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# ON CHROMATIC NUMBERS OF GRAPHS

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## Abstract

A *k-coloring* of a graph $G$ is a mapping $f$ from the vertex set $V(G)$ of $G$ to the set $\mathbb{Z}_k$ such that $f(x) \neq f(y)$ if $x$ and $y$ are adjacent in $G$. The minimum value $k$ such that there exists a $k$-coloring of $G$ is called the *chromatic number* of $G$. Different choices of the mapping $f$ and of the image set of $f$, give rise to different concepts of coloring of graphs. For example, if the image set of $f$ is the set $\mathcal{U}_r$ of unit length arcs of a circumference of length $r$, an $r$-circular coloring is a mapping from $V(G)$ to $\mathcal{U}_r$ such that $f(x) \cap f(y) = \emptyset$ for $xy \in E(G)$. The value $\min\{r \in \mathbb{R} : G \text{ has r-circular coloring}\}$ is called the *circular chromatic number* of $G$.

In this paper, we survey results on different concepts of coloring, concentrating on the relations among different "chromatic numbers". We present several open problems in this setting and we prove results towards the settlement of some well-known conjectures in the theory of coloring of graphs.

KEYWORDS: *chromatic number, circular chromatic number, star chromatic number, k-tuple chromatic number, fractional chromatic number.*

## 1. Introduction

Graph coloring and network flows are two major topics in graph theory. The concept of flow in graph theory is essentially the same as the concept of a current in an electrical network. Despite their little (if any) connection with the physical reality of flows, many combinatorial problems can be formulated and hence, efficiently solved as network flow problems. It is thus not surprising that the study of flows became a classical topic in graph theory and led to important results in combinatorial optimization, polyhedral combinatorics and matroid theory.

Apparently, the dual concept of tension did not receive the same attention in literature, and appears mainly in scheduling and shortest-path problems. However, as Tutte observed in [21], the whole theory of graph coloring can be formulated in terms of tension. Let us explain how the two concepts are related in the case of planar graphs where the duality between flow and tension corresponds to the geometric duality of graphs represented in tha plane.

A *$k$-coloring* of a graph $G$ is a mapping $f$ from the vertex set $V(G)$ of $G$ to the set $\mathbb{Z}_k$ such that $f(x) \neq f(y)$ if $x$ and $y$ are adjacent in $G$. The minimum value $k$ for which $G$ has a $k$-coloring is the *chromatic number* $\chi(G)$ of $G$.

A *$k$-flow* of a directed graph $G$ is a mapping $f : E(G) \to \mathbb{Z}_k$ such that

$$\sum_{e\in\delta^+(X)} f(e) = \sum_{e\in\delta^-(X)} f(e) \qquad \forall X \subseteq V, \tag{1}$$

where $\delta^+(X)$ denotes the set of arcs from $X$ to $V - X$ and $\delta^-(X) = \delta^+(V - X)$.

Consider a directed graph $G$ which is 2-cell-embedded in an orientable surface $\mathcal{S}$. Assume that the embedding is *face-colorable* with $k$ colors - that is, no two adjacent faces have the same color. For each $e$, let $r_e$ be the color of the face on the right of $e$, and let $l_e$ be the color of the face on the left of $e$. It is not difficult to see that the mapping $r_e - l_e$ is a $k$-flow of $G$. The face-coloring property is equivalent to the fact that this flow is nowhere-zero. Actually, something more holds:

**Theorem 1.1.** *(Tutte) A plane graph is $k$-face-colorable if and only if it has a nowhere-zero $k$-flow.*

Next, consider the geometric dual $G^*$ of a plane graph $G$. A $k$-flow in $G$ corresponds to an assignments of *tensions* $x_i$ to the vertices of $G^*$ such that $x_i - x_j = f_e$ for any $ij \in E(G^*)$ with $i = right(e)$ and $j = left(e)$. Since $f_e \neq 0$ and $f_e \in \mathbb{Z}_k$, these tensions represent a proper *$k$-coloring*, $k \geq 2$, of the graph $G^*$ - that is, an assignment of $k$ colors to the vertices of a graph such that no two adjacent vertices get the same color.

