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STOCHASTIC CONTROL OF BILINEAR SYSTEMS: THE OPTIMAL QUADRATIC CONTROLLER

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Abstract

For a bilinear stochastic system described by Ito equations, the following problem is considered: find the optimal feedback control law in a class of quadratic controllers. The optimality criterion is the classical quadratic one for a fixed-interval state-regulation problem. It will be shown that the solution is a linear map of optimal-quadratic state estimate, which can be obtained using the quadratic filter for bilinear system available in literature. Moreover, the matrix function that solves the control problem results to be equal to the one of the linear optimal control (LOC) problem.

Key words: stochastic systems, stochastic control, LQG optimal control, Kalman-Bucy filter, nonlinear filtering, separation principle, Brownian motion, Ito formula.

1. Introduction

Very general methodologies are available in the literature and can be often successfully used to solve the stochastic optimal control problem in particular cases. The dynamic programming alghoritm [1] and the Hamilton-Jacobi-Bellman equation [3] are the main tools at this purpose. A nice solution of this problem is given for a linear and Gaussian system with a quadratic cost criterion (LQG control problem)[1,3]. In this case the solution is given by such a controller which results to be optimal among all the measurable functions of the observed path, and it is a linear function of the optimal state-estimate. The latter can be recursively computed by means of the Kalman filter, whereas the matrix performing the linear map of the state-estimate can be computed by means of a well defined backward Riccati equation.

This solution is quite attractive from two points of view. First of all, the solution is a closed-loop one. Moreover, the controller (which is given in general by a matrix function evolving on the control time-interval) can be computed off-line by means of the above mentioned, and easily implementable, backward differential Riccati equation. The on-line computational burden is entirely loaded on the Kalman filter, so that we can say that, in the linear-Gaussian case, from a computational point of view the control problem is reduced to a filtering one.

Unfortunately, these nice properties of the controller hold no more in the more general case of a nonlinear and/or non-Gaussian system. In [19],[20] it has been given a solution for the stochastic optimal regulator problem with a quadratic cost criterion for discrete time non-Gaussian linear systems. In [10] we have already shown that for a bilinear continuous time system and a quadratic cost criterion, the *linear-optimal* controller, that is to say the controller minimizing the cost functional among all the *linear* functions of the observed path, has the same nice structure of the controller in the LQG case. More exactly, it is a linear map of the *linear-optimal estimate* of the state process. The latter can be computed by means of the linear-optimal filter for bilinear systems described in [8], whereas the linear map performing the control action is a matrix time-function given again by a backward Riccati differential equation similar to the one of the LQG control. Even thought the solution presented in [10] is a suboptimal one, nevertheless it is a closed loop and readily implementable one. Since it is suboptimal, improvements are possible by searching the optimal control in a wider class of controllers than the linear one.

The aim of this technical report is to perform a first step in this direction by searching for quadratic optimal controllers for an optimal control problem with quadratic cost functional and a bilinear system. The result is quite impressive: the quadratic-optimal controller is a linear map of the quadratic-optimal state estimate. The linear map *is the same* as in the Linear Optimal Control (LOC) problem and hence the improvement in the control performance has to be entirely ascribed to the improvement in the state-estimate obtained using a quadratic-filter instead of a linear one.

The paper is organized as follows. In §2 the precise setting of the problem and the proposed solution is presented. Two appendices are enclosed for reader convenience. The first one concerns a brief survey on Kronecker Algebra. In the second one the vector Ito formula in the Kronecker formalism is recalled. The reader is referred to [8] and [13]-[15] for more details about the topics presented in these appendices.

4.

2. Setting of the quadratic optimal control problem

First of all we introduce the basic notations and symbols that will be used throughout the paper. $(\Omega, \mathcal{F}, P)$ will denote the basic probability triple. $\mathbf{E}\{\cdot\}$ denotes the expectation operator. $L^2(\mathcal{E})$, with $\mathcal{E}$ linear space, denotes the Hilbert space of all the $\mathcal{E}$-valued square-integrable random variables defined on $(\Omega, \mathcal{F}, P)$. Let $\mathcal{I}$ be a linear space endowed with some inner product, and $\xi, \eta \in \mathcal{I}$. We use the notation $(\xi \, , \, \eta)$ to denote the inner product between ξ and η. For any matrix M, the notation $M_{i,j}$ will be used to denote its (i, j)-entry. For the identity matrix in $\mathbb{R}^n$ it will be used the symbol I_n and it will be indicated with $0_{n \times m}$ the null matrix in $\mathbb{R}^{n \times m}$.

Let I be a real interval and $\xi : I \to L^2(\mathbb{R}^i)$ an $\mathbb{R}^i$-valued stochastic process; we shall denote with $\mathcal{F}_t^\xi$ the σ-algebra generated by $\{\xi_s; \ s \in I, s \leq t\}$. For a vector-valued process $\{\xi_t\}$, the notation ξ_t^j shall indicate the j-th entry. If $\{\xi_t\}$ and $\{\eta_t\}$ are two second-order scalar stochastic processes, the notation $\{\langle \xi, \eta \rangle_t\}$ will be used to indicate the mutual quadratic variation process. The notation $\langle \xi \rangle_t$ will be also used in place of $\langle \xi, \xi \rangle_t$. When ξ and η are vector-valued, the same notation $\langle \xi, \eta \rangle$ will denote the matrix whose (i, j) entry is given by $\langle \xi^i, \eta^j \rangle$.

