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A COMPACT LINEAR PROGRAM FOR TESTING OPTIMALITY OF PERFECT MATCHINGS

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Abstract

It is a longstanding open problem whether there exists a polynomial size description of the perfect matching polytope. We give a partial answer to this question by proving the following result. The polyhedron defined by the constraints of the perfect matching polytope which are active at a given perfect matching can be obtained as the projection of a compact polyhedron. Thus there exists a compact linear program which is unbounded if and only if the perfect matching is not optimal with respect to a given edge weight. This result provides a simple reduction of the maximum weight perfect matching problem to compact linear programming.

Keywords: Matching, linear programming compact linear programming.

Key words: stable set polytope, graph composition, claw-free graphs.

1. Introduction

A *perfect matching* of a graph $G = (V, E)$ is a set M of edges, such that each node of G is incident with exactly one edge in M. A central problem in algorithms is to find a perfect matching M of G with maximal weight $c(M) = \sum_{e \in M} c_e$ with respect to given edge weights $c \in \mathbb{Q}^E$. This problem was solved by Edmonds [4, 3] with his blossom algorithm. The *characteristic vector* $\chi^M \in \{0,1\}^E$ of a perfect matching M is a 0/1 vector which indicates whether an edge $e \in E$ is a member of the matching or not via the value of its components, i.e., $\chi^M_e = 1$ if $e \in M$ and $\chi^M_e = 0$ otherwise. The *convex hull* of the characteristic vectors of perfect matchings of G is called the *perfect matching polytope* $P(G)$ of G. Edmonds showed that $P(G)$ is described by the following set of linear inequalities.

$$\begin{array}{rcll} \sum_{e \in \delta(u)} x_e & = & 1 & \text{for all } u \in V, \\ x_e & \geq & 0 & \text{for all } e \in E, \\ \sum_{e \in \delta(U)} x_e & \geq & 1 & \text{for all } U \subset V,\ |U| \text{ odd}. \end{array} \tag{1}$$

Edmonds's proof of this fact was algorithmic. Simple non algorithmic proofs of this fact were later given by Lovász [8] and Schrijver [10]. Padberg and Rao [9] showed that the *odd cut inequalities* $\sum_{e \in \delta(U)} x_e \geq 1$ of (1) can be *separated in polynomial time* by describing an algorithm which solves the minimum weight odd cut problem. Via the equivalence of separation and optimization [7] this implies that the *ellipsoid method* for linear programming can be used to optimize a linear function over $P(G)$ in polynomial time. Since the algorithm for computing minimum weight odd cuts in graphs is simpler than Edmonds's blossom algorithm, their result yielded a simpler proof that the maximum weight perfect matching problem is polynomially solvable than the original argument of Edmonds.

It is easy to see that each *odd cut inequality* in (1) induces a facet of $P(G)$ if G is a complete graph with an even number of nodes. Thus an irredundant linear inequality formulation of $P(G)$ must be exponential in the size of G in general. It is a very important question, whether the perfect matching polytope can be obtained as a *projection* of a polyhedron in a higher space, which is described with a polynomial amount of variables and inequalities. This would imply that there exists a *compact linear program*, i.e., a linear program with polynomially many constraints and variables for the maximum weight perfect matching problem. Examples of polytopes stemming from optimization problems which require an exponential number of inequalities and allow such a polynomial representation in a higher space are the *subtour elimination polytope* for the traveling salesman problem [13] or the *stable set polytope* for t-perfect graphs [15]. Yannakakis [15] provided a partially negative result concerning this question by showing that the perfect matching polytope cannot be obtained as the projection of a polynomial size *symmetric* polytope. Barahona [1] presented a polynomial description of the perfect matching polytope for planar graphs. Moreover, Barahona [2] showed that the Chinese postman problem, that is the maximum T-join problem, can be solved by a polynomial number of augmenting steps. Each of these steps can be achieved by solving a minimum mean cycle problem, for which he proposed a compact linear formulation. This formulation is based on a result of Seymour [12] describing the cone of edge disjoint cycles of a graph. A constraint $a^T x \leq \beta$ is *tight* or *active* at a point x^*, if x^* satisfies the constraint with equality, i.e., if $a^T x^* = \beta$. It follows from linear programing duality that a feasible solution x^* of a linear program $\max\{c^T x \mid Ax \leq b\}$ is an optimal solution, if and only if the linear program $\max\{c^T x \mid A^* x \leq b^*\}$ is bounded, where $A^* x \leq b^*$ is the subsystem of $Ax \leq b$ consisting of the active constraints at x^*, see [11, p. 95].