In the following, we survey different concepts of coloring and the relations among them.

## 2. The circular chromatic number

In an ordinary coloring, colors $1, 2, \ldots, k$ are assigned to the vertices of a graph $G$ so that the colors on adjacent vertices are at least one unit apart. A possible generalization of such a coloring might be to assign colors $1, 2, \ldots, k$ to the vertices so that the colors on adjacent vertices are as far apart as possible. More formally,

**Definition 2.1.** *A $(k, d)$-coloring of a graph $G$ is a function $c : V(G) \to \mathbb{Z}_k$ such that*

$$c(x) - c(y) \in \{\pm d, \ldots, \pm(k - d)\} \qquad \forall xy \in E(G).$$

The *star chromatic number* $\chi^*(G)$ can be thought as the best possible such coloring:

$$\chi^*(G) = inf\{\frac{k}{d} : G \quad has \quad (k,d) - coloring\}$$

Notice that, when $d = 1$, we find the usual definition of proper $k$-coloring. This generalization of the chromatic number was introduced by Vince [22] who also proved that the infimum is attained, thus implying that it is actually a minimum. Moreover, he showed that the star chromatic number cannot be too far from the ordinary chromatic number:

**Theorem 2.2.** *For every graph $G$, $\chi(G) - 1 < \chi^*(G) \leq \chi(G)$.*

A combinatorial proof of the above statement is due to Bondy and Hell [3], who actually, provided an alternative definition of circular chromatic number in terms of *homomorphisms*. A *homomorphism* of a graph $G$ to a graph $H$ is a mapping $f : V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. For any rational number $k/d \geq 2$, we denote by $C_k^d$ the circulant graph whose vertex set is $\{1, 2, \ldots, k\}$ and edge set is $\{ij : d \leq |i - j| \leq k - d\}$.

If $G$ admits a homomorphism to $H$ we write $G \to H$. It has been proved in [3] that a graph $G$ is $(k,d)$-colorable if and only if $G$ is homomorphic to $C_k^d$.

A $(k,d)$-coloring of a plane graph $G$ induces a nowhere-zero $k$-flow in $G^*$ by setting $f(xy) = c(x) - c(y)$ for every $xy \in E(G)$, whose values range in absolute value from $d$ to $k - d$. Hence, we may define:

**Definition 2.3.** *A $(k,d)$-flow of a graph $G$ is an orientation of $G$ together with a $k$-flow $f : E(G) \to \mathbb{Z}_k$ such that*

$$f_e \in \{\pm d, \ldots, \pm(k-d)\} \qquad \forall e \in E(G).$$

A $(k,1)$-flow is simply a nowhere-zero $k$-flow and the *star flow number* is defined as

$$\phi^*(G) = inf\{\frac{k}{d} : G \quad has \quad (k,d) - flow\}.$$

Again it can be proved that $\phi^*$ is a minimum and it is a refinement of the well studied *flow number* $\phi(G) := \lceil \phi_c(G) \rceil$, introduced by Tutte as a dual of the chromatic number.

In the following we give other equivalent formulations of $\chi^*(G)$ and $\phi^*(G)$.

Let $\mathcal{U}_r$ be the set of unit length arcs of a circumference $\mathcal{C}_r$ of length $r$. An $r$-circular coloring of a graph $G$ is a mapping $\gamma : V(G) \to \mathcal{U}_r$ such that

$$\gamma(x) \cap \gamma(y) = \emptyset \qquad \forall xy \in E(G).$$

If we interpret a circumference $\mathcal{C}_r$ of length $r$ as the interval $[0,r) \subseteq \mathbb{R}$, the mapping $\gamma$ sends each vertex of $V(G)$ to a unit length open sub-interval of the interval $[0,r)$ such that adjacent vertices are sent to disjoint sub-intervals and each sub-interval can be identified by the initial point of $\gamma(x)$. So, the above coloring is equivalent to the following definition:

**Definition 2.4.** *An $r$-interval coloring of a graph $G$ is a mapping $g : V(G) \to [0,r)$ such that*

$$|g(x) - g(y)| \in [1, r-1] \qquad \forall xy \in E(G).$$

Thus, define the *circular chromatic number* $\chi_c(G)$ as

$$\chi_c(G) = \inf\{r \in \mathbb{R} : G \quad has \quad r - interval \quad coloring\}$$

Dually, we define the *circular flow number* $\phi_c(G)$ of a graph $G$ as

$$\phi_c(G) = \inf\{r \in \mathbb{R} : \text{ some orientation } \vec{G} \text{ admits a circulation } f : E(\vec{G}) \to [1, r-1]\}.$$

It can be proved that if $G$ has an $r$-interval coloring, then $G$ has an $r'$-interval coloring such that $r' = p/q$, for some integers $p$ and $q$, where $p$ is at most the length of the longest cycle of $G$ and $q$ is at most the independence number of $G$ ([24], [8]). Since there are finitely many such numbers, the infimum is attained and

$$\chi_c(G) = \min\{\frac{p}{q} : G \quad has \quad \frac{p}{q} - interval \quad coloring\}.$$

It is then not difficult to see that:

**Lemma 2.5.** *For every graph $G$, $\chi_c(G) = \chi^*(G)$.*

*Proof.* Let $g$ be a $\frac{p}{q}$-interval coloring of $G$. Then the mapping $c(x) = \lfloor g(x)q \rfloor$ is a $(p,q)$-coloring of $G$. On the other hand, if $c$ is a $(k,d)$-coloring of $G$, then the mapping $g(x) = c(x)/d$ from $V(G)$ to the interval $[0, \frac{k}{d})$ corresponds to a $\frac{k}{d}$-interval coloring, and the lemma follows. ∎

The following question, raised by Vince [22], is still open:

**Question 2.6.** *For which graph is $\chi_c(G) = \chi(G)$ ?*

An interesting characterization of the chromatic number of a graph was given by Minty [16] in 1962 and leads to an equivalent formulation of the circular chromatic number.

**Theorem 2.7.** *A graph $G$ is $k$-colorable if and only if it can be oriented in such a way that for any circuit $C$ of $G$ and any given orientation on $C$, at least $\frac{|C|}{k}$ edges are oriented in this direction.*

In other words, Minty proved that a graph $G$ is $k$-colorable if and only if $G$ has an orientation such that

$$\max_C\{\frac{|C|}{|C^+|}, \frac{|C|}{|C^-|}\} \le k. \tag{2}$$

This leads to a definition of the circular chromatic number in terms of orientations (an explicit proof can be found in [6]):

$$\chi_c(G) = \min_{\vec{G}} \max_C\{\frac{|C|}{|C^+|}, \frac{|C|}{|C^-|}\}. \tag{3}$$

Replacing "cycle" by "edge-cuts" in the formulation of Minty's theorem, we get a characterization of those graphs having a $k$-flow and a new formula for $\phi_c$:

$$\phi_c(G) = \min_{\vec{G}} \max_{\emptyset \neq X \subset V} \frac{|\delta(X)|}{|\delta^+(X)|} \tag{4}$$

where $\vec{G}$ ranges over the strong orientations of $G$. (Here $\delta(X) = \delta^+(X) \cup \delta^+(V - X)$.)

A "Minty-like" formula for $\phi_c$ arises directly from Hoffman's circulation condition [12]. The extensions of the above statements to regular matroids have been proved in [6].

