ISTITUTO DI ANALISI DEI SISTEMI ED INFORMATICA
CONSIGLIO NAZIONALE DELLE RICERCHE

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OPTIMAL LINEAR FILTERING FOR BILINEAR STOCHASTIC DIFFERENTIAL SYSTEMS WITH UNKNOWN INPUTS

R. 541 Dicembre 2000

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ISSN: 1128–3378

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Abstract

This work investigates the problem of state estimation for bilinear stochastic multivariable differential systems, in presence of an additional disturbance, whose statistics are completely unknown. Most part of the paper is devoted to the derivation of the best linear filter, i.e. the one that gives the minimum error variance among all linear filtering algorithms. No *a priori* information on the disturbance is required for the filter implementation. The proposed filter is robust with respect to the unknown input, in that the covariance of the estimation error is not affected by such input. The results presented in this paper are innovative not only for bilinear systems but also for continuous-time linear stochastic systems forced by unknown inputs, in that for these systems a suitable extension of the Kalman-Bucy filter has not been presented in literature. This is the reason why this particular case is also treated in detail. Numerical simulations show the effectiveness of the proposed filter.

Key words: Bilinear systems, singular systems, unknown-input systems, robust filtering, Kalman-Bucy filtering

1. Introduction

In many fields of applications, the mathematical model describing the dynamic relationships among the state variables, the inputs and the measurements is given by the following nonlinear stochastic differential system, described by the Ito equations:

$$\begin{aligned} dx(t) &= A(t)x(t)dt + B(t)du(t) + \mathcal{B}_1\big(x(t), dW(t)\big), \\ dy(t) &= C(t)x(t)dt + D(t)du(t) + \mathcal{B}_2\big(x(t), dW(t)\big), \end{aligned} \tag{1.1}$$

where $x(t) \in I\!R^n$ is the state, $u(t) \in I\!R^p$ is the input, $y(t) \in I\!R^q$ is the measured output, $W(t) \in I\!R^b$ is a Wiener process with respect to some increasing family of σ-algebras, namely $\{\mathcal{F}_t\}$, referred to a probability space $(\Omega, \mathcal{F}, P)$; $A(t), B(t), C(t), D(t)$ are matrices of suitable dimensions and $\mathcal{B}_1, \mathcal{B}_2$ are bilinear forms (see [1,2,3,23,29,30] for more details on discrete and continuous-time bilinear systems and filtering problems related to them).

The unknown-input $u(t)$ in system (1.1) may model the presence of an additive noise with no *a priori* statistical informations (deterministic disturbance). The unknown-input can be used also to describe uncertainties in the system equations, for instance derived from linearization errors, or it can be used to model *failure systems*. Among applications, unknown-inputs systems are of great interest in the geophysical and environmental framework, as shown by Kitanidis in [18].

This paper investigates the problem of estimating the state of a time-varying bilinear stochastic differential system, affected by additive disturbances that involve both the state and measurement equations. No a priori knowledge is assumed on the disturbances.

A great deal of literature is available in the field of filtering a discrete-time stochastic linear systems with unknown inputs: a first recursive algorithm, consisting of an optimization technique can be found in [18], where Kitanidis developed an unbiased Kalman filter by minimizing the trace of the error covariance matrix. This technique has been recently parameterized by Darouach and Zasadzinski in [9] and by Keller, Darouach and Caramelle in [17] to extend previous results. Many contributes treat the loss of information by modeling the unknown-input system as a descriptor system and then applying a previously developed filtering algorithm for this class of systems (see [6,7,8,10,16,24, 25]). Others contributions (see [5,11,12,19]) take inspiration from an algorithm, also used for the construction of unknown-input observers, which is able to remove the influence of the disturbance by a clever use of the measurement process. In a recent paper, [14], Hsieh proposes a robust two-stage Kalman filter [13], optimal with respect to the minimum variance, which is shown to be equivalent to the one of Kitanidis [18].

Unfortunately, neither the descriptor system nor the decoupling approach can be directly applied to the continuous-time case, in that they would require the knowledge of the noisy output derivatives. An hypothesis that in the stochastic framework can not be assumed.

This work investigates the problem of defining a robust linear filter for stochastic bilinear differential systems forced by completely unknown inputs. Moreover, the solution here presented is the minimum variance one among the chosen robust structure. Finally, the case of linear systems with unknown inputs, an important subclass of bilinear systems, is also treated in detail.

The sections are structured as follows: in section two is defined the class of the system to be filtered; in section three a filtering algorithm is proposed; section four deals with the time-invariant case; section five treats the Kalman-Bucy filtering for linear systems and finally some simulation results show the effectiveness of the proposed algorithm.

2. Bilinear systems with unknown inputs

Let $(\Omega, \mathcal{F}, P)$ a probability space and $\{\mathcal{F}_t,\ t \geq t_0\}$ a family of nondecreasing sub-σ-algebras of $\mathcal{F}$. As well known, a bilinear stochastic differential system in the Ito formulation is described by the equations:

$$\begin{aligned} dx(t) &= A(t)x(t)dt + B(t)du(t) + \sum_{k=1}^{b} \big(N_k(t)x(t) + F_k(t)\big)dW_k(t), \\ x(t_0) &= x_0, \\ dy(t) &= C(t)x(t)dt + D(t)du(t) + \sum_{k=1}^{b} \big(M_k(t)x(t) + G_k(t)\big)dW_k(t), \end{aligned} \tag{2.1}$$

with $x(t) \in \mathbb{R}^n$ the state of the system, $u(t) \in \mathbb{R}^p$ an additive unknown input, $y(t) \in \mathbb{R}^q$ the measured output, x_0 a random variable with given mean $m_0 = m_x(t_0) = \mathbb{E}[x_0]$ and covariance matrix $\Psi_0 = \Psi_x(t_0) = \mathrm{Cov}(x_0)$, $W_k(t)$ the k-th component of a standard Wiener process $(W(t), \mathcal{F}_t)$, $W(t) \in \mathbb{R}^b$ and the matrices $A(t), N_k(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times p}$, $C(t), M_k(t) \in \mathbb{R}^{q\times n}$, $D(t) \in \mathbb{R}^{q\times p}$, $F_k(t) \in \mathbb{R}^{n\times 1}$, $G_k(t) \in \mathbb{R}^{q\times 1}$. Moreover, it will be assumed that $D(t)$ is full column rank, $\forall t \geq t_0$.