We prove that, given the characteristic vector χ^M of a perfect matching M, the polyhedron which is defined by the active constraints of (1) is the projection of a polynomial sized polyhedron P into the equivalent to the *facial optimality test* of a given feasible 0/1 point. Applied to the maximum weight perfect matching problem, facial optimality is the problem: Given a perfect matching M, a weight vector $c \in \mathbb{R}^E$ and a subset $J \subseteq E$ of the edges of G, decide whether M has maximal weight among all matchings $\widetilde{M}$ with $\widetilde{M} \cap J = M \cap J$. It is easy to see that our result implies that there exists a compact linear program which is bounded, if and only if a given matching M is facial optimal with respect to a weight c and a subset $J \subseteq E$. As a byproduct of our polyhedral study we get a new proof that the maximum weight perfect matching problem can be solved via a polynomial number of queries to compact linear programming problems.

2. A compact formulation of the active constraints

A *polyhedron* P is a set of points $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, for some matrix $A \in \mathbb{R}^{m\times n}$ and some vector $b \in \mathbb{R}^m$. If P is bounded, then P is called a *polytope*. If P is defined as $P = \{\binom{x}{y} \in \mathbb{R}^{n_1+n_2} \mid A\binom{x}{y} \leq b\}$, then the *projection* of P into the x-space is the polyhedron $\Pi_x(P) = \{x \in \mathbb{R}^{n_1} \mid \exists y \in \mathbb{R}^{n_2} \text{ with } \binom{x}{y} \in P\}$.

A *perfect matching* of a graph $G = (V, E)$ is a subset of the edges $M \subseteq E$ such that $m_1 \cap m_2 = \emptyset$ holds for all $m_1 \neq m_2 \in M$. For $U \subseteq V$, the *cut* $\delta(U)$ is the set of edges $e \in E$ with $|e \cap U| = 1$. The cut $\delta(U)$ is an *odd cut*, if $|U|$ or $|V \setminus U|$ is odd. Let s and t be nodes of G. An (s,t)-cut of G is a cut $\delta(U)$ with $|U \cap \{s,t\}| = 1$.

Throughout, let χ^M denote the characteristic vector of the perfect matching M of the graph $G = (V, E)$. An odd cut inequality $\sum_{e\in\delta(U)} x_e \geq 1$, for a subset $U \subseteq V$ with $|U|$ odd is active at χ^M if and only if there exists exactly one edge $m \in M \cap \delta(U)$. Notice that the number of such cuts is exponential. Eisenbrand et al. [6] applied this observation to derive a separation algorithm for the active odd cuts of a given matching which solely rests on simple min-cut computations. Let $m = \{s,t\}$ be an edge of the matching M and consider an odd cut $\delta(U)$ which crosses the matching M in exactly this edge $m = \{s,t\}$. Since this cut $\delta(U)$ does not intersect any other matched edge, $\delta(U)$ is an (s,t)-cut of the graph G_m, which is obtained from G via contracting the edges into single nodes (see Figure 1). On the other hand, each (s,t)-cut of G_m induces an odd cut $\delta(U)$ of G which crosses m and does not cross any other edge in the matching M. More formally, let G_m be the graph with node set V^m consisting of the singleton sets $\{s\}$ and $\{t\}$ and the edges $M - m$ and with edge set E^m consisting of $\{\{s\},\{t\}\}$ and all edges

$$\begin{array}{cl} \{\{s\},\{u,v\}\} & \text{where } \{s,u\} \in E \text{ and } \{u,v\} \in M - m, \\ \{\{t\},\{u,v\}\} & \text{where } \{t,u\} \in E \text{ and } \{u,v\} \in M - m, \\ \{\{a,b\},\{u,v\}\} & \text{where } \{a,u\} \in E \text{ and } \{a,b\},\ \{u,v\} \in M - m. \end{array}$$