## 3. The fractional chromatic number

Let $\mathcal{C}_k$ be the family of $k$-subsets of a given set $\{1, 2, \ldots, l\}$. A *$k$-tuple coloring* of a graph $G$ is a mapping $C : V(G) \to \mathcal{C}_k$ such that $C(x) \cap C(y) = \emptyset$ for $xy \in E(G)$. So, the *$k$-tuple chromatic number* is defined as

$$\chi_k(G) = \min\{l \in \mathbb{Z} : G \quad \text{admits a } k\text{-tuple coloring}\}. \tag{5}$$

Since the vertices $x$ which get the same elements into $C(x)$ form an independent set, we may view a $k$-tuple coloring as a collection of independent sets of $G$ (repetitions allowed) which cover each vertex exactly $k$ times. So, if we denote by $\mathcal{I}_G$ the set of all maximal independent sets of a graph $G$, we may state equivalently that:

$$\chi_k(G) = \{\mathbf{1}y : \sum_{v \in I \in \mathcal{I}_G} y_I = k, y_I \in \mathbb{Z}_+^{|\mathcal{I}_G|}, \forall v \in V(G)\}.$$

Clearly $\chi(G) = \chi_1(G)$ is the ordinary chromatic number. Observe that:

$$\frac{\chi_k(G)}{k} = \{\mathbf{1}y : \sum_{v \in I \in \mathcal{I}_G} y_I = 1, ky_I \in \mathbb{Z}_+^{|\mathcal{I}_G|}, \forall v \in V(G)\}. \tag{6}$$

Now, if we denote by $P_I(G)$ the polytope $\{y \in \mathbb{Z}_+^{\mathcal{I}_G} : \sum_{v \in I \in \mathcal{I}} y_I = 1, \forall v \in V(G)\}$, then the chromatic number will simply be the optimal solution of the integer linear program $min\{\mathbf{1}y : y \in P_I(G)\}$. Hence, we may define the *fractional chromatic number* as the optimal solution of the linear relaxation of the above program, i.e.,

$$\chi_f(G) = min\{\mathbf{1}y : y \in P(G)\}, \tag{7}$$

where $P(G) = \{y \in \mathbb{Q}_+^{\mathcal{I}_G} : \sum_{v \in I \in \mathcal{I}} y_I = 1, \forall v \in V(G)\}$. It is easy to see that $\frac{\chi_k(G)}{k} \geq \chi_f(G)$, because of (6). But the connection between the fractional chromatic number and the k-tuple chromatic number is even stronger. In fact, if $y^*$ is an optimal solution of (7), then there exists an integer $k^*$ such that $k^* y^* \in \mathbb{Z}_+^{|\mathcal{I}_G|}$, and so, $\frac{\chi_{k^*}(G)}{k^*} = \mathbf{1}y^* = \chi_f(G)$. It follows that:

$$\chi_f(G) = \min_k \frac{\chi_k(G)}{k} \leq \frac{\chi_k(G)}{k} \leq \chi(G).$$

It is worth noticing that the parameter $\min_k \frac{\chi_k(G)}{k}$, also known as set-chromatic number [2], or ultimate chromatic number [9], or multicoloring number [11], has been widely studied. We mention here one of its alternative definitions which uses the *lexicographic product* $G \circ K_k$ of a graph $G$ and a clique $K_k$. The graph $G \circ K_k$ has vertex set $V(G) \times V(K_k)$ and edge set $E(G \circ K_k) = \{(u_1, u_2)(v_1, v_2) :$ either $u_1 v_1 \in E(G)$ or $u_1 = v_1$ and $u_2 v_2 \in E(K_k)\}$, i.e., $G \circ K_k$ is obtained by replacing each vertex of $G$ with a clique of size $k$. Hence, a coloring of $G \circ K_k$ is an assignment to each vertex of $G$ of an $k$-tuple of colors so that adjacent vertices of $G$ get disjoint $k$-tuples. Hence,

$$\chi_f(G) = \min_k \frac{\chi(G \circ K_k)}{k} = \min_k \frac{\chi_k(G)}{k}.$$

It is also not difficult to see that $\chi_f(G) \leq \chi_c(G)$. In fact, if $\chi_c(G) = k/d$ and $c$ is a $(k,d)$-coloring of $G$, then each of the sets $S_i = \bigcup_{j=i}^{i+d-1} c^{-1}(j)$, $i = 1, \dots, k$ is an independent set of $G$. The vector $y_{S_i} = \frac{1}{d}$, $\forall S_i$, is a fractional coloring of value $k/d$.

