h_e(X) = h(X_2, X_1), \qquad v_e(k) = \begin{bmatrix} v(k) \\ 0 \end{bmatrix} \tag{2.2}$$

according to the natural decomposition of the extended vector

$$X(k) = \begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix}, \qquad X_1(k) \in I\!R^n, \qquad X_2(k) \in I\!R^m. \tag{2.3}$$

The Carleman approximation, here applied to system (2.1) in order to implement the filtering algorithm, is based on the following steps. Choose an integer μ and consider the sequences $X^{[i]}(k)$ and $y^{[i]}(k)$ of the Kronecker powers of the extended state and output of system (2.1) for $i = 1, \dots, \mu$ (here superscripts in square brackets denote the Kronecker powers of vectors and matrices; for a quick survey on the Kronecker algebra see [2]). The update equations for these sequences are

$$X^{[i]}(k+1) = \Big(f_e\big(X(k)\big) + v_e(k)\Big)^{[i]}, \qquad y^{[i]}(k) = \Big(h_e\big(X(k)\big) + w(k)\Big)^{[i]}. \tag{2.4}$$

Under standard analyticity hypotheses the nonlinear functions $(f_e + v_e)^{[i]}$ and $(h_e + w)^{[i]}$ can be approximated in a suitable neighborhood of a given point $\widetilde{X}$ using Taylor polynomials of degree μ:

$$\Big(f_e\big(X(k)\big) + v_e(k)\Big)^{[i]} \approx \sum_{j=0}^{\mu} F_{i,j}(\widetilde{X})\big(X(k) - \widetilde{X}\big)^{[j]} + \sum_{j=0}^{\mu} \varphi_{i,j}\big(\widetilde{X}, v_e(k)\big)\big(X(k) - \widetilde{X}\big)^{[j]}; \tag{2.5}$$

$$\Big(h_e\big(X(k)\big) + w(k)\Big)^{[i]} \approx \sum_{j=0}^{\mu} H_{i,j}(\widetilde{X})\big(X(k) - \widetilde{X}\big)^{[j]} + \sum_{j=0}^{\mu} \psi_{i,j}\big(\widetilde{X}, w(k)\big)\big(X(k) - \widetilde{X}\big)^{[j]}; \tag{2.6}$$

where $\varphi_{i,j}\big(\widetilde{X}, v_e(k)\big)$ and $\psi_{i,j}\big(\widetilde{X}, w(k)\big)$ are suitably defined polynomials of $v_e(k)$ and $w(k)$ (see [5]), and

$$F_{i,j}(X) = \frac{1}{j!}\left(\nabla_X^{[j]} \otimes f_e^{[i]}\right), \quad H_{i,j}(X) = \frac{1}{j!}\left(\nabla_X^{[j]} \otimes h_e^{[i]}\right); \tag{2.7}$$

the operator $\nabla_x^{[j]}\otimes$ applied to a function $\alpha = \alpha(x) : I\!R^n \mapsto I\!R^i$ is defined as

$$\nabla_x^{[0]} \otimes \alpha = \alpha, \qquad \nabla_x^{[j+1]} \otimes \alpha = \nabla_x \otimes \nabla_x^{[j]} \otimes \alpha, \qquad j \geq 1, \tag{2.8}$$

with $\nabla_x = [\partial/\partial x_1 \ \cdots \ \partial/\partial x_n]$. Note that $\nabla_x \otimes \alpha$ is the standard Jacobian of the vector function α.

Remark 2.1. From a computational point of view, the Carleman coefficients $F_{1,s}$ and $H_{1,s}$ can be achieved directly from the original maps:

$$F_{1,s}(X) = \frac{1}{s!}\left(\nabla_X^{[s]} \otimes f_e(X)\right) = \frac{1}{s!}\left(\nabla_X^{[s]} \otimes \begin{bmatrix} f(X_2, X_1) \\ X_2 \end{bmatrix}\right) = \frac{1}{s!}\begin{bmatrix} \nabla_X^{[s]} \otimes f(\theta, x) \\ \nabla_X^{[s]} \otimes I_d(\theta) \end{bmatrix}_{(\theta,x)=(X_2,X_1)}. \tag{2.9}$$

$$H_{1,s}(X) = \frac{1}{s!}\left(\nabla_X^{[s]} \otimes h_e(X)\right) = \frac{1}{s!}\left(\nabla_X^{[s]} \otimes h(X_2, X_1)\right) = \frac{1}{s!}\left(\nabla_X^{[s]} \otimes h(\theta, x)\right)_{(\theta,x)=(X_2,X_1)}. \tag{2.10}$$

with

$$\nabla_X^{[s]} \otimes I(\theta) = \left[\frac{\partial}{\partial x} \ \ \frac{\partial}{\partial \theta}\right]^{[s]} \otimes I_d(\theta) = \begin{cases} \theta, & \text{for } \ s = 0, \\ \left[O_{m\times n} \ \ I_m\right], & \text{for } \ s = 1, \\ O_{m\times n_e^s}, & \text{for } \ s > 1. \end{cases} \tag{2.11}$$

As it will be clearer from the Appendix, all the Carleman coefficients related to the powers of f_e and h_e can be achieved from $F_{1,s}$ and $H_{1,s}$, and so from (2.9) and (2.10). ∎

The expansion of the powers of the binomials in the summations in eq.s (2.5) and (2.6) allows to write these as polynomials of $X(k)$ of degree μ (see [5]). The substitution of the j-th power of $X(k)$ in the summations with a vector $X_j^\mu(k)$ of the same dimension (recall that $X^{[j]}(k) \in I\!R^{n_e^j}$), and of the i-th power of $y(k)$ with a vector $Y_i^\mu(k) \in I\!R^{q^i}$ in the output equations, yeld the recursive equations of the Carleman approximation of order μ around $\widetilde{X}$:

$$\begin{aligned} X_i^\mu(k+1) &= \sum_{j=1}^{\mu} A_{i,j}^\mu(k, \widetilde{X}) X_i^\mu(k) + u_m^\mu(k, \widetilde{X}) + v_i^\mu(k, \widetilde{X}), \\ Y_i^\mu(k) &= \sum_{j=1}^{\mu} C_{i,j}^\mu(k, \widetilde{X}) X_j^\mu(k) + \gamma_i^\mu(k, \widetilde{X}) + w_i^\mu(k, \widetilde{X}), \end{aligned} \qquad \begin{aligned} &i = 1, \dots, \mu, \\ &X_i^\mu(0) = X_0^{[i]}. \end{aligned} \tag{2.12}$$

The 2μ equations (2.12) of the Carleman approximation of system (1.1) can be put in the following compact form

$$\begin{aligned} X^\mu(k+1) &= \mathcal{A}^\mu(k, \widetilde{X}) X^\mu(k) + \mathcal{U}^\mu(k, \widetilde{X}) + V^\mu(k, \widetilde{X}), \\ Y^\mu(k) &= \mathcal{C}^\mu(k, \widetilde{X}) X^\mu(k) + \Gamma^\mu(k, \widetilde{X}) + W^\mu(k, \widetilde{X}), \end{aligned} \tag{2.13}$$

where

$$X^\mu(k) = \begin{bmatrix} X_1^\mu(k) \\ \vdots \\ X_\mu^\mu(k) \end{bmatrix} \in I\!R^{n_\mu}, \quad Y^\mu(k) = \begin{bmatrix} Y_1^\mu(k) \\ \vdots \\ Y_\mu^\mu(k) \end{bmatrix} \in I\!R^{q_\mu}, \quad \begin{aligned} n_\mu &= \sum_{j=1}^{\mu} n_e^j, \\ q_\mu &= \sum_{j=1}^{\mu} q^j, \end{aligned} \tag{2.14}$$

$$\mathcal{A}^\mu = \begin{bmatrix} A_{1,1}^\mu & \cdots & A_{1,\mu}^\mu \\ \vdots & \ddots & \vdots \\ A_{\mu,1}^\mu & \cdots & A_{\mu,\mu}^\mu \end{bmatrix}, \qquad \mathcal{U}^\mu = \begin{bmatrix} u_1^\mu \\ \vdots \\ u_\mu^\mu \end{bmatrix}, \qquad V^\mu = \begin{bmatrix} v_1^\mu \\ \vdots \\ v_\mu^\mu \end{bmatrix}, \tag{2.15}$$

$$\mathcal{C}^\mu = \begin{bmatrix} C_{1,1}^\mu & \cdots & C_{1,\mu}^\mu \\ \vdots & \ddots & \vdots \\ C_{\mu,1}^\mu & \cdots & C_{\mu,\mu}^\mu \end{bmatrix}, \qquad \Gamma^\mu = \begin{bmatrix} \gamma_1^\mu \\ \vdots \\ \gamma_\mu^\mu \end{bmatrix}, \qquad W^\mu = \begin{bmatrix} w_1^\mu \\ \vdots \\ w_\mu^\mu \end{bmatrix}, \tag{2.16}$$

From eq. (A.9), in Appendix, the noises $V^\mu\big(k,\widetilde{X}\big)$ and $W^\mu\big(k,\widetilde{X}\big)$ are bilinear functions of the extended state $X^\mu(k)$ and of zero-mean random vectors uncorrelated with $X^\mu(k)$ of the type $(v_e^{[h]}(k) - E\{v_e^{[h]}(k)\})$ and $(w^{[h]}(k) - E\{w^{[h]}(k)\})$ (note that these are white sequences). This fact allows to state that the Carleman approximation (2.13) has a bilinear structure with respect to an extended white noise sequence. Moreover, exploiting the same arguments used in [1, 2], it is not difficult, though tedious, to prove that $V^\mu\big(k,\widetilde{X}\big)$ and $W^\mu\big(k,\widetilde{X}\big)$ are *uncorrelated* sequences of zero mean *uncorrelated* random vectors, and that the extended state $X^\mu(k)$ is *uncorrelated* with $W^\mu\big(j,\widetilde{X}\big)$ $\forall j$ and with $V^\mu\big(j,\widetilde{X}\big)$ for $k \le j$ (this result is a direct consequence of the fact that the noises $v(k)$ and $w(k)$ in the original system (1.1) are independent and white, and that the original state $x(k)$ is independent of $w(j)$ $\forall j$ and independent of $v(j)$ for $k \le j$).

In order to ensure that all random vectors in (2.13) ($X^\mu(k)$, $Y^\mu(k)$, $V^\mu\big(k,\widetilde{X}\big)$ and $W^\mu\big(k,\widetilde{X}\big)$) have finite means and covariances, it is necessary to assume that the noises and the initial extended state X_0 have finite and available moments up to order 2μ:

$$I\!E\big\{X_0^{[i]}\big\} < \infty, \qquad \begin{aligned} I\!E\big\{v^{[i]}(k)\big\} &= \xi_i^v(k) < \infty, \\ I\!E\big\{w^{[i]}(k)\big\} &= \xi_i^w(k) < \infty, \end{aligned} \qquad i = 1, \ldots, 2\mu; \tag{2.17}$$

as a matter of fact, it comes that the moments of both the initial original state x_0 and the unknown parameters vector have to be finite and available up to order 2μ. Such moments are needed for the recursive computation of the covariances $\Psi^{V^\mu}\big(k,\widetilde{X}\big)$ and $\Psi^{W^\mu}\big(k,\widetilde{X}\big)$ of the extended noises $V^\mu\big(k,\widetilde{X}\big)$ and $W^\mu\big(k,\widetilde{X}\big)$. The mean and covariance of the extended state $X^\mu(k)$, also needed for the computation of $\Psi^{V^\mu}\big(k,\widetilde{X}\big)$ and $\Psi^{W^\mu}\big(k,\widetilde{X}\big)$, can be recursively computed using standard formulas for bilinear systems. Details on the computation of the system matrices and sequences involved in (2.12), and of the extended noise statistics are reported in Appendix.