For the sequel let us introduce the matrices $T_1(t)$, $T_2(t)$, so defined:

$$T_1(t) = \big(D^T(t)D(t)\big)^{-1}D^T(t) \in \mathbb{R}^{p\times q}, \tag{2.2}$$

while $T_2(t)$ is chosen in such a way that $\mathcal{R}\big(T_2^T(t)\big) = \mathcal{N}\big(D^T(t)\big)$. In other words $T_2(t) \in \mathbb{R}^{(q-p)\times q}$ is a matrix with $(q-p)$ independent rows that are a basis for the left null-space of $D(t)$. Matrix $T_2(t)$ allows to define the post-processed output:

$$dz(t) = T_2(t)dy(t) \tag{2.3}$$

With respect to the output dz, the bilinear stochastic system (2.1) can be represented in the robust form (independent of the unknown input) given by the following lemma

Lemma 2.1. *The bilinear stochastic differential system described by* (2.1) *can be rewritten as:*

$$\begin{aligned} dx(t) &= \mathcal{A}(t)x(t)dt + \mathcal{B}(t)dy(t) + \sum_{k=1}^{b} \big(\mathcal{N}_k(t)x(t) + \mathcal{F}_k(t)\big)dW_k(t) \\ x(t_0) &= x_0 \\ dz(t) &= \mathcal{C}(t)x(t)dt + \sum_{k=1}^{b} \big(\mathcal{M}_k(t)x(t) + \mathcal{G}_k(t)\big)dW_k(t) \end{aligned} \tag{2.4}$$

with:

$$\begin{aligned} \mathcal{A}(t) &= A(t) - B(t)T_1(t)C(t) \in \mathbb{R}^{n\times n}, & \mathcal{C}(t) &= T_2(t)C(t) \in \mathbb{R}^{(q-p)\times n}, \\ \mathcal{B}(t) &= B(t)T_1(t) \in \mathbb{R}^{n\times q}, & \mathcal{M}_k(t) &= T_2(t)M_k(t) \in \mathbb{R}^{(q-p)\times n}, \\ \mathcal{N}_k(t) &= N_k(t) - B(t)T_1(t)M_k(t) \in \mathbb{R}^{n\times n}, & \mathcal{G}_k(t) &= T_2(t)G_k(t) \in \mathbb{R}^{(q-p)\times 1}, \\ \mathcal{F}_k(t) &= F_k(t) - B(t)T_1(t)G_k(t) \in \mathbb{R}^{n\times 1}. \end{aligned} \tag{2.5}$$

Proof. Note first that, by definition, the matrices $T_1(t)$ and $T_2(t)$ are such that $T_1(t)D(t) = I_p$ and $T_2(t)D(t) = 0_{(q-p)\times p}$. From these, it follows

$$T_1(t)dy(t) = T_1(t)C(t)x(t)dt + du(t) + T_1(t)\sum_{k=1}^{b}\big(N_k(t)x(t) + F_k(t)\big)dW_k(t), \tag{2.6}$$

$$dz(t) = T_2(t)dy(t) = T_2(t)C(t)x(t)dt + T_2(t)\sum_{k=1}^{b}\big(M_k(t)x(t) + G_k(t)\big)dW_k(t), \tag{2.7}$$

This last equation gives back the new measure equation, while explicating $du(t)$ from (2.6) and substituting it into the state equation of (2.1), the thesis immediately follows. ∎

Remark 2.2. The structure (2.4) for the bilinear system with unknown inputs is obtained suitably exploiting the information brought by the output on the unknown inputs. The "new" measurement $z(t)$ is what remains after all information available on the unknown input has been exploited, and it is the piece of output that will be used for filtering. Note that the new measure equation vanishes if $q = p$. It follows that the filtering approach presented in this work requires at least $p + 1$ measurements.

3. The optimal linear filtering algorithm

In this section the new measurement vector $z(t)$ defined in (2.3) is used to develop a state estimator for system (2.4). In order to properly take into account the presence of the original output $y(t)$ as a forcing term in the state equation (2.4), a suitable decomposition of the system is required, as given by the following proposition:

Proposition 3.1. *The system (2.4) can be written in the form*

$$\begin{aligned} x(t) &= x_d(t) + x_s(t), \\ dz(t) &= dz_s(t) + \mathcal{C}(t)x_d(t)dt \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} dx_d(t) &= \mathcal{A}(t)x_d(t)dt + \mathcal{B}(t)dy(t), \\ x_d(t_0) &= \mathbb{E}\big[x_0\big], \end{aligned} \tag{3.2}$$

$$\begin{aligned} dx_s(t) &= \mathcal{A}(t)x_s(t)dt + \sum_{k=1}^{b}\Big(\mathcal{N}_k(t)\big(x_d(t) + x_s(t)\big) + \mathcal{F}_k(t)\Big)dW_k(t), \\ x_s(t_0) &= x_0 - \mathbb{E}\big[x_0\big], \\ dz_s(t) &= \mathcal{C}(t)x_s(t)dt + \sum_{k=1}^{b}\Big(\mathcal{M}_k(t)\big(x_d(t) + x_s(t)\big) + \mathcal{G}_k(t)\Big)dW_k(t). \end{aligned} \tag{3.3}$$

The proof is readily obtained by direct computation.