The gap between $\chi_f(G)$ and $\chi(G)$ may be arbitrarly large, and so does the one between $\chi_f(G)$ and $\chi_c(G)$. But something more can be said about the ratio between integral and fractional colorings; in fact, an upper bound for $\chi(G)$ in terms of $\chi_f(G)$ was proved by Lovász in 1975 [15]:

**Theorem 3.1.** $\chi(G) \leq (1 + \log \alpha(G))\chi_f(G)$.

(This result has been used recently to prove the hardness of approximating $\chi_f(G)$ within a factor $n^\epsilon$ [1]).

Many interesting questions remain open as the following:

**Question 3.2.** *For which graph is* $\lceil \chi_f(G) \rceil = \chi(G)$ *?*

**Question 3.3.** *For which graph is* $\chi_f(G) = \chi_c(G)$ *?*

Recent results on the first topic can be found in [13]. The latter one has been studied by Zhu in [26].

Furthermore, it is interesting to observe that

$$\chi_f(G) = \lim_{n \to \infty} \sqrt[n]{\chi(G^n)},$$

where the power of $G$ is the lexicographic product. It is perhaps worth noticing that the optimal solution of the dual program $\omega_f(G) = max\{x \in \mathbb{Q}_+^V : \sum_{v \in I} x_v = 1, \forall I \in \mathcal{I}_G, \forall v \in V\}$ does not always equal $lim_{n \to \infty} \sqrt[n]{\omega(G^n)}$, this limit being the Shannon capacity of the complement of $G$.

## 4. Some questions and partial answers

Let $\mathcal{C}_\ell$ denote the class of all graphs $G$ such that $G \to C_\ell$. It is easy to see that

$$\mathcal{C}_3 \supset \mathcal{C}_5 \supset \mathcal{C}_7 \supset \cdots$$

and that $\bigcap \mathcal{C}_\ell$ is the class of all bipartite graphs. Thus we may view a graph in $\mathcal{C}_\ell$ ($\ell \gg 1$, $\ell$ odd) as being "nearly bipartite", with $\ell$ being a measure of the "bipartiteness" of the graph. This notion can be nicely formulated in terms of circular chromatic numbers.

**Proposition 4.1.** *[22] For any graph $G$ and integer $k$, the following are equivalent:*

*i)* $\chi_c(G) \leq 2 + \frac{1}{k}$

*ii)* $G \to C_{2k+1}$.

Thus a graph is "nearly bipartite" if its circular chromatic number is close to two. It is well known that there exist triangle free graphs of arbitrarily high chromatic number, thus implying that $\chi_c(G) \in [2, \infty)$. More generally, given any two integers $g \geq 2$ and $k \geq 2$, there exists a graph with girth $g$ and chromatic number $k$. On the other hand, the girth proved to be a good constraint for the growth of the chromatic number in the case when the graph $G$ has some "embeddability" properties. The first result of this flavour is probably due to Grötzsch [7], who proved that every planar graph with no triangle can be 3-coloured. Some years later, Kronk and White [14] showed that every toroidal graph of girth at least six is 3-colourable, and Thomassen [20] improved this bound by showing that every toroidal graph of girth at least five can be 3-coloured.

In [5], it has been proved that no minor-closed class of graphs whose members have unbounded girth can have its circular chromatic numbers bounded away from two, i.e.:

**Theorem 4.2.** *For every graph $H$, $\sup\{\chi_c(G) : G \text{ is } H\text{-minor free}\} \to 2 \quad \text{as } girth(G) \to \infty$.*

In particular, graphs of bounded genus are nearly bipartite provided that their girth is sufficiently large. On the other hand, every rational number between 2 and 4 is the circular chromatic number of a planar graph [17, 25]. It is worth noticing that Moser's construction provides planar graphs whose circular chromatic number ranges between 2 and 3 and these graphs are triangle-free but their degree is unbounded; Zhu's construction provides planar graphs whose circular chromatic number ranges between 3 and 4 and these graphs have unbounded degree and contain triangles.