We associate with $x^* \in \mathbb{R}^E$ the vector $x^{*m} \in \mathbb{R}^{E^m}$ whose components $x^{*m}_{\{A,B\}}$ for $\{A,B\} \in E^m$ are defined as

$$x^{*m}_{\{A,B\}} = \sum_{\substack{a\in A,\, b\in B,\\ \{a,b\}\in E}} x^*_{\{a,b\}}.$$

Then we can infer the following crucial observation.

Observation 2.1. *Let M be a perfect matching of the graph $G = (V,E)$ and let $m = \{s,t\} \in M$. The point $x^* \in \mathbb{R}^E$ satisfies all inequalities $\sum_{e\in\delta(U)} x_e \geq 1$ with $\delta(U) \cap M = \{s,t\}$ if and only if there exists an (s,t)-flow in G_m of value at least 1, where the capacities on the edges of G_m are given by the corresponding edge weights x^{*m}.*

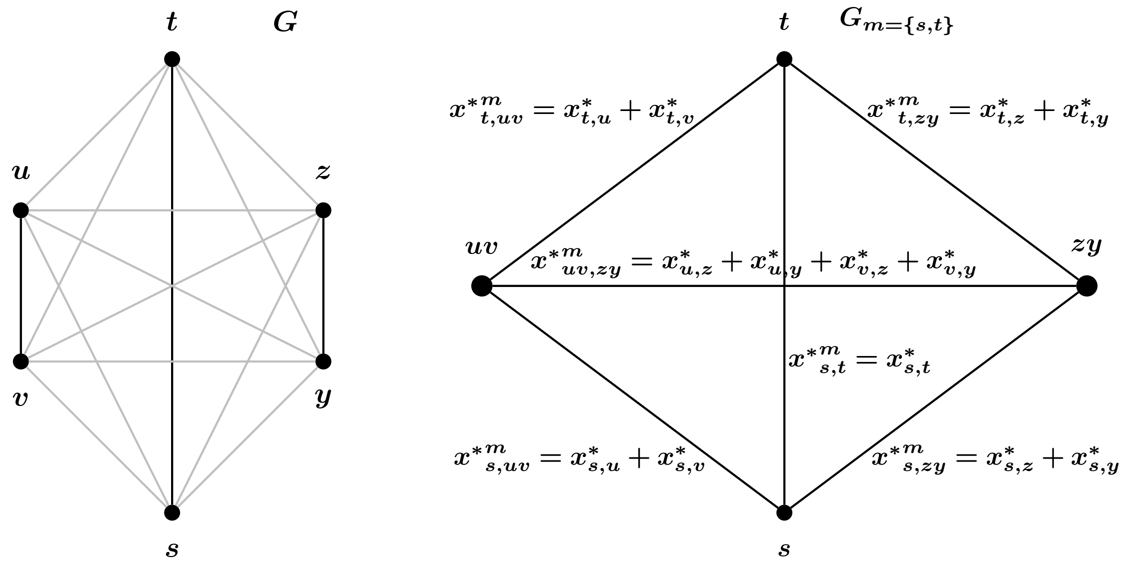

Figure 1: The black edges of graph G represent a perfect matching M. The graph $G_{m=\{s,t\}}$ is obtained from G by contracting all the edges of $M - m$.

of value 1 from s to t in the graph $G_m = (V_m, E_m)$.

$$v \in V_m: \quad \sum_{\substack{e \in \delta^+(v) \\ e \in E_m}} f^m_e - \sum_{\substack{e \in \delta^-(v) \\ e \in E_m}} f^m_e = \begin{cases} -1 & \text{if } v = s, \\ 1 & \text{if } v = t, \\ 0 & \text{otherwise}. \end{cases} \tag{2}$$