Remark 3.2. Proposition 3.1 shows the decomposition of the system state $x(t)$ in two terms: $x_d(t)$ is the totally observed component and $x_s(t)$ is the partially observed, zero-mean component. Denoting $\mathcal{F}_t^Y$ the σ-algebra generated by the measurement process up to time t, it comes out that $x_d(t)$ is $\mathcal{F}_t^Y$-adapted.

It must be stressed that the evolution of system (3.2) is completely determined by the measurements $y(t)$ and does not depend on $x_s(t)$. On the contrary, system (3.3), forced also by the evolution of $x_d(t)$, admits the representation:

$$\begin{aligned} dx_s(t) &= \mathcal{A}(t)x_s(t)dt + \sum_{k=1}^{b} \big(\mathcal{N}_k(t)x_s(t) + \widetilde{\mathcal{F}}_k(t)\big)dW_k(t), \\ x_s(t_0) &= x_0 - I\!E\big[x_0\big], \\ dz_s(t) &= \mathcal{C}(t)x_s(t)dt + \sum_{k=1}^{b} \big(\mathcal{M}_k(t)x_s(t) + \widetilde{\mathcal{G}}_k(t)\big)dW_k(t), \end{aligned} \tag{3.4}$$

where:

$$\widetilde{\mathcal{F}}_k(t) = \mathcal{F}_k(t) + \mathcal{N}_k(t)x_d(t) \in I\!R^{n\times 1}, \qquad \widetilde{\mathcal{G}}_k(t) = \mathcal{G}_k(t) + \mathcal{M}_k(t)x_d(t) \in I\!R^{(q-p)\times 1}.$$

Remark 3.3. The best estimate of $x_d(t)$ is $\hat{x}_d(t) = I\!E[x_d(t)|\mathcal{F}_t^Y]$, that gives the minimum error variance among all the Borel functions of the measurements. Being $x_d(t)$ $\mathcal{F}_t^Y$-measurable, it follows that $\hat{x}_d(t) = x_d(t)$.

Definition 3.4. The robust minimum variance estimate of a bilinear stochastic differential system with unknown inputs like (2.1) is intended to be the following:

$$\hat{x}(t) = x_d(t) + I\!E\big[x_s(t)|\mathcal{F}_t^{Z_s}\big], \tag{3.5}$$

where $\mathcal{F}_t^{Z_s}$ is the σ-algebra generated by the new measurement process z_s.

Remark 3.5. The error covariance matrices of the estimate $\hat{x}(t)$ and of $\hat{x}_s(t)$ are the same:

$$\mathrm{Cov}(x(t) - \hat{x}(t)) = \mathrm{Cov}(x_s(t) - \hat{x}_s(t))$$

As is well known, the optimal filter for the system (3.4) is an infinite dimensional one, because $I\!E\big[x_s(t)|\mathcal{F}_t^{Z_s}\big]$ coincides with the projection of $x_s(t)$ onto the Hilbert space of all the Borel functions of the new measurement process z_s. Nevertheless, from an applicative point of view, it is useful to look for finite-dimensional approximations of the optimal filter, that is estimates deriving from stochastic differential equations of the form:

$$\begin{aligned} d\xi(t) &= f\big(\xi(t)\big)dt + g\big(\xi(t)\big)dz_s(t), \\ \tilde{x}_s(t) &= h\big(\xi(t)\big), \end{aligned} \tag{3.6}$$

where $\big\{\xi(t),\ t \geq t_0\big\}$ is a process taking its values on a finite-dimensional space. The finite-dimensional system (3.6) is an optimal linear filter for the random process $x_s(t)$, if:

$$\tilde{x}_s(t) = \Pi\big[x_s(t)|L_t(z_s)\big],$$

where $\Pi[\cdot|L_t(z_s)]$ denotes the projection onto the space $L_t(z_s)$ linearly spanned by the family of random variables $\big\{z_s(\tau),\ t_0 \leq \tau \leq t\big\}$. In order to achieve the goal of an optimal linear filter for the bilinear system (3.4), it will be considered an important result from reference [4] according to which, the following notation will be adopted: let $M \in I\!R^{\alpha\times\alpha}$ be a symmetric positive semidefinite matrix, such that $\mathrm{rank}(M) = \rho \leq \alpha$. Then there exists a full rank matrix $N \in I\!R^{\alpha\times\rho}$ such that $NN^T = M$. Matrix N will be indicated by $M^{\left(\frac{1}{2}\right)}$.

Theorem 3.6. *Suppose the covariance matrix $\Psi_{x_s}(t) = \mathrm{Cov}\big(x_s(t)\big)$ is nonsingular for all $t \geq t_0$; moreover, let $\mathcal{M}_k(t)$ and $\widetilde{\mathcal{G}}_k(t)$ full row rank matrices for some k and any $t \geq t_0$. Then the optimal linear estimate of $x_s(t)$, is given by:*

$$
\begin{aligned}
d\tilde{x}_s(t) &= \mathcal{A}(t)\tilde{x}_s(t)dt + \left(\sum_{k=1}^{b} \widetilde{\mathcal{F}}_k(t)\widetilde{\mathcal{G}}_k^T(t) + P(t)\mathcal{C}^T(t)\right) R(t)^{-1}\Big(dz_s(t) - \mathcal{C}(t)\tilde{x}_s(t)dt\Big), \\
\tilde{x}_s(t_0) &= 0,
\end{aligned} \tag{3.7}
$$

where $R(t) = \displaystyle\sum_{k=1}^{2p} \widetilde{\mathcal{M}}_k(t)\widetilde{\mathcal{M}}_k^T(t)$, *with*

$$
\widetilde{\mathcal{M}}_k(t) = \begin{cases} \Big(\mathcal{M}_k(t)\Psi_{x_s}(t)\mathcal{M}_k^T(t)\Big)^{\left(\frac{1}{2}\right)}, & 1 \leq k \leq p, \\ \widetilde{\mathcal{G}}_{k-p}(t), & p+1 \leq k \leq 2p, \end{cases} \tag{3.8}
$$