The only upper bounds which are known for $\chi_c(G)$ concern special classes of graphs. In particular, it has been proved [10] that $K_4$-minor free graphs with girth at least $2\lfloor(3k-1)/2\rfloor$ have $\chi_c(G) \leq 2 + 2/(2k-1)$. Hence, a $K_4$-minor free graph with no triangle has circular chromatic number smaller than $8/3$. Similarly, if $G$ is a $K_3^2$-minor free graph with girth at least $3k$ then $\chi_c(G) \leq 2 + 1/(k-1)$ [5]. So, one might ask whether all planar graphs with degree bounded by 3 and girth greater than 3 have circular chromatic number bounded by $8/3$. This is not true, and in Fig. 4 it is shown a planar graph of maximum degree 3 and girth 5 which has circular chromatic number greater than $8/3$. On the other hand, a similar conjecture (mentioned by R. Thomas at Levico 2000) concerning the fractional chromatic number is still open:

**Question 4.3.** *$G$ planar, $\Delta(G) \leq 3$ and $G$ triangle-free, is it $\chi_f(G) \leq 8/3$?*

The dual parameter $\phi_c(G)$ shows a rather different behaviour. First of all, $\phi_c(G)$ ranges in a bounded interval; in fact,

**Theorem 4.4.** *(Seymour, 1981) If $G$ is 2-edge connected then $2 \leq \phi_c(G) \leq 6$.*

**Theorem 4.5.** *(Jaeger, 1975) if $G$ is 4-edge connected then $2 \leq \phi_c(G) \leq 4$.*

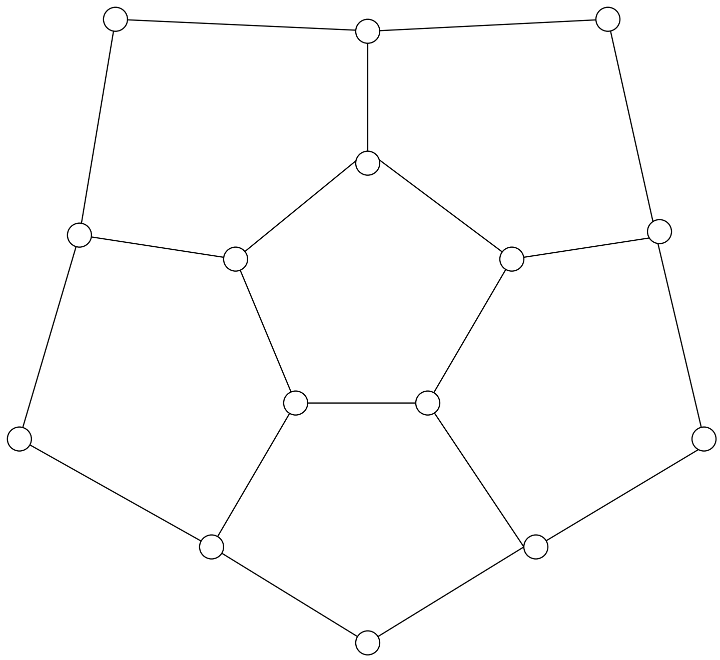
In analogy with the circular chromatic number, the circular flow number may be regarded as a measure of how close a graph is to being eulerian, since $\phi_c(G) \geq 2$ with equality if and only if $G$ is eulerian.