The following constraints link the edge space of the original graph with the flow, by defining capacities on the edges.

$$e = \{A, B\} \in E_m: \quad -\sum_{\substack{\{a,b\} \in E, \\ a \in A,\, b \in B}} x_{\{a,b\}} \quad \leq \quad f^m_e \quad \leq \quad \sum_{\substack{\{a,b\} \in E, \\ a \in A,\, b \in B}} x_{\{a,b\}}. \tag{3}$$

The polyhedron which is defined by the constraints (2) and (3) and variables is denoted by $\mathcal{P}_m$. Consider the system defined by the inequalities (2) and (3) for each $m \in M$. This system has the original variables $x \in \mathbb{R}^E$ and variables $f^m \in \mathbb{R}^{E_m}$ for each $m \in M$ and is thus defined by a polynomial number of variables and inequalities. Denote the polyhedron defined by this system by $\mathcal{P}_M$.

Proposition 2.1. *The projection of $\mathcal{P}_M$ onto the x-variables is the polyhedron C_{odd} which is defined by the inequalities $\sum_{e\in\delta(U)} x_e \geq 1$ of (1) which are active at χ^M.*

Proof. If x^* is not a member of C_{odd}, then there exists an an odd cut $\delta(U)$ which crosses M in exactly one edge $m0\{s,t\} \in M$ and satisfies $\sum_{e\in\delta(U)} x^*_e < 1$. This implies that the minimum weight (s,t)-cut in the graph G_m which is obtained from G with edge weights x^* by contracting all matching edges except m, has value strictly less than 1. Since the value of an (s,t)-cut is an upper bound on the feasible (s,t)-flows, there cannot exist an (s,t)-flow of value 1 in G_m. In other words, there cannot exist a vector f^{m^*} such that (x^*, f^{m^*}) satisfies the constraints (2) and (3) for m.

If x^* is a member of C^M_{odd}, then each odd cut $\delta(U)$ that crosses the matching M in exactly one edge m has value at least 1. This means that each (s,t)-cut in G_m has value at least 1. By the max-flow min-cut theorem we can conclude that there exists an (s,t)-flow in G_m of value 1. This flow gives rise to the vector f^* such that (x^*, f^{m^*}) satisfies the constraints (2) for each $m \in M$. ∎

3. Computing maximum weight perfect matchings

In this section we will apply the previous result to derive an algorithm for the maximum weight perfect matching problem which is based on compact linear programming. Such a result was already obtained by Barahona [2] who reduced the maximum weight matching problem to a sequence of minimum mean cycle problems. Barahona showed that the minimum mean cycle problem has a compact linear programming formulation. In contrast to this, our approach works directly on the constraints of the perfect matching polytope which are active at a given perfect matching.

The idea is to use the compact formulation of the active constraints of χ^M to test whether a matching M is optimal with respect to a given objective function under the fixation of certain variables to 0 or 1. A feasible point $\bar{x} \in \{0,1\}^n$ of a 0/1 polytope $P \subseteq [0,1]^n$ is *facial optimal* with respect to a linear function $c^T x$ and a subset $J \subseteq \{1,\ldots,n\}$, if the point $\bar{x}$ is an optimal solution to the linear programming problem $\max\{c^T x \mid x \in P,\, x_j = \bar{x}_j,\, j \in J\}$. Eisenbrand et al. [6] have shown that a polynomial algorithm for testing this facial optimality property yields a polynomial algorithm for the 0/1 optimization problem. Facial optimality for perfect matching is the following.

Problem 1. *Given a graph $G = (V,E)$, a matching M of G an edge weight $w \in R^E$ and an edge set $J \subseteq E$. Decide whether the matching M has maximum weight among all matchings $\widetilde{M}$ of G with $\widetilde{M} \cap J = M \cap J$.*

As we reminded in Section 1, given a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, a vector $x^* \in P$ is an optimal solution to the linear programming problem $\max\{c^T x \mid x \in P\}$ if and only if x^* is an optimal solution to the linear programming problem $\max\{c^T x \mid A^* x \leq b^*\}$, where $A^* x \leq b^*$ is the subset of the constraints of $Ax \leq b$ which are active at x^*.