$\Psi_{x_s}(t)$, the covariance matrix of $x_s(t)$, is given by the following equations:

$$
\begin{aligned}
\dot{\Psi}_{x_s}(t) &= \mathcal{A}(t)\Psi_{x_s}(t) + \Psi_{x_s}(t)\mathcal{A}^T(t) + \sum_{k=1}^{b} \mathcal{N}_k(t)\Psi_{x_s}(t)\mathcal{N}_k^T(t) + \sum_{k=1}^{b} \widetilde{\mathcal{F}}_k(t)\widetilde{\mathcal{F}}_k^T(t), \\
\Psi_{x_s}(t_0) &= \Psi_0
\end{aligned} \tag{3.9}
$$

and, at last, $P(t) = \mathbb{E}\big[(x_s(t) - \tilde{x}_s(t))(x_s(t) - \tilde{x}_s(t))^T\big]$ is the error covariance matrix:

$$
\begin{aligned}
\dot{P}(t) &= \mathcal{A}(t)P(t) + P(t)\mathcal{A}^T(t) + R(t) \\
&\quad - \left(\sum_{k=1}^{b} \widetilde{\mathcal{F}}_k(t)\widetilde{\mathcal{G}}_k^T(t) + P(t)\mathcal{C}^T(t)\right) R(t)^{-1} \left(\sum_{k=1}^{b} \widetilde{\mathcal{F}}_k(t)\widetilde{\mathcal{G}}_k^T(t) + P(t)\mathcal{C}^T(t)\right)^T, \\
P(t_0) &= \Psi_0.
\end{aligned} \tag{3.10}
$$

Proof. The proof can be found in Theorem 4.4, reference [3].

∎

Theorem 3.7. *Under hypotheses of Theorem 3.6, according to Definition 3.4, the minimum variance estimate of a bilinear stochastic differential system (2.1) is given by the following algorithm:*

$$
\begin{aligned}
d\tilde{x}(t) &= \mathcal{A}(t)\tilde{x}(t)dt + \mathcal{B}(t)dy(t) \\
&\quad + \left(\sum_{k=1}^{b} \widetilde{\mathcal{F}}_k(t)\widetilde{G}_k^T(t) + P(t)C^T(t)\right) \mathcal{R}(t)\Big(dy(t) - C(t)\tilde{x}(t)dt\Big), \\
\tilde{x}(t_0) &= \mathbb{E}\big[x_0\big],
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
\dot{P}(t) &= \mathcal{A}(t)P(t) + P(t)\mathcal{A}^T(t) + R(t) \\
&\quad - \left(\sum_{k=1}^{b} \widetilde{\mathcal{F}}_k(t)\widetilde{G}_k^T(t) + P(t)C^T(t)\right) \mathcal{R}(t) \left(\sum_{k=1}^{b} \widetilde{\mathcal{F}}_k(t)\widetilde{G}_k^T(t) + P(t)C^T(t)\right)^T, \\
P(t_0) &= \Psi_0.
\end{aligned} \tag{3.12}
$$

where:

$$\widetilde{G}_k(t) = G_k(t) + M_k(t)x_d(t), \tag{3.13}$$

$$\mathcal{R}(t) = T_2^T(t)R^{-1}(t)T_2(t) \tag{3.14}$$

and $R(t)$ *is built using* (3.8) *and* (3.9) *of Theorem* 3.6.

Proof. The proof is directly obtained taking into account the decomposition of Proposition 3.1. ∎

4. Optimal linear filtering: the time-invariant case

From an applicative point of view a very important case occurs when the dynamic matrices in (2.1) are time-invariant. However, owing to the decomposition proposed in Proposition 3.1, the system to be filtered, that is system (3.4), is not completely stationary, in that:

$$\begin{aligned} dx_s(t) &= \mathcal{A}x_s(t)dt + \sum_{k=1}^{b}\big(\mathcal{N}_k x_s(t) + \widetilde{\mathcal{F}}_k(t)\big)dW_k(t),\\ x_s(t_0) &= x_0 - I\!\!E\big[x_0\big],\\ dz_s(t) &= \mathcal{C}x_s(t)dt + \sum_{k=1}^{b}\big(\mathcal{M}_k x_s(t) + \widetilde{\mathcal{G}}_k(t)\big)dW_k(t), \end{aligned} \tag{4.1}$$

with:

$$\widetilde{\mathcal{F}}_k(t) = \mathcal{F}_k + \mathcal{N}_k x_d(t) \in I\!\!R^{n\times 1}, \qquad \widetilde{\mathcal{G}}_k(t) = \mathcal{G}_k + \mathcal{M}_k x_d(t) \in I\!\!R^{(q-p)\times 1}.$$

Lemma 4.1. *A sufficient condition to ensure the non singularity of the covariance matrix* $\Psi_{x_s}(t)$ *for any* $t \ge t_0$ *is that at least one of the two following hypotheses has to be satisfied:*
i) Ψ_0 *is a nonsingular matrix;*
ii) the pair $\big(\mathcal{A}, \mathcal{F}_k\big)$ *is controllable for some* $k = 1, \ldots, b$.