Let $\mathcal{S} \subset L^2(\mathcal{E})$ be a linear space and $X \in L^2(\mathcal{E})$; then the symbol $\mathbf{\Pi}\left\{X / \mathcal{S}\right\}$ will denote the orthogonal projection of X onto $\mathcal{S}$. Anytime the underlying space is understood we will use the notation $\widehat{X}$ to denote the orthogonal projection. As well known, the projection $\widehat{X}$ represents the best (in the sense of the error-variance) estimate of X using estimates $\alpha \in \mathcal{S}$, and it is characterized by the following property:

$$\mathbf{E}\left\{(X - \widehat{X}, \alpha)\right\} = 0, \quad \forall \alpha \in \mathcal{S}. \tag{2.1}$$

Let be given the following controlled bilinear system

$$dX_t = A(t)X_t dt + H(t)u_t dt + \sum_{k=1}^{p} \left(B^k(t)X_t + f^k(t)\right) dW'_{k,t}, \quad X_{t_0} = \overline{X}, \tag{2.2}$$

$$dY_t = C(t)X_t dt + dW_t, \qquad\qquad Y_{t_0} = 0, \tag{2.3}$$

where $t \in I$, $I = [t_0, t_f] \subset \mathbb{R}$, $X_t \in \mathbb{R}^n$, $Y_t \in \mathbb{R}^m$ and $u_t \in \mathbb{R}^q$.
As shown in [8, Theorem 4.1], eq. (2.2) is equivalent to

$$dX_t = A(t)X_t dt + H(t)u_t dt + \widetilde{F}(t)d\widetilde{W}_t, \tag{2.4}$$

where $\widetilde{F}$ is the following block-matrix:

$$\widetilde{F}(t) = \left[\ \widetilde{F}^1(t) \ \vdots \ \dots \ \vdots \ \widetilde{F}^m(t) \ \vdots \ \widetilde{f}^1(t) \ \vdots \ \dots \ \vdots \ \widetilde{f}^m(t) \ \right], \tag{2.5}$$

$\widetilde{F}^k(t) \in \mathbb{R}^{n \times \rho_k}$, $\widetilde{f}^k(t) \in \mathbb{R}^n$, for $k = 1, ..., m$ where

$$\rho_k \doteq \mathrm{rank}\left\{B^k(t)\Psi_X(t)B^k(t)^T\right\},$$

$$\widetilde{F}^k(t) \doteq \left(B^k(t)\Psi_X(t)B^k(t)^T\right)^{\left(\frac{1}{2}\right)}, \quad \widetilde{f}^k(t) \doteq B^k(t)\mathbf{E}\{X_t\} + f^k(t).$$

$\widetilde{W}_t$ is a wide-sense-Wiener (WSW) process given by $\widetilde{W}_t^T = [\widetilde{W}_t^{1\,T} \ \dots \ \widetilde{W}_t^{m\,T} \ W_t^T]$, where $\widetilde{W}_t^k \in \mathbb{R}^{\rho_k}$, $k = 1, ..., m$ are mutually uncorrelated standard WSW processes.

We will seek the control law u_t in the class of admissible controls, namely $\widehat{\mathcal{U}}$, defined as:

$$\widehat{\mathcal{U}} = \left\{ u_t \in \mathcal{L}_t^p(\mathcal{Y}) \ / \ u_t = L(t)\widehat{X}_t, \quad \widehat{X}_t = \mathbf{\Pi}\left(X_t/\mathcal{L}_t^n(\mathcal{Y})\right), \quad \mathcal{Y}_s = \begin{bmatrix} Y_s \\ Y_s^{[2]} \end{bmatrix}, \ s \in I, s \leq t \right\} \quad (2.6)$$

where the symbol $\mathbf{\Pi}$ denote the projection operator and $\mathcal{L}_t^i(\mathcal{Y})$ is the set of $\mathrm{I\!R}^i$-valued linear transformations of $\{\mathcal{Y}_s; \ s \in I, s \leq t\}$.

The optimal control problem we will solve can be precisely defined as follows:

$$\min_{u_t \in \widehat{\mathcal{U}}} J(u),$$

$$J(u) = \frac{1}{2}\mathbf{E}\left\{ \left(X_{t_f} \ , \ FX t_f\right) + \int_{t_0}^{t_f} \left\{ \left(X_t \ , \ Q(t)X_t\right) + \left(u_t \ , \ R(t)u_t\right) \right\} dt \right\},$$

where $\forall t$, $Q(t) = Q(t)^T \geq 0$, $R(t) = R(t)^T > 0$, and $F = F^T \geq 0$, under the differential constraints represented by system (2.2) (or (2.4)), (2.3).