The following conjectures (see [19] for references) suggest that the edge connectivity might constrain the growth of the circular flow number as the girth does for the circular chromatic number (note that the two concepts are dual in the matroidal sense):

**Conjecture 4.6.** *(Tutte, 1954) Every 2-edge connected graph $G$ has $\phi_c(G) \leq 5$.*

**Conjecture 4.7.** *(Tutte, 1966) Every 4-edge connected graph $G$ has $\phi_c(G) \leq 3$.*

In analogy to Theorem 4.2 for circular chromatic numbers, the following has been asked [5]:



**Question 4.8.** *Is it true that* $\sup\{\phi_c(G) : G \text{ is } k\text{-edge connected}\} \to 2$ *as* $k \to \infty$*?*

The answer is "yes" for graphs of bounded genus [23], but little progress has been made for general graphs. A small step towards the settlement of Conjecture 4.7 was made in [4], where the following refinement of Jaeger's result was proved:

**Theorem 4.9.** *Every* $6$*-edge connected graph* $G$ *has* $\phi_c(G) < 4$.

For sake of completeness, we present here the proof of the above result.

Let $T \subseteq V(G)$. A $T$*-join* in $G$ is a subset $J \subseteq E(G)$ such that $T$ is the set of odd-degree vertices in the induced subgraph $G[J]$. An $\emptyset$-join is usually called a *cycle* or *even subgraph* of $G$. We use two standard results regarding trees and $T$-joins. The first is folklore, and the second was first proved by Nash-Williams [18].

**Lemma 4.10.** *Any tree* $H$ *contains a* $T$*-join, for any* $T \subseteq V(H)$ *of even cardinality.*

**Lemma 4.11.** *Any* $2k$*-edge connected graph contains* $k$ *edge-disjoint spanning trees.*

**Lemma 4.12.** *Let* $H_1$ *and* $H_2$ *be edge-disjoint spanning trees of a graph* $G$ *and let* $T$ *be an even subset of* $V(G)$*. Then* $H_1 \cup H_2$ *contains a* $T$*-join which is spanning and connected.*

*Proof.* Let $V_1$ be the set of odd-degree vertices in $H_1$. The symmetric difference $V_1 \Delta T$ has even cardinality so by Lemma 4.10, $H_2$ contains a $(V_1 \Delta T)$-join $J_2$. Let $F = H_1 \cup J_2$. Since $H_1$ and $J_2$ are edge disjoint, $F$ is a $T$-join. Furthermore $E(H_1) \subseteq F$ so $F$ spans $G$ and is connected. ∎

**Lemma 4.13.** *Consider the polyhedron* $P = \{x \in \mathbb{R}^8 : Ax = b,\ x \geq 0\}$ *where*

$$[A|b] = \left[\begin{array}{cccccccc|c} 1 & 1 & 1 & 0 & -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right].$$

*Then the linear program* $z^* = \min\{[1\ 1\ 1\ 1\ 0\ 0\ 0\ 0]x : x \in P\}$ *has a unique optimum solution* $x^* = \left[\frac{1}{4}\ 0\ 0\ 0\ 0\ 0\ \frac{1}{4}\ \frac{1}{2}\right]^T$ *with value* $z^* = \frac{1}{4}$.

*Proof.* It is routine to check that $x^*$ is $P$-feasible and that the vector $y^* = \left[\frac{1}{2}\ \frac{1}{4}\ \frac{1}{4}\right]$ is a feasible solution to the dual linear program $\max\{y[0\ 0\ 1]^T : yA \leq [1\ 1\ 1\ 1\ 0\ 0\ 0\ 0]\}$. Both objective values equal $\frac{1}{4}$ so $(x^*, y^*)$ is an optimal dual pair. To show uniqueness we demonstrate that the primal objective vector is in the strict interior of a full dimensional cone generated by normals of active (tight) constraints at $x^*$. Writing $x = [x_1\ x_2\ \ldots\ x_8]^T$, the active constraints are the three equations $Ax = b$ and the five equations $x_i = 0$, $i = 2, 3, 4, 5, 6$. Indeed we have the positive linear combination

$$[1\ 1\ 1\ 1\ 0\ 0\ 0\ 0] = \left[\frac{1}{2}\ \frac{1}{4}\ \frac{1}{4}\right]A + \frac{1}{4}e_2 + \frac{1}{2}e_3 + \frac{1}{2}e_4 + \frac{1}{2}e_5 + \frac{1}{4}e_6$$

where $e_i$ is the $i$-th standard unit vector in $\mathbb{R}^8$. The cone is full dimensional since the first, seventh and eighth columns of $A$ are linearly independent. ∎