Proof. As it has been previously seen, $x_s(t)$ is a zero-mean random process, so that its covariance matrix is $\Psi_{x_s}(t) = \mathrm{Cov}\big(x_s(t)\big) = I\!\!E\big[x_s(t)x_s^T(t)\big]$. Explicating the solution of system (4.1):

$$x_s(t) = e^{\mathcal{A}(t-t_0)}x_s(t_0) + \sum_{k=1}^{b}\left(\int_{t_0}^{t} e^{\mathcal{A}(t-\tau)}\mathcal{N}_k x_s(\tau)dW_k(\tau) + \int_{t_0}^{t} e^{\mathcal{A}(t-\tau)}\widetilde{\mathcal{F}}_k(\tau)dW_k(\tau)\right) \tag{4.2}$$

from which it is easy to verify that:

$$\begin{aligned} \Psi_{x_s}(t) &= e^{\mathcal{A}(t-t_0)}\Psi_0 e^{\mathcal{A}(t-t_0)^T} + \sum_{k=1}^{b}\Bigg(\int_{t_0}^{t} e^{\mathcal{A}(t-\tau)}\mathcal{N}_k\Psi_{x_s}(\tau)\mathcal{N}_k^T e^{\mathcal{A}(t-\tau)^T}d\tau\\ &\qquad + \int_{t_0}^{t} e^{\mathcal{A}(t-\tau)}\widetilde{\mathcal{F}}_k(\tau)\widetilde{\mathcal{F}}_k(\tau)^T e^{\mathcal{A}(t-\tau)^T}d\tau\Bigg)\\ &= e^{\mathcal{A}(t-t_0)}\Psi_0 e^{\mathcal{A}(t-t_0)^T} + \sum_{k=1}^{b}\Bigg(\int_{t_0}^{t} e^{\mathcal{A}(t-\tau)}\mathcal{N}_k\Psi_{x_s}(\tau)\mathcal{N}_k^T e^{\mathcal{A}(t-\tau)^T}d\tau\\ &\qquad + \int_{t_0}^{t} e^{\mathcal{A}(t-\tau)}\big(\mathcal{F}_k + \mathcal{N}_k x_d(\tau)\big)\big(\mathcal{F}_k + \mathcal{N}_k x_d(\tau)\big)^T e^{\mathcal{A}(t-\tau)^T}d\tau\Bigg). \end{aligned}$$

Then $\Psi_{x_s}(t)$ is the sum of $2p+1$ positive semidefinite matrices. If Ψ_0 is nonsingular, the first term is positive definite, and so is the whole $\Psi_{x_s}(t)$, whereas the hypothesis of controllability of the pair $(\mathcal{A}, \mathcal{F}_k)$ for some k implies the positive definitiveness of at least one term of the last sum. ∎

5. Kalman-Bucy filtering for linear systems with unknown inputs

This section is devoted to solve the filtering problem for a linear stochastic differential system forced by unknown inputs. The reason for considering this particular case lies on the fact that it was unsolved up to now in the general framework of stochastic differential systems described by Ito equations. The proposed methodology presented as an application of the previously mentioned algorithm for bilinear systems is not comparable to other techniques used in discrete-time systems, as each of them would produce a quite-difficult-to-implement filter in that the use of measurements and noises derivatives is required.

Consider the system described by equations (1.1), with $\mathcal{B}_1$ and $\mathcal{B}_2$ linear operators with respect to $dW(t)$ in $I\!R^{n\times b}$ and $I\!R^{q\times b}$ respectively, that is:

$$\begin{aligned} dx(t) &= A(t)x(t)dt + B(t)du(t) + F(t)dW(t), \\ x(t_0) &= x_0 \\ dy(t) &= C(t)x(t)dt + D(t)du(t) + G(t)dW(t). \end{aligned} \tag{5.1}$$

According to lemma 2.1 defining a suitable post-processed output as in (2.3), equations (5.1) can be rewritten as

$$\begin{aligned} dx(t) &= \mathcal{A}(t)x(t)dt + \mathcal{B}(t)dy(t) + \mathcal{F}(t)dW(t), \\ x(t_0) &= x_0, \\ dz(t) &= \mathcal{C}(t)x(t)dt + \mathcal{G}(t)dW(t), \end{aligned} \tag{5.2}$$

with the new state matrices defined as in (2.5).

Following the same steps of the bilinear case, system (5.2) is split into the following two systems:

$$\begin{aligned} dx_d(t) &= \mathcal{A}(t)x_d(t)dt + \mathcal{B}(t)dy(t), \\ x_d(t_0) &= I\!E\big[x_0\big], \end{aligned} \tag{5.3}$$

$$\begin{aligned} dx_s(t) &= \mathcal{A}(t)x_s(t)dt + \mathcal{F}(t)dW(t), \\ x_s(t_0) &= x_0 - I\!E\big[x_0\big], \\ dz(t) - \mathcal{C}(t)x_d(t)dt &= dz_s(t) = \mathcal{C}(t)x_s(t)dt + \mathcal{G}(t)dW(t). \end{aligned} \tag{5.4}$$

Definition 5.1. The minimum variance estimate of a linear stochastic differential system with unknown inputs like (5.1) is intended to be the following:

$$\tilde{x}(t) = x_d(t) + I\!E\big[x_s(t)|\mathcal{F}_t^{Z_s}\big] \tag{5.5}$$

Remark 5.2. The best estimate of $x_d(t)$ with respect to all the Borel functions of the first p components of the measurements is $x_d(t)$ itself, so equation (5.5).