3. The filtering algorithm

The previous section has described the μ-th order Carleman approximation of a stochastic non-linear system. The result is a bilinear system driven by white noise, given by eq.'s (2.13). For the filter construction it is assumed that the output of the original system (1.1) is generated in fact by the approximate model (2.13), and thus in the filter equations $Y_i^\mu(k)$ will coincide with $y^{[i]}(k)$. For a system of the type (2.13) the optimal linear filter (linear w.r.t. the *extended* measurements Y^μ) provides the optimal μ-degree polynomial filter w.r.t. the *original* measurements, and can be constructed without any further approximation (see [1, 2]). Since the extended noises $V^\mu(k,\tilde{x})$ and $W^\mu(k,\tilde{x})$ in eq.'s (2.13) are uncorrelated sequences of uncorrelated zero-mean vectors, as discussed in the previous section, the optimal linear filter is implemented by the standard Kalman filter equations. According to the same philosophy of the standard EKF, the system matrices and the covariances needed in the Riccati equations are computed using, at each step, the equations of the Carleman approximation around the current state estimate and prediction. In particular, the state estimate is used instead of $\widetilde{X}$ for the computation of matrices $\mathcal{A}^\mu$, $\mathcal{U}^\mu$ and Ψ^{V^μ}, while the state prediction is used for the computation of matrices $\mathcal{C}^\mu$, Γ^μ and Ψ^{W^μ}, according to the formulas reported in the Appendix. Note that the estimate $\widehat{X}(k)$ and prediction $\widehat{X}(k+1|k)$ of $X(k)$ are computed by selecting from the estimate and prediction of the extended state, $\widehat{X}^\mu(k)$ and $\widehat{X}^\mu(k+1|k)$, respectively, the first n_e components:

$$\widehat{X}(k) = [I_{n_e} \;\; O_{n_e\times(n_\mu-n_e)}]\widehat{X}^\mu(k), \quad \widehat{X}(k+1|k) = [I_{n_e} \;\; O_{n_e\times(n_\mu-n_e)}]\widehat{X}^\mu(k+1|k); \tag{3.1}$$

the original state and parameter estimate, $\hat{x}(k)$ and $\hat{\theta}(k)$ respectively, are then simultaneously achieved by selecting from $\widehat{X}(k)$ the first n and last m components:

$$\hat{x}(k) = [I_n \;\; O_{n\times m}]\widehat{X}(k), \qquad \hat{\theta}(k) = [O_{m\times n} \;\; I_m]\widehat{X}(k). \tag{3.2}$$

The steps of the PEKF algorithm are summarized below:

The Polynomial Extended Kalman Filter (PEKF)

I) Computation of the initial conditions of the filter:

$$\begin{aligned} &\widehat{X}^\mu(0|-1) = I\!E\{X^\mu(0)\}, && \textit{a priori} \text{ estimate of } X^\mu(0), \text{ from eq. (A.24)} \\ &P_P(0) = \mathrm{Cov}(X^\mu(0)), && \text{covariance of the } \textit{a priori} \text{ estimate from eq. (A.24)} \\ &k = -1, && \text{inizialization of the counter;} \end{aligned}$$

II) computation of the matrices of the μ-th degree approximation of the extended output equation around $\widehat{X}(k+1|k) = [I_{n_e} \;\; O_{n_e\times(n_\mu-n_e)}]\widehat{X}^\mu(k+1|k)$:

$$\begin{aligned} \bar{\mathcal{C}}^\mu(k+1) &= \mathcal{C}^\mu(k+1, \widehat{X}^\mu(k+1|k)), && \text{from eq.'s (2.16) and (A.5),} \\ \bar{\Gamma}^\mu(k+1) &= \Gamma^\mu(k+1, \widehat{X}^\mu(k+1|k)), && \text{from eq.'s (2.16) and (A.8),} \\ \bar{\Psi}^{W^\mu}(k+1) &= \Psi^{W^\mu}(k+1, \widehat{X}^\mu(k+1|k)); && \text{from eq. (A.13),} \end{aligned} \tag{3.3}$$

III) computation of the prediction of the extended output:

$$\widehat{Y}^\mu(k+1|k) = \bar{\mathcal{C}}^\mu(k+1)\widehat{X}^\mu(k+1|k) + \bar{\Gamma}^\mu(k+1); \tag{3.4}$$

IV) computation of the Kalman gain:

$$K(k+1) = P_P(k+1)\overline{\mathcal{C}}^{\mu}(k+1)^T\Big(\overline{\mathcal{C}}^{\mu}(k+1)P_P(k+1)\overline{\mathcal{C}}^{\mu}(k+1)^T + \overline{\Psi}^{W^{\mu}}(k+1)\Big)^{\dagger}; \quad (3.5)$$

V) computation of the error covariance matrix:

$$P(k+1) = \Big(I_{n_\mu} - K(k+1)\overline{\mathcal{C}}^{\mu}(k+1)\Big)P_P(k+1); \quad (3.6)$$

VI) computation of the extended state estimates $\widehat{X}^{\mu}(k+1)$ and $\widehat{X}(k+1)$ and of the estimates $\hat{x}(k+1)$, $\hat{\theta}(k+1)$ of the original state and of the unknown parameter:

$$\begin{aligned}\widehat{X}^{\mu}(k+1) &= \widehat{X}^{\mu}(k+1|k) + K(k+1)\Big(Y^{\mu}(k+1) - \widehat{Y}^{\mu}(k+1|k)\Big),\\ \widehat{X}(k+1) &= [I_{n_e} \;\; O_{n_e\times(n_\mu - n_e)}]\widehat{X}^{\mu}(k+1),\\ \hat{x}(k) &= [I_n \;\; O_{n\times m}]\widehat{X}(k), \qquad \hat{\theta}(k) = [O_{m\times n} \;\; I_m]\widehat{X}(k);\end{aligned} \quad (3.7)$$

VII) increment of the counter: $k = k+1$;

VIII) computation of the matrices of the μ-th degree approximation of the extended state equation around $\widehat{X}(k)$:

$$\begin{aligned}\overline{\mathcal{A}}^{\mu}(k) &= \mathcal{A}^{\mu}\big(k, \widehat{X}(k)\big), && \text{from eq.'s (2.15) and (A.4)},\\ \overline{\mathcal{U}}^{\mu}(k) &= \mathcal{U}^{\mu}\big(k, \widehat{X}(k)\big), && \text{from eq.'s (2.15) and (A.8)},\\ \overline{\Psi}^{V^{\mu}}(k) &= \Psi^{V^{\mu}}\big(k, \widehat{X}(k)\big); && \text{from eq. (A.12)},\end{aligned} \quad (3.8)$$

IX) computation of the extended state prediction:

$$\widehat{X}^{\mu}(k+1|k) = \overline{\mathcal{A}}^{\mu}(k)\widehat{X}^{\mu}(k) + \overline{\mathcal{U}}^{\mu}(k); \quad (3.9)$$

X) computation of the one-step prediction error covariance matrix:

$$P_P(k+1) = \overline{\mathcal{A}}^{\mu}(k)P(k)\overline{\mathcal{A}}^{\mu}(k)^T + \overline{\Psi}^{V^{\mu}}(k); \quad (3.10)$$

XI) GOTO STEP **II**.

Remark 3.1. For consistency with all the developments made in the paper, the PEKF algorithm has been here presented in a form that is not computationally optimized, in that the Kronecker powers contain redundant components (if $X \in \mathbb{R}^{n_e}$ then $X^{[i]} \in \mathbb{R}^{n_e^i}$, but only $\tilde{n}_i = \binom{n_e+i-1}{i}$ monomials are independent). Such redundancies can be avoided through the definition of *reduced Kronecker powers*, containing the independent components of ordinary Kronecker powers (see [1]). More in detail, denoting with $X^{(i)} \in \mathbb{R}^{n_i}$ the reduced i-th Kronecker power of X, it is always possible to define a selection matrix $T_i(n_e) \in \mathbb{R}^{\tilde{n}_i \times n^i}$ made of 0's and 1's, such that:

$$X^{(i)} = T_i(n_e)X^{[i]} \quad (3.11)$$

(note that the choice of matrix $T_i(n_e)$ is not univocal). Similarly, the ordinary Kronecker power $X^{[i]}$ is recovered from the reduced power $X^{(i)}$ through multiplication with a suitable matrix $\widetilde{T}_i(n_e) \in \mathbb{R}^{n^i \times \tilde{n}_i}$. Straightforward but tedious substitutions in the above PEKF algorithm provide a filter with a reduced computational burden, and this last should be considered for efficient implementations.

4. Simulation results

Some significative results are here reported in order to show the effectiveness of the proposed algorithm. Consider the following nonlinear system:

$$\begin{aligned} x_1(k+1) &= \alpha(\theta)x_1(k) + x_1(k)x_2(k) + 0.1 + 0.01v_1(k), \\ x_2(k+1) &= 1.5x_2(k) - x_1(k)x_2(k) + 0.1 + 0.01v_2(k), \\ y(k) &= x_2(k) + 0.04w(k), \end{aligned} \tag{4.1}$$

with the zero-mean white noises v_1, v_2, w independent and obeying the following discrete distributions:

$$\begin{aligned} &P\{v_1(k)=-1\}=0.6, && P\{v_2(k)=-1\}=0.8, && P\{w(k)=-7\}=0.3, \\ &P\{v_1(k)=0\}=0.2, && P\{v_2(k)=4\}=0.2, && P\{w(k)=3\}=0.7. \\ &P\{v_1(k)=3\}=0.2, \end{aligned} \tag{4.2}$$

The initial state $x(0)$ is also a random variable, independent of both the state and output noises, with distribution:

$$\begin{aligned} &P\{x_1(0)=0.4\}=0.2, && P\{x_2(0)=0.1\}=0.2, \\ &P\{x_1(0)=0.8\}=0.8, && P\{x_2(0)=0.4\}=0.8. \end{aligned} \tag{4.3}$$

The coefficient α in the first state equation of (4.1) is unknown: it is known that it is close to its nominal value 0.8, so that a parameterization could be as follows:

$$\alpha(\theta) = 0.8\frac{\theta}{\sqrt{1+\theta^2}}, \qquad \theta \in [0,10], \tag{+}$$

with θ assuming the uniform distribution in its interval constraint (i.e. the unknown coefficient α is internal with respect to the open interval $[0, 0.8)$). Such a parameterization is useful, in that it avoids the filter to overestimates α so reducing the possibility of divergences for the algorithm. A further consequence is that there is a great probability that α is close to the nominal value 0.8: more than 90% to belong to $[0.7, 0.8)$.