*Proof.* Let $V_1$ be the vertices of odd degree in $G$. By Lemma 4.11, $G$ has three edge disjoint spanning trees. So by Lemmas 4.10 and 4.12, $G$ has two edge-disjoint $V_1$-joins, $J_1$, $J_2$, such that $J_2$ spans $G$ and is connected. Let $\vec{C_1}$ and $\vec{C_2}$ be eulerian orientations of the complementary cycles $C_1 = E - J_1$ and $C_2 = E - J_2$. Note that $C_1 \cup C_2 = E(G)$. Let $\vec{G}$ be the *lexicographic orientation* of $G$ induced by $(\vec{C_1}, \vec{C_2})$. That is, we orient each edge $e \in E(G) \cap C_1$ as it is oriented in $\vec{C_1}$, and we orient each $e \in E(G) - C_1$ as it is oriented in $\vec{C_2}$.

Let $X$ be a proper nonempty subset of $V(\vec{G})$. We shall show that $\frac{|\delta^+X|}{|\delta X|} > \frac{1}{4}$ and the result follows from (4). We associate with every edge $e \in \delta X$ an ordered pair $\sigma\tau \in \{+, -, 0\}^2$ where

$$\sigma = \begin{cases} + & \text{if } \vec{C_1} \text{ traverses } e \text{ from } X \text{ to } V - X \\ - & \text{if } \vec{C_1} \text{ traverses } e \text{ from } V - X \text{ to } X \\ 0 & \text{if } e \notin C_1 \end{cases}$$

and where $\tau$ is defined similarly using $C_2$ in place of $C_1$. The pair $\sigma\tau$ is called the *type* of $e$.

Let $x_{\sigma\tau}$ denote the proportion of edges in $\delta X$ having type $\sigma\tau$. Since $C_1 \cup C_2 = E(G)$, no edge has type 00. We consider the 8-dimensional column vector

$$x = [x_{++}\ x_{+0}\ x_{+-}\ x_{0+}\ x_{--}\ x_{-0}\ x_{-+}\ x_{0-}]^T.$$

Since each $\vec{C_i}$ traverses $\delta X$ the same number of times in each direction, $x$ is a feasible point in the polyhedron $P$ of Lemma 4.13. Because $\vec{G}$ is defined lexicographically, we have

$$\frac{|\delta^+X|}{|\delta X|} = [1\ 1\ 1\ 1\ 0\ 0\ 0\ 0]x$$

so this ratio is bounded below by the optimum value of the linear program of Lemma 4.13. By that lemma, the unique optimum solution is $x^* = \left[\frac{1}{4}\ 0\ 0\ 0\ 0\ 0\ \frac{1}{4}\ \frac{1}{2}\right]^T$ with value $z^* = \frac{1}{4}$. Since

$J_2$ is spanning and connected, we have $\delta X - C_2 \neq \emptyset$, so $x_{+0} + x_{-0} > 0$. Thus $x \neq x^*$ and $[1\ 1\ 1\ 1\ 0\ 0\ 0\ 0]x > \frac{1}{4}$ as claimed. ∎

We have not proved that $\sup\{\phi_c(G) : G \text{ is 6-edge connected }\} < 4$.

To do so by our method would require finding disjoint $V_1$-joins $J_1$, $J_2$ such that $|J_1 \cap \delta X| \geq c|\delta X|$ for all $X \subseteq V$ and some $c > 1/2$. If this is the case, then, by orienting the edges of $G$ as in the proof of Theorem 4.9, we ensure that

$$|\delta^+ X| \geq \frac{|J_1 \cap \delta X|}{2} \geq c\frac{|\delta X|}{2}.$$

Using only Lemmas 4.10 and 4.11, the best we can ensure at the moment is that $|J_1 \cap \delta X| \geq \left\lfloor \frac{k-4}{2} \right\rfloor$.

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