We now describe a compact linear program which is bounded if and only if the answer to Problem 1 is "No". The linear program is defined as $\max c^T x$ under the constraints (2), (3) and

$$\begin{aligned} x_e &= \chi^M_e && \text{for each } e \in J,\\ \textstyle\sum_{e\in\delta(v)} x_e &= 1 && \text{for each } v \in V,\\ x_e &\geq 0 && \text{for each } e \notin M. \end{aligned} \tag{4}$$

A polynomial time algorithm for the maximum weight perfect matching problem which is based on linear programming now follows with the equivalence of facial optimality testing and 0/1 optimization.

Corollary 2.1. *There exists a polynomial time algorithm for the maximum weighted perfect matching*

Concluding remarks

T-joins

It is easy to see that the construction of Section 2 carries over to the *T-join polyhedron*. We briefly sketch how this can be done. Let $T \subseteq V$ be a subset of the nodes of G with $|T|$ even. A *T-join* J is a multiset of the edges of G such that $|\delta_J(u)|$ is odd if $u \in T$ and even otherwise.

The T-join polyhedron is the set of points that can be expressed as a convex combination of the characteristic vectors of a finite set of T-joins. Here, the characteristic vector of a T-join χ^J records the number of times that a particular edge e is contained in J with the component χ^J_e. Edmonds and Johnson [5] showed that the T-join polyhedron is described by the system

$$\begin{array}{rcll} \sum_{e\in\delta(U)} x_e & \geq & 1 & \text{for all } U \subseteq V,\ |U \cap T| \text{ odd}, \\ x_e & \geq & 0 & e \in E. \end{array} \tag{5}$$

To represent the active constraints of χ^J, one considers only those cuts $\delta(U)$ of (5) which cross exactly one edge $m = \{s,t\}$ of J with $\chi^J_e = 1$ and no other edge of J. For each such edge $m = \{s,t\}$, one then models a flow of value 1 in the graph $\widetilde{G}_m$ obtained from G by contracting each set of nodes defining a connected component in $G - m$ (see [6]). Since it is possible to formulate a general matching problem as a simple perfect b-matching problem via a polynomial reduction proposed by Tutte [14], one has a pseudopolynomial algorithm for the maximum weight general matching problem which is based on compact linear programming. At this point we would like to stress that this compact linear formulation is achieved without Seymour's description of the cycle cone [12].

A compact formulation of $P(G)$?

It is not clear whether this idea to represent the active constraints of a perfect matching in a higher space can be extended to a compact representation of the complete perfect matching polytope. The number $N(M)$ of odd cut inequalities for a graph G with n nodes which are active at χ^M is

$$\begin{array}{rcl} N(M) & = & (n/2) \displaystyle\sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n/2}{k} \\ & = & O(2^{n/2}). \end{array}$$

On the other hand, the number of facets $F(n)$ of the matching polytope of a complete graph with an even number n of vertices is equal to the number of odd subsets of $\{1,\dots,n\}$ of size at most $n/2$. This can be bounded from below by the number of subsets of $\{1,\dots,n\}$ of size at most $n/2-1$, which is

$$\begin{array}{rcl} F(n) & = & \Omega\left(\displaystyle\sum_{k=0}^{n/2-1} \binom{n}{k}\right) \\ & = & \Omega(2^n). \end{array}$$

This means that one would need an exponential number of families of active constraints to cover all the facets of the perfect matching polytope of a complete graph with an even number of vertices.

Open problems

This compact formulation for the active cone of a given perfect matching could be given, since the parity could be to find out, whether this primal view can be helpful to find compact linear formulations of active cones of polyhedra for other classes of combinatorial problems. Interesting candidates might be the stable-set polyhedron of a claw-free graph or the odd-cut polyhedron, which is the blocker of the T-join polyhedron.

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