Simulations are reported comparing the standard first and second order EKF with the quadratic ($\mu = 2$) and cubic ($\mu = 3$) version of the proposed PEKF.

The sample error variances computed in a typical simulation over a 500 points horizon, are reported in table 4.1.

Table 4.1. Steady state error variances

	EKF	2nd ord. EKF	$\text{PEKF}_{\mu=2}$	$\text{PEKF}_{\mu=3}$
$\sigma^2_{x_1}$	$9.15\cdot 10^{-3}$	$1.00\cdot 10^{-2}$	$7.45\cdot 10^{-3}$	$4.75\cdot 10^{-3}$
$\sigma^2_{x_2}$	$1.75\cdot 10^{-3}$	$1.76\cdot 10^{-3}$	$1.30\cdot 10^{-3}$	$4.28\cdot 10^{-4}$

In this example the quadratic and cubic PEKF perform better than the EKF and 2nd order EKF, which have a very similar behavior. In particular, the quadratic PEKF achieves 18% and 25% reduction of the error variance of the two state components, respectively, w.r.t. standard EKF, while the cubic PEKF achieves 48% and 75% variance reduction. Figures 4.1 and 4.2 report the true states and their estimates using the 2nd order EKF, the quadratic PEKF and the cubic PEKF (for the clarity of the representation, only the last 80 time steps is reported). The EKF estimates is not reported in the figures because they are extremely similar to those provided by the 2nd order EKF and can not be distinguished.

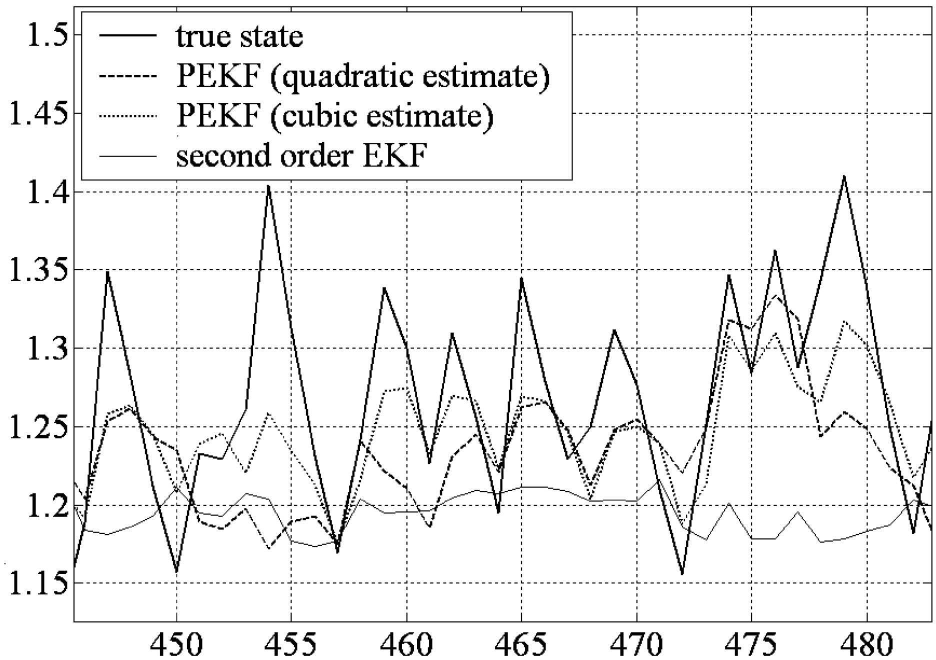

Fig. 4.1 – True and estimated state: the first component.

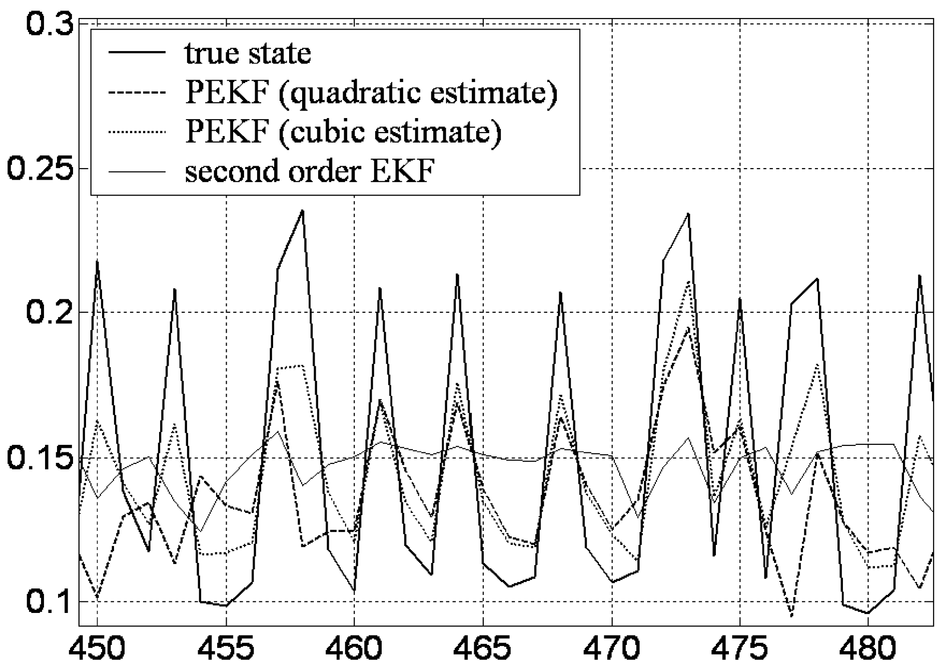

Fig. 4.2 – True and estimated state: the second component.

As far as what concerns the parameter identification, figure 4.3 shows the improvements from the 2nd order EKF up to the quadratic and cubic PEKF:

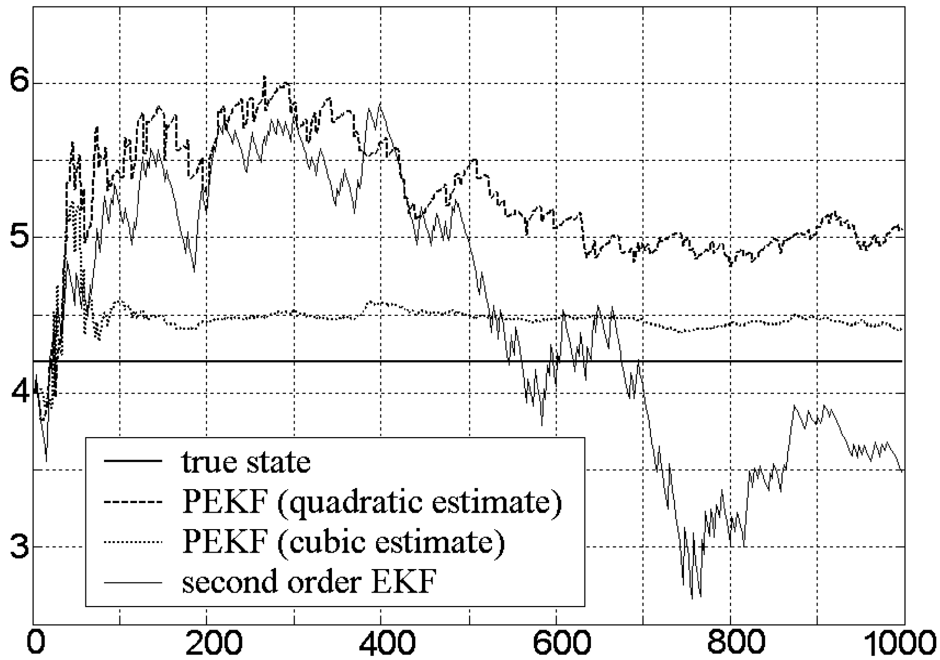

Fig. 4.3 – True and estimated parameter θ.

5. Conclusions

The problem of state estimation for a nonlinear system affected by additive noises, not necessarily Gaussian, has been investigated in this paper. The filtering algorithm here proposed is based on two steps: first the nonlinear system is approximated using the Carleman bilinearization approach, taking into account all the powers of the series expansion up to a fixed degree μ; next, the minimum variance filter of the approximating system in the Hilbert space of all the μ^{th}-degree polynomial transformations of the measurements is computed. This step is based on a well known literature concerning suboptimal polynomial estimates for linear and bilinear state space representations [1, 2]. When $\mu = 1$, the proposed algorithm gives back the standard Extended Kalman Filter.

Appendix: the Kronecker algebra

This Appendix reports the expressions of all the terms needed for the PEKF implementation. The derivation of these equations exploits the rules of the Kronecker algebra (see [2] for a quick survey) and take advantage of a formalism that allows to expand Kronecker powers of sums of vectors. Consider a multiindex $t = \{t_0, t_1, \cdots\ t_\nu\} \in (\mathbb{Z}^+)^{\nu+1}$. Its modulus, denoted $|t|$, is defined as the sum of its entries, i.e. $|t| = t_0 + \cdots + t_\nu$. The i-th Kronecker power of a sum of $\nu+1$ vectors $z_i \in \mathbb{R}^p$, $i = 0, 1, \ldots, \nu$, can be expressed as

$$(z_0 + z_1 + ... + z_\nu)^{[i]} = \sum_{|t|=i} M_t^p \Big(z_0^{[t_0]} \otimes z_1^{[t_1]} \otimes \cdots \otimes z_\nu^{[t_\nu]} \Big), \tag{A.1}$$

with a suitable definition of the matricial coefficients $M_t^p \in \mathbb{R}^{p^i \times p^i}$ (see [2]). Note that for $t \in (Z^+)^2$ it is $M^1_{t_0,t_1} = \binom{t_0+t_1}{t_0}$. The Kronecker product of n matrices A_h, $h = 1, \ldots n$, is denoted as

$$\bigotimes_{h=1}^{n} A_h = A_1 \otimes A_2 \otimes \cdots \otimes A_n. \tag{A.2}$$

With this definition, equation (A.1) can be put in the more compact form

$$\Big(\sum_{h=0}^{\nu} z_h \Big)^{[i]} = \sum_{|t|=i} M_t^p \bigotimes_{h=0}^{\nu} z_h^{[t_h]}. \tag{A.3}$$

Let us recall that the stack of a matrix $A \in \mathbb{R}^{r \times c}$ is the vector in $\mathbb{R}^{r \cdot c}$ that piles up all the columns of matrix A, and is denoted $\mathrm{st}(A)$. The inverse operation is denoted $\mathrm{st}^{-1}_{r,c}(\cdot)$, and transforms a vector of size $r \cdot c$ in a $r \times c$ matrix.