The controlled system (2.4), (2.3) can be written also in the following way:

$$dX_t = A(t)X_t dt + \mathcal{E}\Lambda(t)\widehat{\mathcal{X}}_t dt + \widetilde{F}(t)d\widetilde{W}_t, \quad X_{t_0} = \overline{X}, \quad (2.7)$$

$$dY_t = C(t)X_t dt + dW_t, \quad Y_{t_0} = 0, \quad (2.8)$$

where

$$\widehat{\mathcal{X}}_t = \mathbf{\Pi}\left(\mathcal{X}_t/\mathcal{L}_t^\alpha(\mathcal{Y})\right), \ \alpha = n(1 + n + m), \quad (2.9)$$

$$\mathcal{X}_t = \begin{bmatrix} X_t \\ X_t^{[2]} \\ X_t \otimes Y_t \end{bmatrix}, \quad \Lambda(t) = \begin{bmatrix} H(t)L(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix} I_n & 0_{n \times n^2} & 0_{n \times nm} \end{bmatrix}. \quad (2.10)$$

Let us now consider the *uncontrolled* system

$$dX_t^0 = A(t)X_t^0 dt + \widetilde{F}(t)d\widetilde{W}_t, \quad X_{t_0}^0 = \overline{X}, \quad (2.11)$$

$$dY_t^0 = C(t)X_t^0 dt + dW_t, \quad Y_{t_0}^0 = 0, \quad (2.12)$$

and suppose, for simplicity, that $f_k = 0$, $k = 1, ..., p$ in (2.2). We can state the following lemma:

Lemma 2.1. *Let $\mathcal{Y}_t^0$ be the aggregate vector of the first and second Kronecker powers of the uncontrolled system output vector:*

$$\mathcal{Y}_t^0 = \begin{bmatrix} Y_t^0 \\ Y_t^{0[2]} \end{bmatrix}. \quad (2.13)$$

Moreover, let us define the $(p + m)$ disturbances vector as the aggregate vector of the state and output noises:

$$\overline{W}_t = \begin{bmatrix} W_t' \\ W_t \end{bmatrix}. \quad (2.14)$$

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Then, it is possible to construct the uncontrolled extended system as follows:

$$d\mathcal{X}_t^0 = \mathcal{A}(t)\mathcal{X}_t^0 \, dt + \sum_{k=1}^{p+m} \mathcal{B}_k(t)\mathcal{X}_t^0 \, d\overline{W}_{k,t}, \qquad \mathcal{X}_{t_0}^0 = \overline{\mathcal{X}}, \qquad (2.15)$$

$$d\mathcal{Y}_t^0 = \left(\mathcal{C}(t)\mathcal{X}_t^0 + \mathcal{U}(t)\right) dt + \sum_{k=1}^{p+m} \left(\mathcal{D}_k\mathcal{Y}_t^0 + \mathcal{G}_k\right) d\overline{W}_{k,t}, \quad \mathcal{Y}_{t_0}^0 = 0, \qquad (2.16)$$

where

$$\mathcal{X}_t^0 = \begin{bmatrix} X_t^0 \\ X_t^{0[2]} \\ X_t^0 \otimes Y_t^0 \end{bmatrix} \in \mathbb{R}^\alpha, \qquad (2.17)$$

and

$$\mathcal{A}(t) = \begin{bmatrix} A(t) & 0 & 0 \\ 0 & A^{(1)}(t) & 0 \\ 0 & I_n \otimes C(t) & A(t) \otimes I_m \end{bmatrix}, \qquad (2.18)$$

$$A^{(1)}(t) = U_n^2(A(t) \otimes I_n) + \sum_{k=1}^p B^k(t)^{[2]}, \qquad (2.19)$$

$$B_k^{(2)}(t) = U_n^2(B^k(t) \otimes I_n), \qquad (2.20)$$

$$\mathcal{C}(t) = \begin{bmatrix} C(t) & 0 & 0 \\ 0 & 0 & C^{(1)}(t) \end{bmatrix}, \qquad (2.21)$$

$$C^{(1)}(t) = U_m^2(C(t) \otimes I_m), \qquad (2.22)$$

$$\mathcal{U}(t) = \begin{bmatrix} 0 \\ st(I_m) \end{bmatrix}, \qquad (2.23)$$

$$\mathcal{B}_k(t) = \begin{bmatrix} B^k(t) & 0 & 0 \\ 0 & B_k^{(2)}(t) & 0 \\ 0 & 0 & (B^k(t) \otimes I_m) \end{bmatrix}, \quad k = 1, ..., p, \qquad (2.24)$$

$$\mathcal{B}_k(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \overline{B}^k & 0 & 0 \end{bmatrix}, \quad k = p+1, ..., p+m, \qquad (2.25)$$

$$\overline{B}^k = \begin{bmatrix} \overline{B}_1^k \\ \overline{B}_2^k \\ \vdots \\ \overline{B}_k^k \\ \vdots \\ \overline{B}_m^k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I_n \\ \vdots \\ 0 \end{bmatrix}, \quad \overline{B}_i^k \in \mathbb{R}^{n \times n}, \quad i = 1, ..., m, \qquad (2.26)$$

$$\mathcal{D}_k = 0, \quad \mathcal{G}_k = 0, \quad k = 1, ..., p, \qquad (2.27)$$

$$\mathcal{D}_k = \begin{bmatrix} 0 & 0 \\ U_m^2\overline{D}^k & 0 \end{bmatrix}, \quad \mathcal{G}_k = \begin{bmatrix} g_{k,1} \\ g_{k,2} \\ \vdots \\ g_{k,k} \\ \vdots \\ g_{k,m+m^2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad k = p+1, ..., p+m, \qquad (2.28)$$

$$\overline{D}^k = \begin{bmatrix} \overline{D}^k_1 \\ \overline{D}^k_2 \\ \vdots \\ \overline{D}^k_k \\ \vdots \\ \overline{D}^k_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I_m \\ \vdots \\ 0 \end{bmatrix}, \quad \overline{D}^k_i \in \mathbb{R}^{m \times m}, \quad i = 1, ..., m, \tag{2.29}$$

with U_m^2, U_n^2 as in Lemma B.1.