Remark 5.3. The error covariance matrix referred to $\tilde{x}(t)$ is the same of that referred to $\tilde{x}_s(t)$:

$$\mathrm{Cov}(x(t) - \tilde{x}(t)) = \mathrm{Cov}(x_s(t) - \tilde{x}_s(t))$$

The stochastic component $x_s(t)$ can be filtered by using the Kalman-Bucy algorithm (see [20] for more details) so that the following theorem is derived:

Theorem 5.4. *According to Definition* 5.1*, the minimum variance estimate for a linear stochastic differential system, described by the Ito equations* (5.1) *is given by the following equations:*

$$
\begin{aligned}
d\tilde{x}(t) &= \mathcal{A}(t)\tilde{x}(t)dt + \mathcal{B}(t)dy(t) \\
&\qquad + \Big(\mathcal{F}(t)G^T(t) + P(t)C^T(t)\Big)\Big(\mathcal{G}(t)\mathcal{G}^T(t)\Big)^{-1}\Big(dy(t) - C(t)\tilde{x}(t)dt\Big), \\
x(t_0) &= x_0,
\end{aligned}
\tag{5.6}
$$

with $P(t) = I\!\!E\big[(x(t) - \tilde{x}(t))(x(t) - \tilde{x}(t))^T\big]$*, the error covariance matrix, given by:*

$$
\begin{aligned}
dP(t) &= \mathcal{A}(t)P(t)dt + P(t)\mathcal{A}^T(t)dt + \mathcal{F}(t)\mathcal{F}^T(t)dt \\
&\qquad - \Big(\mathcal{F}(t)G^T(t) + P(t)C^T(t)\Big)\Big(\mathcal{G}(t)\mathcal{G}^T(t)\Big)^{-1}\Big(G(t)\mathcal{F}^T(t) + C(t)P(t)\Big)dt, \\
P(t_0) &= \Psi_0.
\end{aligned}
\tag{5.7}
$$

Proof. It immediately comes by applying the Kalman-Bucy filter to the stochastic system (5.4).

∎

6. Numerical simulations

Simulations have been produced in order to taste the goodness of theoretical results. The state filtered is given by the evolution of the bilinear differential systems described by the Ito equations of (2.1), with the following numerical choices:

- $x(t) \in I\!R^3$, $y(t) \in I\!R^2$;
- the unknown-input $u(t)$ is a scalar piece-wise constant signal, shown in figure 6.1;
- the noise is a standard Wiener process;
- the system matrices are the following:

$$A = \begin{bmatrix} -4 & 0.1 & 0 \\ 0 & -3 & -1 \\ 0.4 & 0 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.5 \\ -1.5 \end{bmatrix}, \quad N = \begin{bmatrix} -1 & 0 & 0.1 \\ 0 & -2 & 0 \\ 0 & 0.4 & 0.5 \end{bmatrix}, \quad F = \begin{bmatrix} -1 \\ 1.2 \\ -2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0.5 & -2 \\ 1 & 1 & -0.2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & -0.5 & -1 \\ 0 & -2 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

Here follow some pictures showing the comparison between the real and the estimated state:

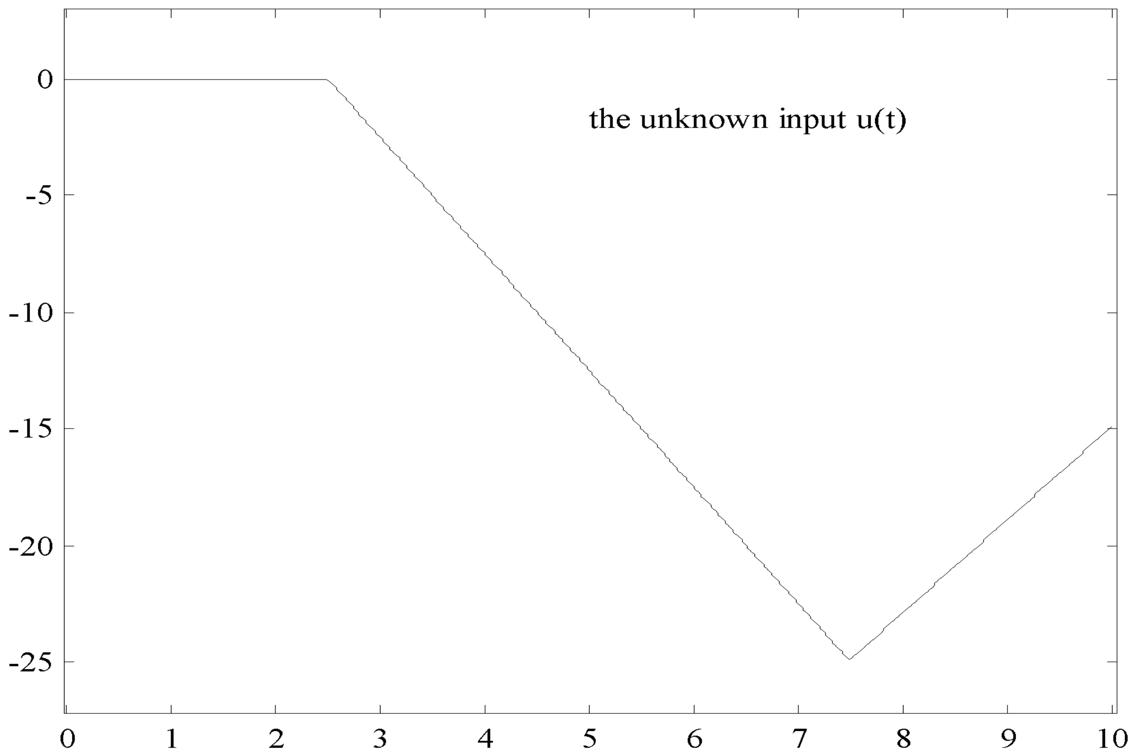

Fig. 6.1 – The unknown input

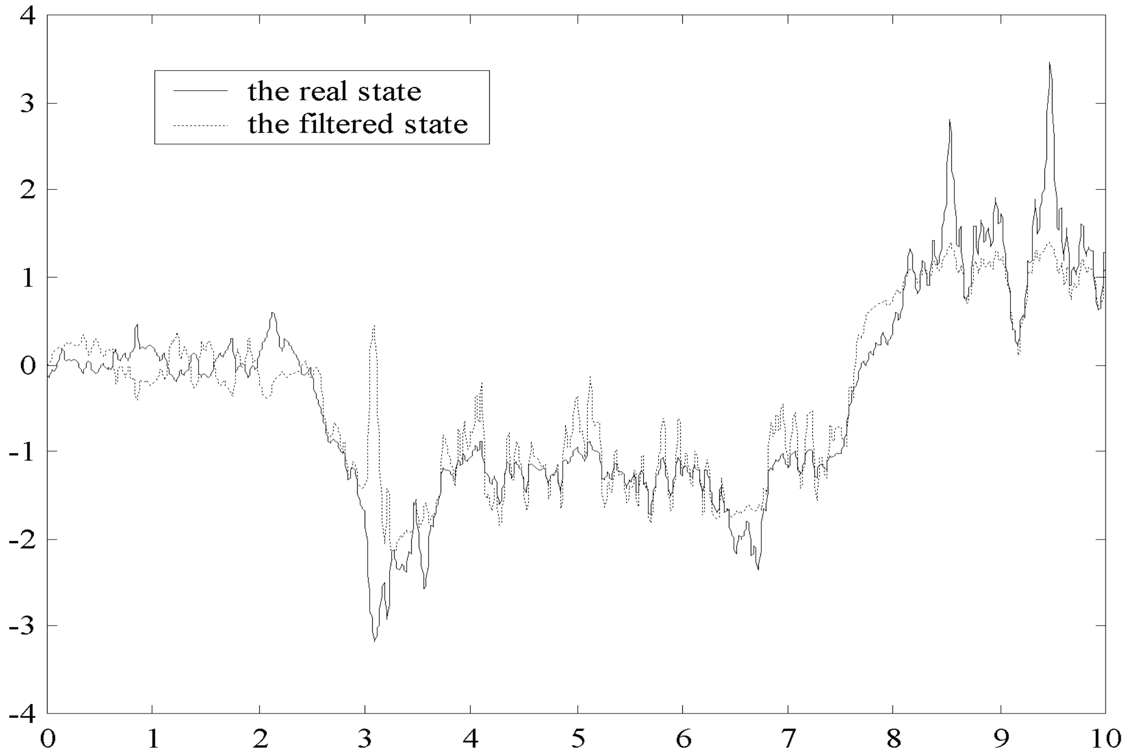

Fig. 6.2 – The first component estimated compared with the real first component

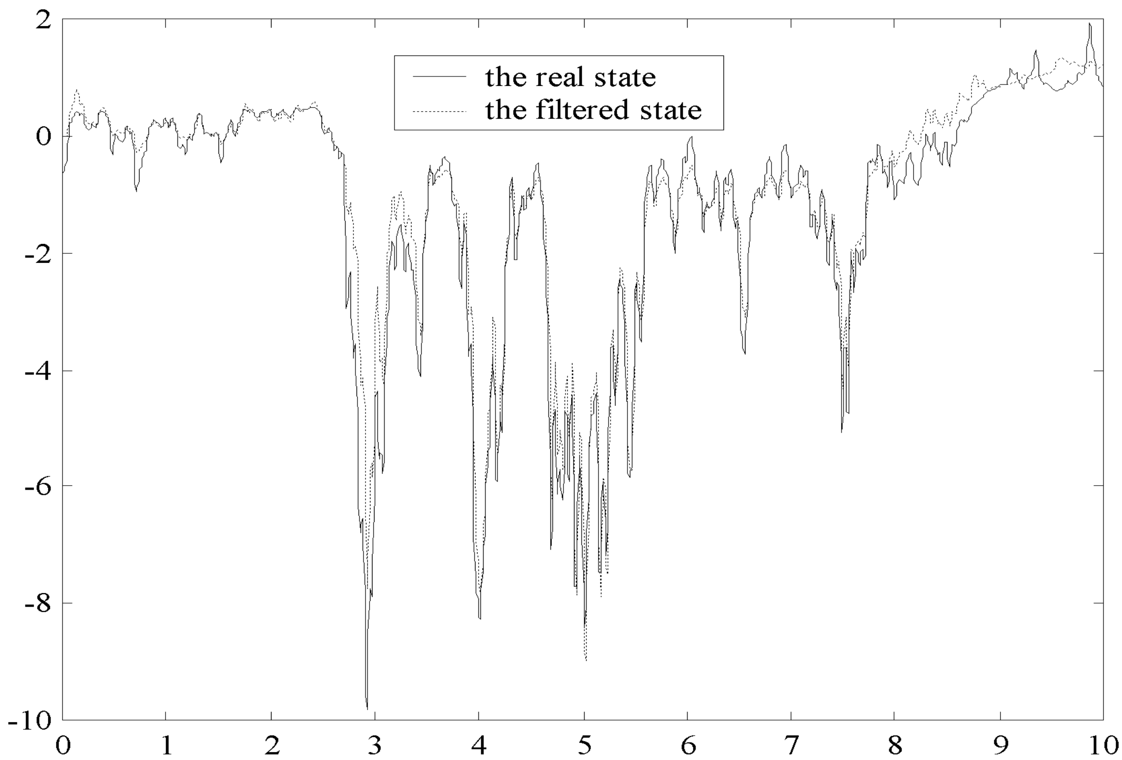

Fig. 6.3 – The second component estimated compared with the real second component