Lemma A.1. *The matrices $A^\mu_{ij}(k, \widetilde{X})$ and $C^\mu_{m,i}(k, \widetilde{X})$ of system* (2.12) *are as follows:*

$$A^\mu_{ij}(k, \widetilde{X}) = \sum_{r \in \mathcal{R}^\mu_{ij}} M_r^{n_e} \mathcal{F}_r(\widetilde{X}) \big(M^{n_e}_{\alpha(r)-j,j} \otimes \xi^v_{r_{\mu+1}}(k) \big) \big(I_{n_e^j} \otimes (-\widetilde{X})^{[\alpha(r)-j]} \big), \tag{A.4}$$

$$C^\mu_{ij}(k, \widetilde{X}) = \sum_{r \in \mathcal{R}^\mu_{ij}} M_r^{q} \overline{H}_r(\widetilde{X}) \big(M^{n_e}_{\alpha(r)-j,j} \otimes \xi^w_{r_{\mu+1}}(k) \big) \big(I_{n_e^j} \otimes (-\widetilde{X})^{[\alpha(r)-j]} \big), \tag{A.5}$$

with $r = \{r_0, \cdots, r_{\mu+1}\}$ a multi-index in $(\mathbb{Z}^+)^{\mu+2}$ and:

$$\alpha(r) = \sum_{s=1}^{\mu} s\, r_s, \qquad \mathcal{R}^\mu_{ij} = \big\{ r \in (\mathbb{Z}^+)^{\mu+2} : |r| = i,\ j \le \alpha(r) \le \mu \big\}; \tag{A.6}$$

the matrices $\mathcal{F}_r$, $\overline{H}_r$ in (A.4), (A.5) *are defined as:*

$$\mathcal{F}_r(\widetilde{X}) = \Big(\bigotimes_{s=0}^{\mu} F^{[r_s]}_{1,s}(\widetilde{X}) \Big) \otimes \begin{bmatrix} I_n \\ O_{m \times n} \end{bmatrix}^{[r_{\mu+1}]}, \qquad \overline{H}_r(\widetilde{X}) = \Big(\bigotimes_{s=0}^{\mu} H^{[r_s]}_{1,s}(\widetilde{X}) \Big) \otimes I_{q^{r_{\mu+1}}}. \tag{A.7}$$

Moreover, the deterministic drifts u^μ_i, γ^μ_i are computed as

$$u^\mu_i(k, \widetilde{X}) = \sum_{r \in \mathcal{R}^\mu_{i0}} M_r^{n_e} \mathcal{F}_r(\widetilde{X}) \big(\widetilde{X}^{[\alpha(r)]} \otimes \xi^v_{r_{\mu+1}}(k) \big), \tag{A.8}$$

$$\gamma_i^\mu(k,\widetilde{X}) = \sum_{r\in\mathcal{R}_{i0}^\mu} M_r^q \overline{H}_r(\widetilde{X})\big(\widetilde{X}^{[\alpha(r)]}\otimes \xi^w_{r_{\mu+1}}(k)\big), \tag{A.9}$$

and the random sequences $\{v_i^\mu\}$, $\{w_i^\mu\}$ *are given by*

$$v_i^\mu(k,\widetilde{X}) = \sum_{r\in\mathcal{R}_{i0}^\mu}\sum_{s=0}^{\alpha(r)} \Delta^r_{i,s}(\widetilde{X})\Big(X_s^\mu(k)\otimes\big(v^{[r_{\mu+1}]}(k)-\xi^v_{r_{\mu+1}}(k)\big)\Big), \tag{A.10}$$

$$w_i^\mu(k,\widetilde{X}) = \sum_{r\in\mathcal{R}_{i0}^\mu}\sum_{s=0}^{\alpha(r)} \Phi^r_{i,s}(\widetilde{X})\Big(X_s^\mu(k)\otimes\big(w^{[r_{\mu+1}]}(k)-\xi^w_{r_{\mu+1}}(k)\big)\Big), \tag{A.11}$$

with

$$\Delta^r_{i,s}(\widetilde{X}) = M_r^{n_e}\mathcal{F}_r(\widetilde{X})\Big(M^{n_e}_{\alpha(r)-s,s}\big(I_{n_e^s}\otimes(-\widetilde{X})^{[\alpha(r)-s]}\big)\otimes I_{n_e^{r_{\mu+1}}}\Big), \tag{A.12}$$

$$\Phi^r_{i,s}(\widetilde{X}) = M_r^q\overline{H}_r(\widetilde{X})\Big(M^{n_e}_{\alpha(r)-s,s}\big(I_{n_e^s}\otimes(-\widetilde{X})^{[\alpha(r)-s]}\big)\otimes I_{q^{r_{\mu+1}}}\Big). \tag{A.13}$$

Proof. The derivation of the Carleman approximation of a generic nonlinear stochastic system, endowed by the powers of the output equation, in the discrete-time framework, has been presented in [5]. Here it is reported the way to define such a derivation for the nonlinear function f_e and h_e defined in (2.2). Taking into account the extended noise v_e defined in (2.2), from [5] and according to (A.6), it comes that:

$$A_{ij}^\mu(k,\widetilde{X}) = \sum_{r\in\mathcal{R}_{ij}^\mu} M_r^{n_e}\overline{F}_r(\widetilde{X})\left(M^{n_e}_{\alpha(r)-j,j}\otimes\left(\begin{bmatrix} I_n \\ O_{m\times n}\end{bmatrix}^{[r_{\mu+1}]}\cdot\xi^v_{r_{\mu+1}}(k)\right)\right)\big(I_{n_e^j}\otimes(-\widetilde{X})^{[\alpha(r)-j]}\big), \tag{A.14}$$

with

$$\overline{F}_r(\widetilde{X}) = \left(\bigotimes_{s=0}^{\mu} F_{1,s}^{[r_s]}(\widetilde{X})\right)\otimes I_{n_e^{r_{\mu+1}}}. \tag{A.15}$$

Taking into account that given four matrices suitably dimensioned, the following identity holds

$$\big(A\cdot B\big)\otimes\big(C\cdot D\big) = \big(A\otimes C\big)\cdot\big(B\otimes D\big), \tag{A.16}$$

then:

$$\begin{aligned}
&\overline{F}_r(\widetilde{X})\left(M^{n_e}_{\alpha(r)-j,j}\otimes\left(\begin{bmatrix} I_n \\ O_{m\times n}\end{bmatrix}^{[r_{\mu+1}]}\cdot\xi^v_{r_{\mu+1}}(k)\right)\right)\\
&\quad=\left(\left(\bigotimes_{s=0}^{\mu} F_{1,s}^{[r_s]}(\widetilde{X})\right)\otimes I_{n_e^{r_{\mu+1}}}\right)\cdot\left(\left(I_{n_e^i}\cdot M^{n_e}_{\alpha(r)-j,j}\right)\otimes\left(\begin{bmatrix} I_n \\ O_{m\times n}\end{bmatrix}^{[r_{\mu+1}]}\cdot\xi^v_{r_{\mu+1}}(k)\right)\right)\\
&\quad=\left(\left(\bigotimes_{s=0}^{\mu} F_{1,s}^{[r_s]}(\widetilde{X})\right)\otimes I_{n_e^{r_{\mu+1}}}\right)\cdot\left(I_{n_e^i}\otimes\begin{bmatrix} I_n \\ O_{m\times n}\end{bmatrix}^{[r_{\mu+1}]}\right)\cdot\left(M^{n_e}_{\alpha(r)-j,j}\otimes\xi^v_{r_{\mu+1}}(k)\right)\\
&\quad=\left(\left(\bigotimes_{s=0}^{\mu} F_{1,s}^{[r_s]}(\widetilde{X})\right)\otimes\begin{bmatrix} I_n \\ O_{m\times n}\end{bmatrix}^{[r_{\mu+1}]}\right)\cdot\left(M^{n_e}_{\alpha(r)-j,j}\otimes\xi^v_{r_{\mu+1}}(k)\right),
\end{aligned} \tag{A.17}$$

so that eq. (A.4) comes with $\mathcal{F}_r(\widetilde{X})$ as in (A.7). Analogously, eq.'s (A.8), (A.10), with (A.12), readily come. The approximate output equation (i.e. its term involved in the eq.'s (A.5), (A.7), (A.9), (A.11), (A.13) are straightforward from [5]. ∎

Lemma A.2. *Consider Ψ^{V^μ} and Ψ^{W^μ}, the covariances of the random vectors V^μ and W^μ defined in eq.'s (2.15) and (2.16), whose entries, by definition, are:*

$$\Psi_{ij}^{V^\mu}(k,\widetilde{X}) = I\!E\Big\{v_i^\mu(k,\widetilde{X})v_j^\mu(k,\widetilde{X})^T\Big\}, \qquad \Psi_{ij}^{W^\mu}(k,\widetilde{X}) = I\!E\Big\{w_i^\mu(k,\widetilde{X})w_j^\mu(k,\widetilde{X})^T\Big\}. \quad \text{(A.18)}$$

These can be computed as follows:

$$\begin{aligned}\Psi_{ij}^{V^\mu}(k,\widetilde{X}) = &\sum_{r\in\mathcal{R}_{j0}^\mu}\sum_{t\in\mathcal{R}_{i0}^\mu}\sum_{s=0}^{\alpha(r)}\sum_{l=0}^{\alpha(t)}\Delta_{i,s}^r(\widetilde{X})\\ &\cdot\bigg(\Big(\Psi_{s,l}^{X^\mu}(k,\widetilde{X})\otimes \mathrm{st}^{-1}_{n_e^i,n_e^j}\Big(\big(\xi^v_{t_{\mu+1}+r_{\mu+1}}(k)-\xi^v_{t_{\mu+1}}(k)\otimes\xi^v_{r_{\mu+1}}(k)\big)\Big)\bigg)\Delta_{j,l}^t(\widetilde{X})^T,\end{aligned} \quad \text{(A.19)}$$

$$\begin{aligned}\Psi_{ij}^{W^\mu}(k,\widetilde{X}) = &\sum_{r\in\mathcal{R}_{j0}^\mu}\sum_{t\in\mathcal{R}_{i0}^\mu}\sum_{s=0}^{\alpha(r)}\sum_{l=0}^{\alpha(t)}\Phi_{i,s}^r(\widetilde{X})\\ &\cdot\bigg(\Big(\Psi_{s,l}^{X^\mu}(k,\widetilde{X})\otimes \mathrm{st}^{-1}_{q^i,q^j}\Big(\big(\xi^w_{t_{\mu+1}+r_{\mu+1}}(k)-\xi^w_{t_{\mu+1}}(k)\otimes\xi^w_{r_{\mu+1}}(k)\big)\Big)\bigg)\Phi_{j,l}^t(\widetilde{X})^T,\end{aligned} \quad \text{(A.20)}$$

with $\Psi_{ij}^{X^\mu}(k,\widetilde{X}) = I\!E\Big\{X_i^\mu(k)X_j^\mu(k)^T\Big\}$ the blocks of the matrix of second order moments of the extended state, $\Psi^{X^\mu}(k,\widetilde{X}) = I\!E\Big\{X^\mu(k)X^\mu(k)^T\Big\}$, computed by the recursive equation

$$\begin{aligned}\Psi^{X^\mu}(k+1,\widetilde{X}) = &\mathcal{A}^\mu(k,\widetilde{X})\Psi^{X^\mu}(k,\widetilde{X})\mathcal{A}^\mu(k,\widetilde{X})^T+\mathcal{U}^\mu(k,\widetilde{X})\mathcal{U}^\mu(k,\widetilde{X})^T+\Psi^{V^\mu}(k,\widetilde{X})\\ &+\mathcal{A}^\mu(k,\widetilde{X})Z^\mu(k)\mathcal{U}^\mu(k,\widetilde{X})^T+\mathcal{U}^\mu(k,\widetilde{X})Z^\mu(k)^T\mathcal{A}^\mu(k,\widetilde{X})^T,\end{aligned} \quad \text{(A.21)}$$

where $Z^\mu(k) = I\!E\big\{X^\mu(k)\big\}$ is the mean value of the extended state, computed as:

$$Z^\mu(k+1) = \mathcal{A}^\mu(k,\widetilde{X})Z^\mu(k)+\mathcal{U}^\mu(k,\widetilde{X}). \quad \text{(A.22)}$$