Proof. First of all, note that system (2.11)-(2.12) is equivalent to

$$dX_t^0 = A(t)X_t^0 dt + \sum_{k=1}^{p} B^k(t)X_t^0 dW'_{k,t}, \quad X_{t_0}^0 = \overline{X}, \tag{2.30}$$

$$dY_t^0 = C(t)X_t^0 dt + dW_t, \qquad\qquad Y_{t_0}^0 = 0. \tag{2.31}$$

To prove (2.15)-(2.27) it is sufficient to compute the equations for $dX_t^{0\,[2]}$, $d(X_t^0 \otimes Y_t^0)$, and $dY_t^{0\,[2]}$.

From (2.30), using the vector Ito formula given by (B.8) and the identities (A.8), we have that

$$dX_t^{0\,[2]} = \left(\frac{d}{dx} \otimes x^{[2]} \right)_{x=X_t^0} \cdot dX_t^0 + \frac{1}{2} \left(\frac{d}{dx} \otimes \frac{d}{dx} \otimes x^{[2]} \right)_{x=X_t^0} \cdot st \left(\sum_{k=1}^{p} (B^k(t)X_t^0)(B^k(t)X_t^0)^T \right) dt$$

$$= U_n^2 \left(I_n \otimes X_t^0 \right) A(t)X_t^0 dt + \sum_{k=1}^{p} U_n^2 \left(I_n \otimes X_t^0 \right) B^k(t)X_t^0 dW'_{k,t}$$

$$+ \sum_{k=1}^{p} I_{n^2} \cdot st \left(B^k(t)X_t^0 X_t^{0\,T} B^k(t)^T \right) dt,$$

where it has been used rules (B.5), (B.6) and (B.7).
Then, taking into account (A.5) and (A.8), it results

$$dX_t^{0\,[2]} = U_n^2 \left(A(t) \otimes I_n \right) X_t^{0\,[2]} dt + \sum_{k=1}^{p} U_n^2 \left(B^k(t) \otimes I_n \right) X_t^{0\,[2]} dW'_{k,t}$$

$$+ \sum_{k=1}^{p} I_{n^2} \cdot \left(B^k(t) \otimes B^k(t) \right) X_t^{0\,[2]} dt$$

$$= \left(A^{(1)}(t)X_t^{0\,[2]} \right) dt + \sum_{k=1}^{p} B_k^{(2)}(t)X_t^{0\,[2]} dW'_{k,t}, \tag{2.32}$$

where $A^{(1)}(t)$, $B_k^{(2)}(t)$ are defined in (2.19),(2.20).
Moreover, Ito formula (B.8) applied with $F(X,Y) = X \otimes Y$ gives:

$$d(X_t^0 \otimes Y_t^0) = \left(\left[\frac{d}{dx} \quad \frac{d}{dy} \right] \otimes (x \otimes y) \right)_{x=X_t^0, y=Y_t^0} \cdot \begin{bmatrix} dX_t^0 \\ dY_t^0 \end{bmatrix}$$

$$+ \frac{1}{2} \left(\left[\frac{d^{[2]}}{dx^{[2]}} \quad \frac{d^{[2]}}{dx \otimes dy} \quad \frac{d^{[2]}}{dy \otimes dx} \quad \frac{d^{[2]}}{dy^{[2]}} \right] \otimes (x \otimes y) \right)_{x=X_t^0, y=Y_t^0} \cdot \begin{bmatrix} B(t)X_t^0 dW'_t \\ dW_t \end{bmatrix}^{[2]}. \tag{2.33}$$

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Since it results

$$\frac{d}{dx} \otimes (x \otimes y) = (\frac{d}{dx} \otimes x) \otimes y + x \otimes (\frac{d}{dx} \otimes y) = I_n \otimes y, \tag{2.34}$$

and

$$\frac{d}{dy} \otimes (x \otimes y) = x \otimes I_m, \tag{2.35}$$

the first addendum in (2.33) can be written as follows

$$\left(\left[\frac{d}{dx} \otimes (x \otimes y) \quad \frac{d}{dy} \otimes (x \otimes y) \right] \right)_{x=X_t^0, y=Y_t^0} \cdot \left[\begin{array}{c} A(t)X_t^0 dt + \widetilde{F}(t)d\widetilde{W}_t \\ C(t)X_t^0 dt + dW_t \end{array} \right]$$

$$= [\, I_n \otimes y \quad x \otimes I_m \,]_{x=X_t^0, y=Y_t^0} \cdot \left[\begin{array}{c} A(t)X_t^0 dt + \widetilde{F}(t)d\widetilde{W}_t \\ C(t)X_t^0 dt + dW_t \end{array} \right]. \tag{2.36}$$

Moreover, taking into account that (see A.8), for any vector v, it results $v^{[2]} = st(vv^T)$, we have:

$$\left[\begin{array}{c} B(t)X_t^0 dW_t' \\ dW_t \end{array} \right]^{[2]} = st \left(\left[\begin{array}{c} B(t)X_t^0 dW_t' \\ dW_t \end{array} \right] \left[\, X_t^{0T} B(t)^T dW_t' \quad dW_t^T \, \right] \right)$$

$$= st \left[\begin{array}{cc} B(t)X_t^0 X_t^{0T} B(t)^T dt & 0 \\ 0 & I_m dt \end{array} \right].$$