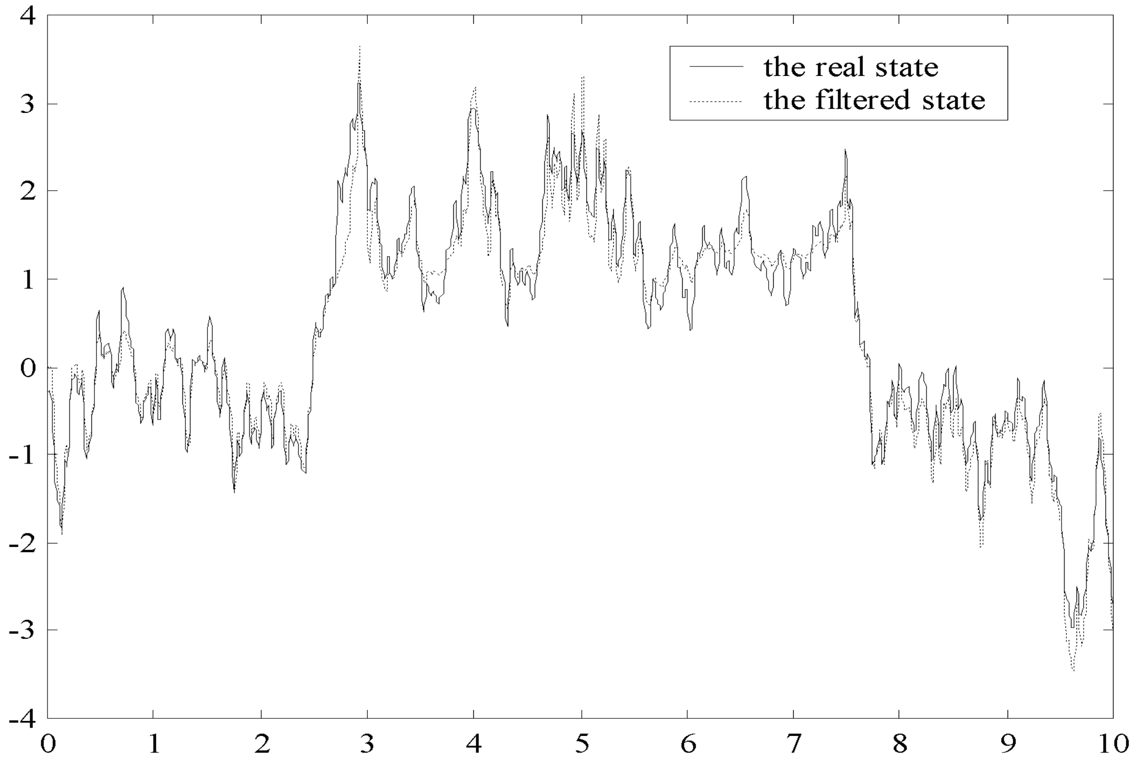

Fig. 6.4 – The third component estimated compared with the real third component

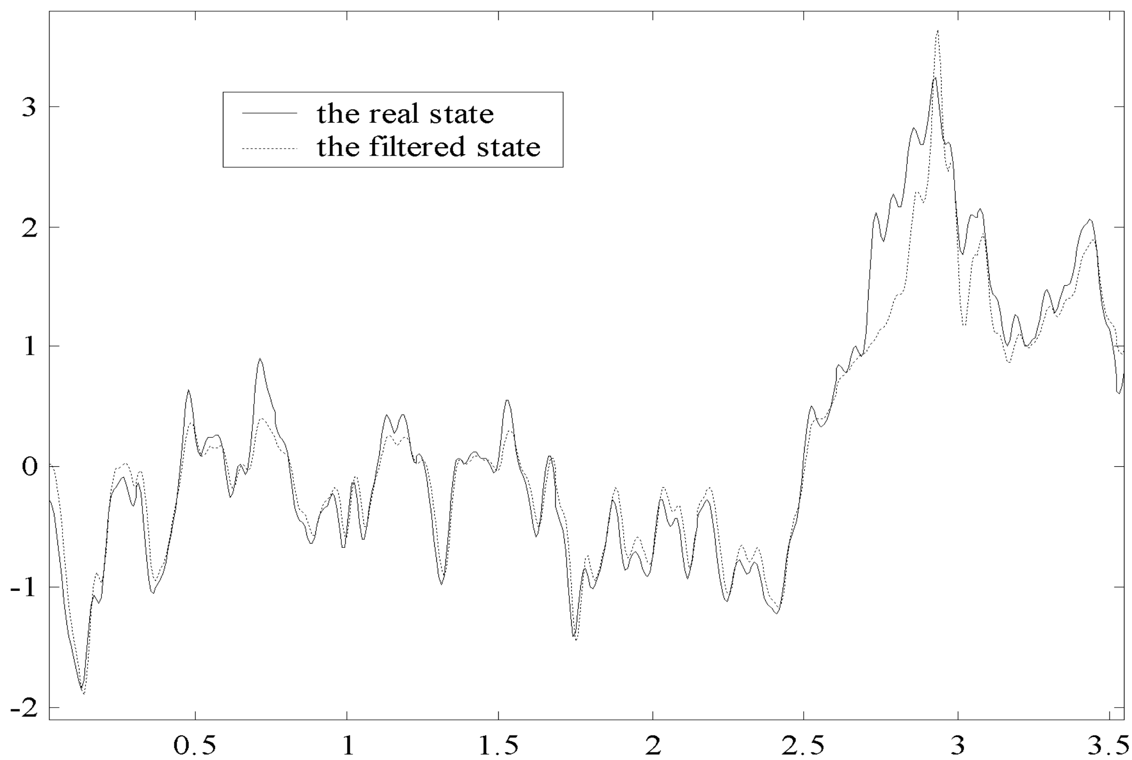

Fig. 6.5 – A particular from fig. 6.4

7. Conclusions

A linear filter has been proposed for a bilinear stochastic differential system, forced by disturbances, whose statistics are completely unknown. The algorithm is robust with respect to the unknown input, in that the covariance of the estimation error is not affected by such input. Moreover, the estimate is obtained by using projections onto the space of all linear transformations of the post-processed output, in order to achieve a minimum variance filter. Numerical simulations support theoretical results.

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