The initialization of (A.21) and (A.22) are as follows

$$\Psi_{ij}^{X^\mu}(0,\widetilde{X}) = E\left\{X_0^{[i]}\big(X_0^{[j]}\big)^T\right\} = \mathrm{st}^{-1}_{n_e^i,n_e^j}\Big(I\!E\big\{X_0^{[i+j]}\big\}\Big), \qquad Z_i^\mu(0) = I\!E\big\{X_0^{[i]}\big\}. \quad (A.23)$$

with

$$I\!E\big\{X_0^{[i]}\big\} = \sum_{j=0}^{i} M^2_{j,i-j}\left(\begin{bmatrix} I_n\\ O_{m\times n}\end{bmatrix}^{[j]}\otimes\begin{bmatrix} O_{n\times m}\\ I_m\end{bmatrix}^{[i-j]}\right)\big(\zeta_j^0\otimes\zeta_{i-j}^\theta\big), \qquad \begin{aligned}\zeta_j^0 &= I\!E\big\{X_0^{[i]}\big\},\\ \zeta_j^\theta &= I\!E\big\{\theta^{[i]}\big\}.\end{aligned} \quad \text{(A.24)}$$

References

[1] F. Carravetta, A. Germani and M. Raimondi, "Polynomial filtering for linear discrete-time non-Gaussian systems," *SIAM J. Contr. Optim.*, Vol. 34, No. 5, pp. 1666–1690, 1996.

[2] F. Carravetta, A. Germani and M. Raimondi, "Polynomial filtering of discrete-time stochastic linear systems with multiplicative state noise," *IEEE Trans. Automat. Contr.*, Vol. 42, pp. 1106–1126, 1997.

[3] H. Cox, "On the estimation of state variables and parameters for noisy dynamic systems," *IEEE Trans. Automat. Contr.*, Vol. 9, pp. 5-12, 1964.

[4] A. Gelb, *Applied Optimal Estimation*, MIT Press, MA: Cambridge, 1984.

[5] Germani, A., C. Manes and P. Palumbo, "Polynomial Extended Kalman Filtering for Discrete-Time Nonlinear Stochastic Systems", *Proc. of 42nd IEEE Conf. on Decision and Control*, Maui, Hawaii, pp. 886–891, 2003.

[6] Jazwinski, A. H. (1970). *Stochastic Processes and Filtering Theory*, Academic, New York.

[7] Julier, S. J. and J. K. Uhlmann (2004). "Unscented filtering and nonlinear estimation," *Proc. of IEEE*, **Vol. 92**, pp. 401–422.

[8] R.E. Kopp and R. J. Oxford, "Linear regression applied to system identification for adaptive control systems," *AIAA Journal*, Vol. 1, No. 10, pp. 2300-2306, 1963.

[9] K. Kowalski and W. H. Steeb, *Nonlinear Dynamical Systems and Carleman Linearization*, World Scientific, Singapore, 1991.

[10] Reif, K., S. Günther, E. Yaz and R. Unbehauen (1999). "Stochastic stability of the discrete-time extended Kalman filter," *IEEE Trans. Autom. Contr.*, **Vol. 44**, pp. 714–728.

[11] Sastry, S. (1999). *Nonlinear systems: Analysis, Stability and Control*, Springer-Verlag, New-York.

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Abstract

This paper investigates the problem of asymptotic state reconstruction for a class of continuous-time systems characterized by linear input-state dynamics and polynomial state-output function. It is shown that the dynamics of systems in this class can be embedded into the dynamics of systems of higher dimension, with time-varying linear state dynamics and linear state-output map. An asymptotic state observer for the original system is presented, whose design is based on the equations of the extended system. The observer gain is computed on-line by solving a Riccati differential equation. The interest in this observer is in its capability of state reconstruction also in cases in which the original system is not drift-observable (observable for zero input) nor uniformly observable (observable for any input).

Key words: nonlinear systems, state observers, Riccati equation, Kronecker algebra.

1. Introduction

Linear time-invariant (LTI) descriptions of dynamical systems are widely used in control and identification theory [8], even though no real-life system can be considered exactly LTI. Nevertheless, LTI models are of enormous value in all of engineering fields, since LTI models may be good approximations of real systems and have proved to be very useful in control and/or identification algorithms design. One step toward reality is to consider dynamic systems described by linear input-state equations and nonlinear state-output functions. This paper deals with the state observation problem for such systems, in the particular case of output functions that are polynomials of the state vector. This class of systems is of particular interest in applications. Consider, as an example, an electrical network in which the powers at some terminals are measured. Such outputs are quadratic functions of the state of the network, typically made of currents and voltages. In electromechanical systems the torque is the product of magnetic fluxes and currents, and the counterelectromotive force is proportional to the product of the magnetic flux and rotor velocity. Note also that any smooth nonlinear output function can be approximated through polynomials.

The observability analysis of nonlinear systems can be made through the so called *drift-observability map* (see [4]): theoretical invertibility of such map guarantees the possibility of state reconstruction for systems with full relative degree or with zero input. However, drift-obsevability does not provide a complete observabilty analysis of a nonlinear system in the case of general relative degree, when the input is not identically zero. Differently from what happens for linear systems, it is well known that in nonlinear systems the forcing input plays an important role in the state observability. The observability for any input (*uniform observability* in [3]) is a strong property that in most applications is not satisfied.

The asymptotic state observer presented in this paper does not assume the drift-observability of the system, nor the uniform observability. This observer can be constructed in all cases in which the system, together with the applied input, satisfies an observability condition based on the observability Gramian of a suitably defined extended system.

The paper is organized as follows: section II introduces the class of systems considered and the concept of drift-obsevability. Also the formalism of Kronecker products and powers is introduced for the system description. In section III an extended state space and an extended system are defined. In section IV the observation algorithm is presented, whose construction is based on the extended system. Simulations results are reported in section V. Conclusions follow. Some results concerning the Kronecker algebra are reported in Appendix.

2. Systems with linear input-state dynamics and polynomial output

The class of systems considered in this paper is described by the following state-space representation:

$$\begin{aligned} \dot{x}(t) &= f\big(x(t)\big) + Bu(t), \\ y(t) &= h\big(x(t)\big), \end{aligned} \qquad t \geq 0, \quad x(0) = x_0, \tag{2.1}$$

where $x(t) \in I\!R^n$ is the system state, $y(t) \in I\!R^q$ is the measured output, $u(t) \in I\!R^p$ is a known input, $f(x) = Ax$, with $A \in I\!R^{n\times n}$, is the (linear) drift term of the system dynamics, $B \in I\!R^{n\times p}$ is the state-input link, and $h(x)$ is the state-output function, whose components are polynomials of the state.

In [4] it is shown that if the *drift-observability property* is satisfied (i.e. observability for $u(t) \equiv 0$), then the system turns out to be observable for a class of *suitably* bounded inputs (the bound depends on some Lipschitz constants of the system equations). If the observation relative degree of the system is *full*, then the drift-observability property allows asymptotic state reconstruction for *any* bounded input.

The observer described in [4] is based on the construction of a drift-observability map, a square function that, in the case of zero input, provides the vector output and a number of its derivatives as a function of the systems-state. In the simple case of scalar output the drift-obervability map provides the output and its derivatives up to the $n-1$ order, and is given by

$$\Phi(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{pmatrix}, \tag{2.2}$$

where $L_f^k h(x)$ is the k-th repeated Lie derivative of the function $h(x)$ along the field $f(x)$. Global/local drift-observability coincides with the global/local invertibility of the drift-observability map. The observer in [4] can be implemented only if the Jacobian $d\Phi/dx$ is nonsingular and the convergence is ensured inside the region of invertibility of $\Phi(x)$.

It is important to stress that, differently from what happens to linear systems, the observability property for nonlinear systems may depend on the input applied: a *favorable input* may allow the state reconstruction for systems that are not drift-observable, while a *bad input* may forbid state reconstruction for systems that are drift-observable.

The approach described in this paper allows the construction of an observer for the class of systems of the type (2.1) that works well when the input is *favorable*.

Before discussing the observer construction, it is useful to describe how polynomial functions of vectors can be conveniently written using linear combinations of Kronecker powers. Recall that the Kronecker power of a matrix M is recursively defined as

$$M^{[0]} = 1, \quad M^{[i]} = M \otimes M^{[i-1]}, \quad i \geq 1, \tag{2.3}$$

with $\otimes$ the standard Kronecker product. Recall that if $M \in I\!R^{a\times b}$, then $M^{[i]} \in I\!R^{a^i \times b^i}$. Since in this paper the Kronecker algebra is intensively used, its main properties are reported in the Appendix, to the ease of the reader. See the Appendix in [5] for a quick survey on the Kronecker algebra. See also [9] for more properties.

A q components polynomial of degree not greater than m of a vector $x \in I\!R^n$ can be written as

$$\sum_{i=0}^{m} D_i v^i, \tag{2.4}$$

where $D_i \in I\!R^{q\times n^i}$ are the coefficients of the polynomial.

Systems of the class (2.1) can be written in the following form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= \sum_{i=1}^{m} C_i x^{[i]}(t), \end{aligned} \qquad t \geq 0, \quad x(0) = x_0. \tag{2.5}$$

where $C_i \in I\!R^{q\times n^i}$, $i = 1, \ldots, m$ are the coefficients of the output polynomial. Note that, without loss of generality, the constant term of the output polynomial is not considered.