Then, we can write:

$$\frac{1}{2} \left(\left[\frac{d^{[2]}}{dx^{[2]}} \quad \frac{d^{[2]}}{dx \otimes dy} \quad \frac{d^{[2]}}{dy \otimes dx} \quad \frac{d^{[2]}}{dy^{[2]}} \right] \otimes (x \otimes y) \right)_{x=X_t^0, y=Y_t^0} \cdot \left[\begin{array}{c} B(t)X_t^0 dW_t' \\ dW_t \end{array} \right]^{[2]}$$

$$= \frac{1}{2} \left(\left[\frac{d}{dx} \quad \frac{d}{dy} \right] \otimes \left[\frac{d}{dx} \quad \frac{d}{dy} \right] \otimes (x \otimes y) \right)_{x=X_t^0, y=Y_t^0} st \left[\begin{array}{cc} B(t)X_t^0 X_t^{0T} B(t)^T dt & 0 \\ 0 & I_m dt \end{array} \right]$$

$$= \frac{1}{2} \left(\left[\frac{d}{dx} \quad \frac{d}{dy} \right] \otimes \left[\frac{d}{dx} \otimes (x \otimes y) \quad \frac{d}{dy} \otimes (x \otimes y) \right] \right)_{x=X_t^0, y=Y_t^0} st \left[\begin{array}{cc} B(t)X_t^0 X_t^{0T} B(t)^T dt & 0 \\ 0 & I_m dt \end{array} \right]$$

$$= \frac{1}{2} \left(\left[\frac{d}{dx} \quad \frac{d}{dy} \right] \otimes [\, I_n \otimes y \quad x \otimes I_m \,] \right)_{x=X_t^0, y=Y_t^0} st \left[\begin{array}{cc} B(t)X_t^0 X_t^{0T} B(t)^T dt & 0 \\ 0 & I_m dt \end{array} \right],$$

where we have used (A.8) and we have taking into account properties of the stochastic differential operator given in Appendix B. Finally, it can be obtained:

$$\frac{1}{2} \left(\left[\frac{d^{[2]}}{dx^{[2]}} \quad \frac{d^{[2]}}{dx \otimes dy} \quad \frac{d^{[2]}}{dy \otimes dx} \quad \frac{d^{[2]}}{dy^{[2]}} \right] \otimes (x \otimes y) \right)_{x=X_t^0, y=Y_t^0} \cdot \left[\begin{array}{c} B(t)X_t^0 dW_t' \\ dW_t \end{array} \right]^{[2]}$$

$$= \frac{1}{2} st \left\{ \left[\frac{d}{dx} \quad \frac{d}{dy} \right] \cdot \left[\begin{array}{cc} B(t)\overline{x} \cdot \overline{x}^T B(t)^T dt & 0 \\ 0 & I_m dt \end{array} \right] \cdot \left[\begin{array}{c} I_n \otimes y^T \\ x^T \otimes I_m \end{array} \right] \right\}_{\overline{x}, x=X_t^0, y=Y_t^0}^T$$

$$= \frac{1}{2} st \left\{ \left[\frac{d}{dx} \quad \frac{d}{dy} \right] \cdot \left[\begin{array}{c} (B(t)\overline{x} \cdot \overline{x}^T B(t)^T)(I_n \otimes y^T) dt \\ I_m (x^T \otimes I_m) dt \end{array} \right] \right\}_{\overline{x}, x=X_t^0, y=Y_t^0}^T$$

$$= 0, \tag{2.37}$$

where we have suitably used rules (A.6), (A.7) of the Kronecker algebra. Hence (2.33) becomes, exploiting (2.35) and (2.37):

$$
\begin{aligned}
d(X_t^0 \otimes Y_t^0) &= \left[\, I_n \otimes y \quad x \otimes I_m \,\right]_{x=X_t^0, y=Y_t^0} \cdot \begin{bmatrix} A(t)X_t^0 dt + \widetilde{F}(t)d\widetilde{W}_t \\ C(t)X_t^0 dt + dW_t \end{bmatrix} \\
&= (I_n \otimes Y_t^0)A(t)X_t^0 dt + (X_t^0 \otimes I_m)C(t)X_t^0 dt \\
&\quad + (I_n \otimes Y_t^0)\widetilde{F}(t)d\widetilde{W}_t + (X_t^0 \otimes I_m)dW_t \\
&= (A(t)X_t^0) \otimes Y_t^0 dt + X_t^0 \otimes (C(t)X_t^0)dt \\
&\quad + \sum_{k=1}^{p}(I_n \otimes Y_t^0)B^k(t)X_t^0 dW_{k,t}' + (X_t^0 \otimes I_m)dW_t \\
&= \Big((A(t) \otimes I_m)(X_t^0 \otimes Y_t^0) + (I_n \otimes C(t))X_t^{0\,[2]}\Big)\, dt \\
&\quad + \sum_{k=1}^{p}(B^k(t) \otimes I_m)(X_t^0 \otimes Y_t^0)dW_{k,t}' + (X_t^0 \otimes I_m)dW_t.
\end{aligned} \tag{2.38}
$$