3. The extended system

In this section it is shown that the state-output dynamics of the nonlinear stationary system (2.5) obeys linear time-varying equations in the state-space form if an extended state $X_m(t)$ is suitably defined as follows

$$X_m(t) = \begin{pmatrix} x(t) \\ x^{[2]}(t) \\ \vdots \\ x^{[m]}(t) \end{pmatrix} \in I\!R^{d(n,m)}, \tag{3.1}$$

where $d(n,m) = \sum_{i=1}^{m} n^i$. With this definition the output $y(t)$ can be written as a linear function of the extended state $X_m(t)$:

$$y(t) = \mathcal{C} X_m(t), \tag{3.2}$$

where $C = \begin{bmatrix} C_1 & C_2 & \cdots & C_m \end{bmatrix}$. It is interesting to show that the dynamics of the extended state obeys the equation described by the following lemma:

Lemma 3.1. *The dynamics of the extended state $X_m(t)$ defined by (3.1) is given by:*

$$\dot{X}_m(t) = \mathcal{A}\big(u(t)\big) X_m(t) + \mathcal{B} u(t), \tag{3.3}$$

with matrix $\mathcal{A}(u)$ is defined as

$$\begin{bmatrix} \mathcal{A}_{1,1} & O & \cdots & 0 & 0 \\ \mathcal{A}_{2,1}(u) & \mathcal{A}_{2,2} & \cdots & 0 & 0 \\ O & \mathcal{A}_{3,2}(u) & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{A}_{m,m-1}(u) & \mathcal{A}_{m,m} \end{bmatrix}, \tag{3.4}$$

where matrices $\mathcal{A}_{i,i}$, $i = 1, \ldots, m$ and $\mathcal{A}_{i,i-1}(u)$, $2 \leq i \leq m$ are recursively defined by

$$\begin{aligned} \mathcal{A}_{i,i} &= A \otimes I_{n^{i-1}} + I_n \otimes \mathcal{A}_{i-1,i-1}, \\ \mathcal{A}_{1,1} &= A, \\ \mathcal{A}_{i,i-1}(u) &= (Bu) \otimes I_{n^{i-1}} + I_n \otimes \mathcal{A}_{i-1,i-2}(u), \\ \mathcal{A}_{2,1}(u) &= Bu, \end{aligned} \tag{3.5}$$

I_{n^k} is the identity matrix of dimension n^k and

$$\mathcal{B} = \begin{bmatrix} B \\ O \\ \vdots \\ O \end{bmatrix}. \tag{3.6}$$

Proof. The state dynamics (3.3) is equivalent to the following equations

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}_{1,1} x(t) + Bu(t), \\ \frac{d}{dt} x^{[i]}(t) &= \mathcal{A}_{i,i} x^{[i]}(t) + \mathcal{A}_{i,i-1}\big(u(t)\big) x^{[i-1]}(t), \\ i &= 2, \ldots, m. \end{aligned} \tag{3.7}$$

The first of (3.7) is readily proved by observing that, by definition, $\mathcal{A}_{1,1} = A$. The second equation, for $i = 2$, is proved with the following passages, by using the Kronecker product properties:

$$\frac{d}{dt}x^{[2]}(t) = x \otimes \dot{x} + \dot{x} \otimes x = x \otimes \big(Ax + Bu\big) + \big(Ax + Bu\big) \otimes x$$
$$= \big(A \otimes I_n + I_n \otimes A\big)\, x^{[2]} + \big((Bu) \otimes I_n + I_n \otimes (Bu)\big)\, x. \quad (3.8)$$

By definitions (3.5)

$$\begin{aligned} \mathcal{A}_{2,2} &= A \otimes I_n + I_n \otimes A \\ \mathcal{A}_{2,1}(u) &= \big(Bu\big) \otimes I_n + I_n \otimes \big(Bu\big) \end{aligned} \quad (3.9)$$

so that the second of (3.7) is proved for $i = 2$.

Now, proceed by induction: assume that (3.7) is true for a given $i \geq 2$ and prove that it is also true for $i + 1$. Indeed

$$\frac{d}{dt}x^{[i+1]}(t) = x \otimes \frac{d}{dt}x^{[i]} + \dot{x} \otimes x^{[i]} = x \otimes \big(\mathcal{A}_{i,i}x^{[i]} + \mathcal{A}_{i,i-1}(u)x^{[i-1]}\big) + \big(Ax + Bu\big) \otimes x^{[i]}$$
$$= \big(A \otimes I_{n^i} + I_n \otimes \mathcal{A}_{i,i}\big)\, x^{[i+1]} + \big((Bu) \otimes I_{n^i} + I_n \otimes \mathcal{A}_{i,i-1}(u)\big)x^{[i]}. \quad (3.10)$$

From definitions (3.5) it follows

$$\begin{aligned} \mathcal{A}_{i+1,i+1} &= A \otimes I_{n^i} + I_n \otimes \mathcal{A}_{i,i} \\ \mathcal{A}_{i+1,i}(u) &= (Bu) \otimes I_{n^i} + I_n \otimes \mathcal{A}_{i,i-1}(u) \end{aligned} \quad (3.11)$$

so that

$$\frac{d}{dt}x^{[i+1]}(t) = \mathcal{A}_{i+1,i+1}x^{[i+1]}(t) + \mathcal{A}_{i+1,i}\big(u(t)\big)x^{[i]}(t), \quad (3.12)$$

and the induction is proved. □

Remark 3.2. Note that the extended state matrix (3.4) is time-varying, due to its dependence on the known input $u(t)$.

Since redundant terms are present in the Kronecker powers, redundant state components are present in the extended state vector X_m, so that the extended state space results to be output-indistinguishable. Such redundancy can be eliminated by considering suitably defined reduction matrices. First of all note that $x^{[i]}$, the i-th Kronecker power of $x \in \mathbb{R}^n$, has n^i components, but only $\binom{n+i-1}{i}$ are distinct terms (the number of ways to choose i elements from a set of n, with repetitions allowed). Defining the following functions of pairs of integers

$$d(n, m) = n\frac{1 - n^m}{1 - n} = \sum_{i=1}^{m} n^i, \quad (3.13)$$

$$c(n, m) = \binom{n + m}{m} - 1 = \sum_{i=1}^{m} \binom{n + i - 1}{i}, \quad (3.14)$$

it is easy to see that the vector X_m has $d(n,m)$ components, but only $c(n,m)$ are distinct (obviously $c(n,m) < d(n,m)$).

A block-diagonal reduction matrix $\overline{T}_{n,m} \in I\!R^{c(n,m)\times d(n,m)}$ can be suitably defined, as described in detail in [6], for the selection of a nonredundant subvector $\overline{X}_m \in I\!R^{c(n,m)}$ from $X_m \in I\!R^{d(n,m)}$. A block-diagonal matrix $T_{n,m} \in I\!R^{d(n,m)\times c(n,m)}$ allows to reconstruct the redundant vector X_m from $\overline{X}_m$. In formulas

$$\overline{X}_m(k) = \overline{T}_{n,m} X_m, \quad X_m = T_{n,m}\overline{X}_m(k). \tag{3.15}$$

Using Lemma (3.1) and the reduction matrices (3.15), system (2.5) can be embedded in the following system of larger dimension

$$\begin{aligned} \dot{\mathcal{X}}(t) &= \bar{\mathcal{A}}(u)\big(u(t)\big)\mathcal{X}(t) + \overline{\mathcal{B}}u(t) \\ y(t) &= \bar{\mathcal{C}}\mathcal{X}(t), \end{aligned} \tag{3.16}$$

where $\mathcal{X}(t) \in I\!R^{c(n,m)}$ and

$$\begin{aligned} \bar{\mathcal{A}}(u) &= T_{n,m}\mathcal{A}(u)\overline{T}_{n,m}, \qquad \overline{\mathcal{B}} = T_{n,m}\mathcal{B}, \\ \bar{\mathcal{C}} &= \mathcal{C}\overline{T}_{n,m}. \end{aligned} \tag{3.17}$$

The embedding of the original system (2.5) into the extended system (3.16) should be intended as follows: if the initial state value of the extended state of (3.16) is set to

$$\mathcal{X}(0) = T_{n,m}\begin{pmatrix} x(0) \\ x^{[2]}(0) \\ \vdots \\ x^{[m]}(0) \end{pmatrix} = T_{n,m}X_m(0), \tag{3.18}$$

then the outputs of the two systems is the same for any input, and the state $x(t)$ of the original system (2.5) is recovered by selecting the first n components of the extended state $\mathcal{X}(t)$:

$$\begin{aligned} x(t) &= \Sigma\mathcal{X}(t), \\ \text{where } \ \Sigma &= \big[\, I_n \ 0_{n\times(n^2+\cdots n^m)} \,\big]. \end{aligned} \tag{3.19}$$

4. The asymptotic state observer

The asymptotic observer for systems of the type (2.5) presented in this section is constructed as an observer for the extended system (3.16), that has the simpler structure of a system linear in the state.

Theorem 4.1. *Consider the system* (2.5) *and the extended system* (3.16). *Assume that the pair* $(\bar{\mathcal{A}}(u), \bar{\mathcal{C}})$ *and the input function* $u(t)$ *are such that there exist positive scalars* α, β, δ, *with* $\alpha < \beta$, *such that for all* $t \geq 0$

$$\alpha I_{c(n,m)} \leq \int_t^{t+\delta} e^{\bar{\mathcal{A}}_u^T(\tau)}\bar{\mathcal{C}}^T\bar{\mathcal{C}}e^{\bar{\mathcal{A}}_u(\tau)}d\tau \leq \beta I_{c(n,m)}, \tag{4.1}$$

where $\bar{\mathcal{A}}_u(t)$ *denotes the matrix* $\bar{\mathcal{A}}\big(u(\tau)\big)$.

Then, for any $\widehat{\mathcal{X}}(0)$*, the system*

$$\dot{\widehat{\mathcal{X}}}(t) = \bar{\mathcal{A}}_u(t)\widehat{\mathcal{X}}(t) + \bar{\mathcal{B}}u(t) + P(t)\bar{\mathcal{C}}^T\left(y(t) - \hat{y}(t)\right), \tag{4.2}$$

$$\dot{P}(t) = \left(\bar{\mathcal{A}}_u(t) - P(t)\bar{\mathcal{C}}^T\bar{\mathcal{C}}\right)P(t) + P(t)\left(\bar{\mathcal{A}}_u(t) - P(t)\bar{\mathcal{C}}^T\bar{\mathcal{C}}\right)^T + Q(t), \tag{4.3}$$

$$\hat{x}(t) = \Sigma\widehat{\mathcal{X}}(t) \tag{4.4}$$

with $Q(t)$ *and* $P(0)$ *symmetric positive definite,* $Q(t) \geq q_m I_{c(n,m)}$ *for some positive* q_m*, is an asymptotic observer for the system* (2.5)*, i.e.*

$$\lim_{t\to\infty} \|x(t) - \hat{x}(t)\| = 0. \tag{4.5}$$

Proof. It is sufficient to show that equations (4.2)–(4.4) are an asymptotic observer for the extended system (3.16), i.e.

$$\lim_{t\to\infty} \|\mathcal{X}(t) - \widehat{\mathcal{X}}(t)\| = 0. \tag{4.6}$$

First, note that the assumption (4.1) coincides with the *uniform observability* of a linear time-varying system, and implies that $P(t)$ admits upper and lower bounds (see [1, 2]), i.e. there exist positive scalars p_m and p_M such that

$$p_m I_{c(n,m)} \leq P(t) \leq p_M I_{c(n,m)}. \tag{4.7}$$

Let $\varepsilon(t) = \mathcal{X}(t) - \widehat{\mathcal{X}}(t)$ be the observation error of the extended system. Subtracting equation (3.16) from (4.2), the error dynamics is obtained

$$\dot{\varepsilon}(t) = \left(\bar{\mathcal{A}}_u(t) - K(t)\bar{\mathcal{C}}\right)\varepsilon(t), \qquad t \geq 0. \tag{4.8}$$

Consider the following function of the observation error

$$V(\varepsilon, t) = \varepsilon^T P^{-1}(t)\varepsilon, \tag{4.9}$$

positive definite for all $t \geq 0$ because (4.7) implies

$$\frac{1}{p_M} I_{c(n,m)} \leq P^{-1}(t) \leq \frac{1}{p_m} I_{c(n,m)}. \tag{4.10}$$

Let $v(t) = V(\varepsilon(t), t)$. The time derivative of $v(t)$ along the error trajectories is

$$\begin{aligned}\dot{v}(t) &= \dot{\varepsilon}^T(t)P^{-1}(t)\varepsilon(t) + \varepsilon^T(t)\dot{P}^{-1}(t)\varepsilon(t) + \varepsilon^T(t)P^{-1}(t)\dot{\varepsilon}(t)\\ &= \varepsilon^T(t)\Big[\left(\bar{\mathcal{A}}_u(t) - P(t)\mathcal{C}^T\mathcal{C}\right)^T P^{-1}(t) + \dot{P}^{-1}(t) + P^{-1}(t)\left(\bar{\mathcal{A}}_u(t) - P(t)\mathcal{C}^T\mathcal{C}\right)\Big]\varepsilon(t).\end{aligned} \tag{4.11}$$

Recalling that $\dot{P}^{-1}(t) = -P^{-1}(t)\dot{P}(t)P^{-1}(t)$, it follows

$$\dot{v}(t) = \varepsilon^T(t)P^{-1}(t)\Big[P(t)\left(\bar{\mathcal{A}}_u(t) - P(t)\mathcal{C}^T\mathcal{C}\right)^T - \dot{P}(t) + \left(\bar{\mathcal{A}}_u(t) - P(t)\mathcal{C}^T\mathcal{C}\right)P(t)\Big]P^{-1}(t)\varepsilon(t). \tag{4.12}$$

From this, recalling (4.3),

$$\dot{v}(t) = -\varepsilon^T(t)P^{-1}(t)Q(t)P^{-1}(t)\varepsilon(t). \tag{4.13}$$

Since, by (4.10), it is

$$\dot{v}(t) \le -\frac{1}{p_M^2} q_m \|\varepsilon(t)\|^2, \tag{4.14}$$

$$v(t) \ge \frac{1}{p_M} \|\varepsilon(t)\|^2. \tag{4.15}$$

It follows

$$\dot{v}(t) \le -\frac{q_m}{p_M} v(t), \tag{4.16}$$

from which

$$v(t) \le e^{-\frac{q_m}{p_M}t} v(0) = \frac{1}{p_m} e^{-\frac{q_m}{p_M}t} \|\varepsilon(0)\|^2, \tag{4.17}$$

and finally

$$\|\varepsilon(t)\|^2 \le \frac{p_M}{p_m} e^{-\frac{q_m}{p_M}t} \|\varepsilon(0)\|^2. \tag{4.18}$$

This proves the convergence (4.5), and therefore (4.6). □

Remark 4.2. Note that it may be difficult to test condition (4.1) before the construction of the observer. Moreover, in the cases in which the input applied to the system is measured, the condition (4.1) can only be checked *on-line*. In practice, the observer (4.2)–(4.4) can be applied without a preliminary check of the condition (4.1). A positive definite initial value of $P(t)$ for $t = 0$ ensures that for small t the matrix $P(t)$ remains nonsingular and bounded. However, it is convenient to monitor the minimum and maximum eigenvalues of $P(t)$ during its evolution. The divergence of $\lambda_{\text{Max}}\big(P(t)\big)$ or the approach to zero of $\lambda_{\min}\big(P(t)\big)$ are caused by a loss of observability of the extended system (3.16), due to a *bad input*. In these cases it may be necessary to reset $P(t)$ to some well conditioned P_0, waiting for the input to become *favorable* again.

Remark 4.3. Equation (4.18) provides a suggestion for the choice of matrix $Q(t)$ in equation (4.3): a faster convergence of the observer can be obtained by increasing q_m, the lower bound of $Q(t)$.

5. Simulation results

Simulations results are here reported in order to show the effectiveness of the proposed observer. Consider the following linear system with cubic output w.r.t. the state:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x \in I\!R^3 \tag{5.1}$$

$$y(t) = \sum_{i=1}^{3} C_i x^{[i]}(t), \quad y \in I\!R^2. \tag{5.2}$$

The values chosen for the simulation are:

$$A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \quad C_2 = \begin{bmatrix} 1 & 0 & 1 & 2 & -1 & 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & 3 & 0 & 1 & 4 & 0 & -2 \end{bmatrix} \quad C_3 = C_1 \otimes C_2. \tag{5.3}$$

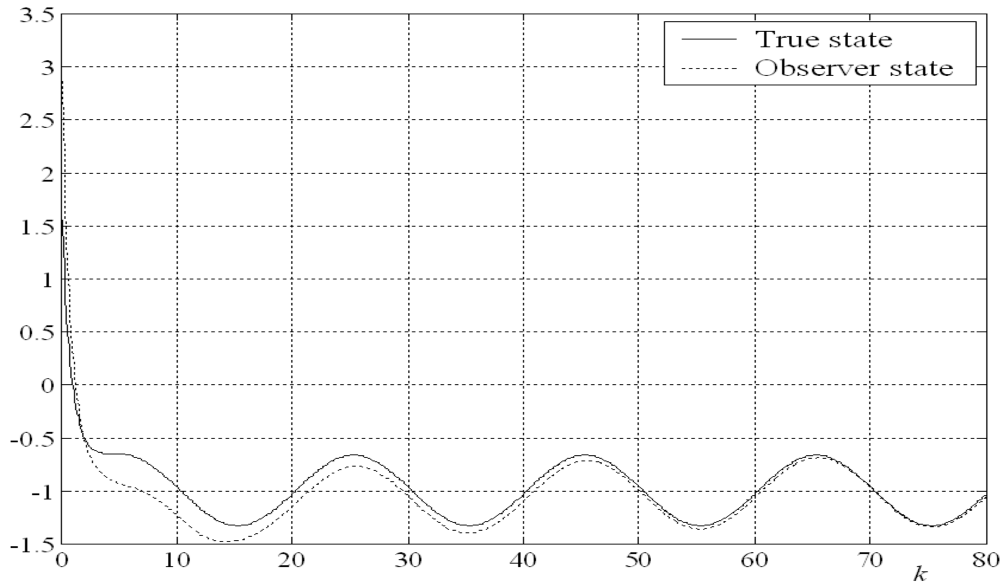

Figure 1: **True and estimated state: the first component.**

The linear approximation of this system around the origin (the pair (A, C_1)) is not observable, not even detectable: A has an eigenvalue in 0 whose eigenvector u_0 lies in the nullspace of C_1:

$$u_0 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} : \quad \begin{bmatrix} A \\ C_1 \end{bmatrix} u_0 = 0 \tag{5.4}$$

This implies that in this example the drift observability map $\Phi(x)$, computed as defined in (2.2), loses rank at the origin, and therefore the considered system is not drift-observable in a neighborhood of the origin. Nevertheless, the presence of an input can allow the state reconstruction.

In the simulations performed the matrix $Q(t)$ and the initial value for the matrix $P(t)$ in the Riccati equation (4.3) have been chosen of the type

$$P(0) = \alpha I_{c(n,m)}, \qquad Q(t) = \beta I_{c(n,m)} \tag{5.5}$$

(remember that in our example $m = n = 3$, so that $c(3,3) = 29$)). The numerical simulations have shown that the convergence speed can be improved by increasing the value of the parameter β, thus confirming what is claimed in Remark (4.3) (note that in our example $\beta = q_m$).

The simulations reported in this section are performed with a sinusoidal input

$$u(t) = \sin(\frac{2\pi}{T} t), \qquad T = 20. \tag{5.6}$$

Figs. 1-3 show the true and the observed state components.

6. Conclusions

The problem of the state observation for the class of systems with linear input-state dynamics and polynomial state-output function has been investigated in this paper, and an asymptotic

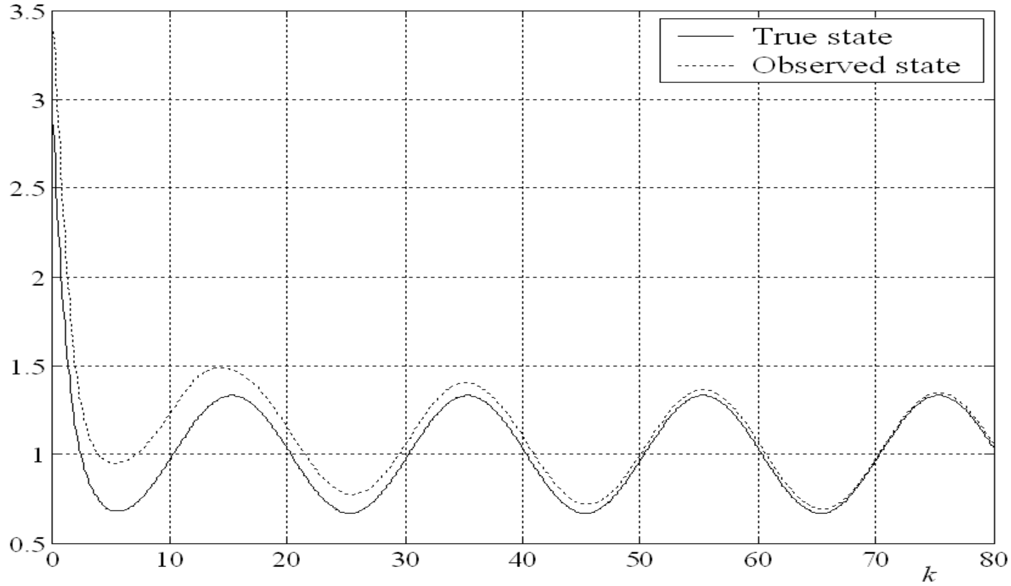

Figure 2: **True and estimated state: the second component.**

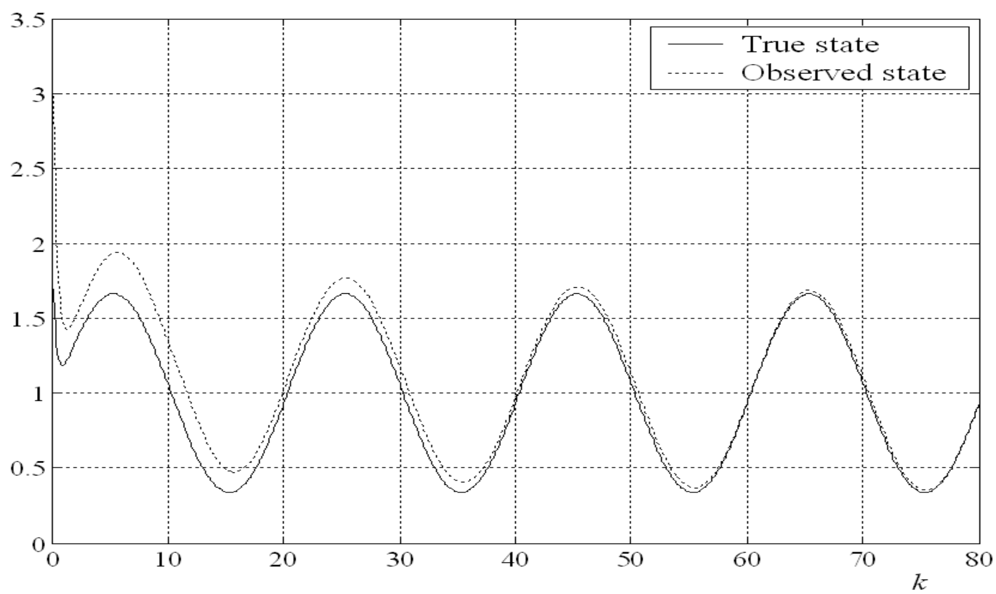

Figure 3: **True and estimated state: the third component.**

observer is presented. It is shown how the original system can be embedded into an extended system, whose state is made of the original state and of some of its Kronecker powers. Next, an observation algorithm is presented, whose structure is derived from the extended system. The observer gain is time varying and is obtained as the solution of a differential Riccati equation. An interesting property of the proposed observer is that it can be implemented and works well also in cases in which the system is not drift-observable nor uniformly-observable, provided that the input applied to the system is *favorable* in a sense that is formalized in theorem (4.1). The observer behavior has been numerically tested on some examples and has always given good results. The simulations here reported refer to a system whose linear approximation around the origin is not observable, not even detectable.