As far as the Kronecker second power of the uncontrolled output is concerned, we have:

$$
\begin{aligned}
dY_t^{0\,[2]} &= \left(\frac{d}{dy} \otimes y_t^{[2]}\right)_{y=Y_t^0} \cdot dY_t^0 + \frac{1}{2}\left(\frac{d}{dy} \otimes \frac{d}{dy} \otimes y_t^{[2]}\right)_{y=Y_t^0} st(I_m)dt \\
&= U_m^2(I_m \otimes Y_t^0)(C(t)X_t^0 dt + dW_t) + \frac{1}{2}O_m^2 st(I_m)dt \\
&= U_m^2(I_m \otimes Y_t^0)(C(t)X_t^0 dt + dW_t) + st(I_m)dt \\
&= \Big(U_m^2(C(t) \otimes I_m)(X_t^0 \otimes Y_t^0) + st(I_m)\Big)\, dt + U_m^2(I_m \otimes Y_t^0)dW_t \\
&= \Big(C^{(1)}(t)(X_t^0 \otimes Y_t^0) + st(I_m)\Big)\, dt + U_m^2(I_m \otimes Y_t^0)dW_t,
\end{aligned} \tag{2.39}
$$

with $C^{(1)}(t)$ given by (2.22). Equations (2.30),(2.31),(2.32),(2.38),(2.39) prove the lemma. ∎

From the uncontrolled extended system, it is possible to obtain the filter, for a suitable matrix $\mathcal{K}(t)$:

$$
d\widehat{\mathcal{X}_t^0} = \mathcal{A}(t)\widehat{\mathcal{X}_t^0}dt + \mathcal{K}(t)d\nu_t^0, \tag{2.40}
$$

$$
d\nu_t^0 = d\mathcal{Y}_t^0 - \Big(\mathcal{C}(t)\widehat{\mathcal{X}_t^0} + \mathcal{U}(t)\Big)dt \tag{2.41}
$$

where

$$
\widehat{\mathcal{X}_t^0} = \mathbf{\Pi}\left(\mathcal{X}_t^0 / \mathcal{L}_t^\alpha(\mathcal{Y}^0)\right), \tag{2.42}
$$

in which α denotes the extended state dimension given by (2.9) (i.e. $\mathcal{X}_t^0 \in \mathbb{R}^\alpha$).

Remark 2.2. Note that the uncontrolled extended system (2.15)-(2.16) can be equivalently rewrite in the following linear representation:

$$
d\mathcal{X}_t^0 = \mathcal{A}(t)\mathcal{X}_t^0 dt + \sum_{k=1}^{p}\widetilde{\mathcal{B}}_k(t)d\widetilde{W}_{k,t}, \qquad \mathcal{X}_{t_0}^0 = \overline{\mathcal{X}}, \tag{2.43}
$$

$$
d\mathcal{Y}_t^0 = \left(\mathcal{C}(t)\mathcal{X}_t^0 + \mathcal{U}(t)\right)dt + \sum_{k=1}^{p}\widetilde{\mathcal{D}}_k(t)d\widetilde{W}_{k,t}, \quad \mathcal{Y}_{t_0}^0 = 0, \tag{2.44}
$$

where $\widetilde{\mathcal{B}}_k(t)$, $\widetilde{\mathcal{D}}_k(t)$ are suitably matrices and $\widetilde{W}_t$ is a standard wide sense Wiener process (see [8]).

We are now in a position to prove the following result (Theorem 2.4), concerning the representation of a *controlled* extended system. At this purpose, we need to state in advance the following Lemma.

Let ξ a continuous, $\mathbb{R}^d$-valued, $\mathcal{F}_t$-adapted, process, $\bar{X} \in \mathbb{R}^n$ a random variable, $f(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$, $g(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^{np}$ functions satisfying some Lipschitz condition that guarantee the existence and unicity of a strong solution for the following stochastic differential equation:

$$dX_t = f(X_t, \xi_t)dt + g(X_t, \xi_t)dW_t, \quad X_0 = \bar{X}. \tag{2.45}$$

Lemma 2.3. *Let η a continuous, $\mathbb{R}^d$-valued, $\mathcal{F}_t$-adapted, process, such that the measures μ_η, μ_ξ induced by η, ξ on $\mathcal{C}([0,T]; \mathbb{R}^d)$ are equivalent. Suppose that there exists a Borel-measurable function $\psi : \mathbb{R}^d \to \mathbb{R}^n$ such that for the solution X of (2.45) one has $X_t = \psi(\xi_t)$. Then, denoting by X' the solution of:*

$$dX_t' = f(X_t', \eta_t)dt + g(X_t', \eta_t)dW_t, \quad X_0' = \bar{X},$$

it results $X_t' = \psi(\eta_t)$.