A. Kronecker Algebra

For the ease of the reader, in this Appendix some useful results on the Kronecker algebra are reported. The proofs and other further details can be found in [6]. Let M and N be matrices of dimensions $r \times s$ and $p \times q$ respectively, then the Kronecker product $M \otimes N$ is defined as the $(r \cdot p) \times (s \cdot q)$ matrix

$$M \otimes N = \begin{bmatrix} m_{11}N & \dots & m_{1s}N \\ \vdots & \ddots & \vdots \\ m_{r1}N & \dots & m_{rs}N \end{bmatrix}, \tag{A.1}$$

where the m_{ij} are the entries of M. The stack of M is an operator that gives the $r \cdot s$ vector:

$$\mathrm{st}(M) = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_s \end{bmatrix}. \tag{A.2}$$

The Kronecker power of M is defined as

$$M^{[0]} = 1 \in \mathbb{R}, \tag{A.3}$$
$$M^{[l]} = M \otimes M^{[l-1]} \quad l \geq 1. \tag{A.4}$$

Some useful properties of the Kronecker product and stack operation are the followings:

$$(A + B) \otimes (C + D) = A \otimes C + A \otimes D + B \otimes C + B \otimes D \tag{A.5}$$
$$A \otimes (B \otimes C) = (A \otimes B) \otimes C \tag{A.6}$$
$$(A \cdot C) \otimes (B \cdot D) = (A \otimes B) \cdot (C \otimes D) \tag{A.7}$$
$$(A \otimes B)^T = A^T \otimes B^T \tag{A.8}$$
$$\mathrm{st}(A \cdot B \cdot C) = (C^T \otimes A) \cdot \mathrm{st}(B) \tag{A.9}$$
$$u \otimes v = \mathrm{st}(v \cdot u^T) \tag{A.10}$$
$$\mathrm{tr}(A \otimes B) = \mathrm{tr}(A) \cdot \mathrm{tr}(B) \tag{A.11}$$

Other useful properties can be found in [9].

A generalized version of (A.7), often used throughout the paper is the following:

$$(A_1 \cdot B_1) \otimes (A_2 \cdot B_2) \otimes (A_3 \cdot B_3) = (A_1 \otimes A_2 \otimes A_3) \cdot (B_1 \otimes B_2 \otimes B_3). \tag{A.12}$$

According to its definition (A.1), the Kronecker product is not commutative. However, the following result holds:

Lemma A.1. *For any given pair of matrices $A \in \mathbb{R}^{r\times s}$, $B \in \mathbb{R}^{n\times m}$, it is:*

$$B \otimes A = C_{r,n}^T (A \otimes B) C_{s,m}, \tag{A.13}$$

where $C_{r,n}$, $C_{s,m}$ are defined so that, denoted $\{C_{u,v}\}_{h,l}$ their (h,l) entries:

$$\{C_{u,v}\}_{h,l} = \begin{cases} 1, & \text{if } l = (|h-1|_v)u + \left(\left[\frac{h-1}{v}\right] + 1\right); \\ 0, & \text{otherwise.} \end{cases} \tag{A.14}$$

Proposition A.2. *For any given matrices A, B, C, D, having dimensions $n_A \times m_A$, $n_B \times m_B$, $n_C \times m_C$, $n_D \times m_D$ respectively:*

$$A \otimes B \otimes C \otimes D = \left(I_{n_A} \otimes C^T_{n_C n_D, n_B}\right)\left(A \otimes C \otimes D \otimes B\right)\left(I_{m_A} \otimes C_{m_C m_D, m_B}\right). \tag{A.15}$$

Proof. By applying property (A.6), (A.7) and lemma A.1:

$$\begin{aligned} A \otimes B \otimes C \otimes D &= \Big(A \otimes \big(B \otimes (C \otimes D)\big)\Big) \\ &= \Big(A \otimes \big(C^T_{n_C n_D, n_B}(C \otimes D \otimes B) C_{m_C m_D, m_B}\big)\Big) \\ &= \big(I_{n_A} \otimes C^T_{n_C n_D, n_B}\big)\Big(A \otimes \big((C \otimes D \otimes B) C_{m_C m_D, m_B}\big)\Big) \\ &= \big(I_{n_A} \otimes C^T_{n_C n_D, n_B}\big)(A \otimes C \otimes D \otimes B)\big(I_{m_A} \otimes C_{m_C m_D, m_B}\big). \end{aligned} \tag{A.16}$$

□

Remark A.3. Observe that $C_{1,1} = 1$, hence in the vector case when $a \in \mathbb{R}^r$ and $b \in \mathbb{R}^n$, (A.13) becomes

$$b \otimes a = C_{r,n}^T (a \otimes b). \tag{A.17}$$

Moreover, in the vector case the commutation matrices satisfy also the following recursive formula.

Lemma A.4. *Let $a, b \in \mathbb{R}^n$ and $l \in \mathbb{N}$. Then*

$$b^{[l]} \otimes a = G_l(n)(a \otimes b^{[l]}), \tag{A.18}$$

with the sequence $\{G_l(n) = C^T_{n,n^l}\}$ given by the following recursive equations

$$G_1(n) = C^T_{n,n}, \tag{A.19}$$

$$G_l(n) = (I_{n,1} \otimes G_{l-1}(n)) \cdot (G_1(n) \otimes I_{n,l-1}),\ \ l > 1, \tag{A.20}$$

where $I_{n,r}$ is the identity matrix in $\mathbb{R}^{n^r}$.

A binomial formula can be found for the Kronecker power, which generalizes the classical Newton one.

Lemma A.5. *Let $a, b \in I\!R^n$. For any integer $h \geq 0$ the matrix coefficients of the following binomial power formula:*

$$(a+b)^{[h]} = \sum_{k=0}^{h} M_k^h(n)(a^{[k]} \otimes b^{[h-k]}) \tag{A.21}$$

constitute a set of matrices $\{M_0^h(n), ..., M_h^h(n);\ M_k^h(n) \in I\!R^{n^h \times n^h}\}$ such that:

$$M_h^h(n) = M_0^h(n) = I_{n,h}, \tag{A.22}$$
$$M_j^h(n) = (M_j^{h-1}(n) \otimes I_{n,1}) + (M_{j-1}^{h-1}(n) \otimes I_{n,1})(I_{n,j-1} \otimes G_{h-j}(n)), \quad 1 \leq j \leq h-1, \tag{A.23}$$

where $G_l(n)$ and $I_{n,l}$ are as in Lemma A.4.

Lemma A.5 can also be generalized to the polynomial case. Obviously, given any polynomial $a_1 + ... + a_p$, $a_i \in I\!R^n$, $1 \leq i \leq p$, $p \in I\!N$, its h-th Kronecker power admits a representation as:

$$(a_1 + a_2 + ... + a_p)^{[h]} = \sum_{\substack{h_1,...,h_p \geq 0 \\ h_1+...+h_p = h}} M^h_{h_1,...,h_p}\big(a_1^{[h_1]} \otimes a_2^{[h_2]} \otimes \cdots \otimes a_p^{[h_p]}\big) \tag{A.24}$$

where $M^h_{h_1,...,h_p}$ are suitable matrices. The definition of symbols $M^l_{l_1,...,l_s}$ is extended, with $l > 0$ when at least one of the l_i's is negative, as

$$M^l_{l_1,...,l_s} = O_{n^l \times n^l}. \tag{A.25}$$

Moreover the following statement can be proved:

Lemma A.6. *The matrices $M^h_{h_1,...,h_p} \in I\!R^{n^h \times n^h}$ in (A.24) satisfy the recursive formula:*

$$M^h_{h_1,...,h_p} = I_1, \qquad h = 1 \tag{A.26}$$
$$M^h_{h_1,...,h_p} = \sum_{1 \leq i \leq p-1} \big(M^{h-1}_{h_1,...,h_i-1,...,h_p} \otimes I_1\big) \cdot \big(I_{h_1+...+h_i-1} \otimes G_{h_{i+1}+...+h_p}\big)$$
$$+ M^{h-1}_{h_1,...,h_p-1} \otimes I_1, \quad h > 1. \tag{A.27}$$

Proof. The proof can be found in [5]. □

References

[1] R.S. Bucy, "Global theory of the Riccati equation," *J. Comput. Syst. Sci.,* Vol. 1, pp. 349–361, 1967.

[2] A.H. Jazwinski, *Stochastic Processes and Filtering Theory.* New York: Academic, 1970.

[3] J.P. Gauthier, and G. Bornard, "Observability for any $u(t)$ of a class of nonlinear systems," *IEEE Trans. A. C.*, Vol. 26, no. 4, pp. 922–926, 1981.

[4] M. Dalla Mora, A. Germani, C. Manes, "Design of State Observers from a Drift-Observability Property," *IEEE Trans. A. C.*, Vol. 45, No. 6, pp. 1536–1540, August 2000.

[5] F. Carravetta, A. Germani, M. Raimondi, "Polynomial filtering of discrete-time stochastic linear systems with multiplicative state noise," *IEEE Trans. A. C.*, Vol. 42, NO. 8, pp. 1106–1126, 1997.

[6] F. Carravetta, A. Germani, M. Raimondi, "Polynomial filtering for linear discrete-time non-Gaussian systems," *SIAM J. Contr. Optim.*, Vol. 34, No. 5, pp. 1666–1690, 1996.

[7] C. Bruni, A. Gandolfi, A. Germani, "Observability of linear-in-the-state systems: a functional approach," *IEEE Trans. A. C.*, Vol. AC–25, No. 3, pp. 566–567, June 1980.

[8] L. Ljung, "Linear time-invariant models of non-linear time-varying Systems," *European Journal of Control,* Vol. 7, issue 2-3, pp. 203–219, 2001.

[9] A. Graham, *Kronecker Products and Matrix Calculus with Applications,* Ellis Horwood ltd. Publishers, Chichester, 1981.