Proof. Let $\Phi_t, \tilde{\Phi}_t, \Psi_t, \tilde{\Psi}_t : \Omega \times \mathcal{C}([0,T]; \mathbb{R}^d) \to \mathbb{R}^n$ nonanticipative functionals such that

$$\Phi_t(\omega, \xi) = \int_0^t f(X_\tau(\omega), \xi_\tau)d\tau$$

$$\tilde{\Phi}_t(\omega, \eta) = \int_0^t f(X_\tau(\omega), \eta_\tau)d\tau.$$

and

$$\Psi_t(\omega, \xi) = \int_0^t g(X_\tau(\omega), \xi_\tau)dW_\tau(\omega)$$

$$\tilde{\Psi}_t(\omega, \eta) = \int_0^t g(X_\tau(\omega), \eta_\tau)dW_\tau(\omega).$$

By the hypotheses one has

$$\psi(\xi_t) = \bar{X} + \Phi(\omega, \xi) + \Psi(\omega, \xi),$$
$$X_t' = \bar{X} + \tilde{\Phi}(\omega, \eta) + \tilde{\Psi}(\omega, \eta). \tag{2.46}$$

From (2.46) we have that the theorem is proven as soon as it is shown that, P-a.s.:

$$\Phi(\omega, \eta) = \tilde{\Phi}(\omega, \eta); \quad \Psi(\omega, \eta) = \tilde{\Psi}(\omega, \eta). \tag{2.47}$$

Under the equivalence hypothesis for the measures μ_η and μ_ξ, the identities in (2.47) can be verified following the same lines as in the proof of Lemma 4.10 of [6, vol. 1]. $\blacksquare$

Theorem 2.4. *Let define the stochastic processes $\mathcal{X}_t'$, $\mathcal{Y}_t'$ which satisfy the following equations:*

$$d\mathcal{X}_t' = \mathcal{A}(t)\mathcal{X}_t'dt + \Lambda(t)\mu_t dt + \sum_{k=1}^p \widetilde{\mathcal{B}}_k(t)d\widetilde{W}_{k,t}, \tag{2.48}$$

$$d\mathcal{Y}_t' = (\mathcal{C}(t)\mathcal{X}_t' + \mathcal{U}(t))\,dt + \sum_{k=1}^p \widetilde{\mathcal{D}}_k(t)d\widetilde{W}_{k,t}, \tag{2.49}$$

where $\mu_t \in \mathbb{R}^\alpha$ is a stochastic process such that:

$$d\mu_t = \mathcal{A}(t)\mu_t dt + \Lambda(t)\mu_t dt + \mathcal{K}(t)d\nu_t^0, \tag{2.50}$$

with $\Lambda(t)$ as in (2.10), $\mathcal{A}(t)$, $\mathcal{C}(t)$, $\mathcal{U}(t)$ given by (2.18),(2.21),(2.23) respectively, and $\widetilde{\mathcal{B}}_k(t)$, $\widetilde{\mathcal{D}}_k(t)$ as in Remark 2.2.
It follows that:

$$\mu_t = \widehat{\mathcal{X}}_t = \mathbf{\Pi}\left(\mathcal{X}_t / \mathcal{L}_t^\alpha(\mathcal{Y})\right), \quad \mathcal{Y}_s = \begin{bmatrix} Y_s \\ Y_s^{[2]} \end{bmatrix}, \quad s \in I, s \leq t. \tag{2.51}$$

Proof. First of all, we prove that

$$\mu_t = \widehat{\mathcal{X}}_t' = \mathbf{\Pi}\left(\mathcal{X}_t' / \mathcal{L}_t^\alpha(\mathcal{Y}')\right). \tag{2.52}$$

To this end, we can state that the following equality is verified:

$$\mathcal{L}_t^\alpha(\mathcal{Y}') = \mathcal{L}_t^\alpha(\mathcal{Y}^0). \tag{2.53}$$

We have, indeed, that:

$$\mathcal{Y}_t' - \mathcal{Y}_t^0 = \int_{t_0}^t \mathcal{C}(\tau)(\mathcal{X}_\tau' - \mathcal{X}_\tau^0)d\tau, \tag{2.54}$$

Denoting with $\Phi(t, t_0)$ the state transition matrix associated to $\mathcal{A}(t)$:

$$\mathcal{X}_\tau' = \Phi(\tau, t_0)\overline{\mathcal{X}} + \int_{t_0}^\tau \Phi(\tau, s)\Lambda(s)\mu_s ds$$
$$+ \sum_{k=1}^p \int_{t_0}^\tau \Phi(\tau, s)\mathcal{B}_k(s)d\widetilde{W}_{k,s}ds, \tag{2.55}$$

$$\mathcal{X}_\tau^0 = \Phi(\tau, t_0)\overline{\mathcal{X}} + \sum_{k=1}^p \int_{t_0}^\tau \Phi(\tau, s)\mathcal{B}_k(s)d\widetilde{W}_{k,s}ds. \tag{2.56}$$

and then

$$\mathcal{Y}_t' - \mathcal{Y}_t^0 = \int_{t_0}^t \mathcal{C}(\tau)\int_{t_0}^\tau \Phi(\tau, s)\Lambda(s)\mu_s ds d\tau. \tag{2.57}$$

Since $\mu_t \in \mathcal{L}_t^\alpha(\mathcal{Y}^0)$, from (2.57) it results that $\mathcal{Y}_t' \in \mathcal{L}_t^\beta(\mathcal{Y}^0)$, $\beta = m + m^2$; moreover, since both ν_t^0 and ν_t', where $d\nu_t' = d\mathcal{Y}_t' - \left(\mathcal{C}(t)\widehat{\mathcal{X}_t'} + \mathcal{U}(t)\right)dt$, are WSW processes, it follows that $\mu_t \in \mathcal{L}_t^\alpha(\mathcal{Y}')$ and then (see again (2.57)) we have that $\mathcal{Y}_t^0 \in \mathcal{L}_t^\beta(\mathcal{Y}')$. So the equality (2.53) has been proven.
From the equations (2.40),(2.50),(2.55),(2.56) we have

$$\mu_t = \widehat{\mathcal{X}_t^0} + U_t, \tag{2.58}$$
$$\mathcal{X}_t' = \mathcal{X}_t^0 + U_t, \tag{2.59}$$

where $U_t = \int_{t_0}^t \Phi(t, s)\Lambda(s)\mu_s ds$. Then, taking into account the orthogonality principle, for all linear transformation Θ:

$$\mathbf{E}\left\{\left(\mathcal{X}_t' - \mu_t, \Theta(\mathcal{Y}_t^0)\right)\right\} = \mathbf{E}\left\{\left(\mathcal{X}_t^0 - \widehat{\mathcal{X}_t^0}, \Theta(\mathcal{Y}_t^0)\right)\right\} = 0,$$

12.

that is
$$\mu_t = \mathbf{\Pi}\left(\mathcal{X}'_t / \mathcal{L}^\alpha_t(\mathcal{Y}^0)\right).$$

From this, and taking into account of (2.53) it results that
$$\mu_t = \mathbf{\Pi}\left(\mathcal{X}'_t / \mathcal{L}^\alpha_t(\mathcal{Y}')\right) = \widehat{\mathcal{X}}'_t. \tag{2.60}$$

Now, let us consider the following partition of vectors $\mathcal{X}'_t$ and $\mathcal{Y}'_t$:

$$\mathcal{X}'_t = \begin{bmatrix} \mathcal{X}'_{t,(1)} \\ \mathcal{X}'_{t,(2)} \\ \mathcal{X}'_{t,(3)} \end{bmatrix}, \quad \mathcal{Y}'_t = \begin{bmatrix} \mathcal{Y}'_{t,(1)} \\ \mathcal{Y}'_{t,(2)} \end{bmatrix}, \tag{2.61}$$

where $\mathcal{X}'_{t,(1)} \in \mathbb{R}^n$, $\mathcal{X}'_{t,(2)} \in \mathbb{R}^{n^2}$, $\mathcal{X}'_{t,(3)} \in \mathbb{R}^{nm}$, $\mathcal{Y}'_{t,(1)} \in \mathbb{R}^m$, $\mathcal{Y}'_{t,(2)} \in \mathbb{R}^{m^2}$. Since $\mu_t = \widehat{\mathcal{X}}'_t$, and from the definition of $\Lambda(t)$ given in (2.10), we have

$$\Lambda(t)\mu_t = \Lambda(t)\widehat{\mathcal{X}}'_t = \begin{bmatrix} H(t)L(t)\widehat{\mathcal{X}}'_{t,(1)} \\ 0 \\ 0 \end{bmatrix},$$

and then rewriting eqs. (2.48)-(2.49) in terms of subvectors (2.60),(2.61):

$$d\mathcal{X}'_{t,(1)} = A(t)\mathcal{X}'_{t,(1)}dt + \sum_{k=1}^{p} B^k(t)\mathcal{X}'_{t,(1)}dW'_{k,t} + H(t)L(t)\widehat{\mathcal{X}}'_{t,(1)}dt, \tag{2.62}$$

$$d\mathcal{X}'_{t,(2)} = \left(A^{(1)}(t)\mathcal{X}'_{t,(2)}\right)dt + \sum_{k=1}^{p} B_k^{(2)}(t)\mathcal{X}'_{t,(2)}dW'_{k,t}, \tag{2.63}$$

$$d\mathcal{X}'_{t,(3)} = \left((A(t) \otimes I_m)\mathcal{X}'_{t,(3)} + (I_n \otimes C(t))\mathcal{X}'_{t,(2)}\right)dt$$
$$+ \sum_{k=1}^{p}(B^k(t) \otimes I_m)\mathcal{X}'_{t,(3)}dW'_{k,t} + (\mathcal{X}'_{t,(1)} \otimes I_m)dW_t, \tag{2.64}$$

$$d\mathcal{Y}'_{t,(1)} = C(t)\mathcal{X}'_{t,(1)}dt + dW_t, \tag{2.65}$$

$$d\mathcal{Y}'_{t,(2)} = \left(C^{(1)}(t)\mathcal{X}'_{t,(3)} + st(I_m)\right)dt + U_m^2(I_m \otimes \mathcal{Y}'_{t,(1)})dW_t. \tag{2.66}$$

By comparing equations (2.32), (2.31) with equations (2.63),(2.65) respectively and using Lemma 2.3, we have that:
$$\mathcal{X}'_{t,(2)} = {\mathcal{X}'_{t,(1)}}^{[2]}, \tag{2.67}$$

so eq. (2.64) can be rewritten as:

$$d\mathcal{X}'_{t,(3)} = \left((A(t) \otimes I_m)\mathcal{X}'_{t,(3)} + (I_n \otimes C(t))(\mathcal{X}'_{t,(1)})^{[2]}\right)dt$$
$$+ \sum_{k=1}^{p}(B^k(t) \otimes I_m)\mathcal{X}'_{t,(3)}dW'_{k,t} + (\mathcal{X}'_{t,(1)} \otimes I_m)dW_t. \tag{2.68}$$

By comparing equations (2.65),(2.68) with equations (2.12),(2.38) respectively and using Lemma 2.3, we see that
$$\mathcal{X}'_{t,(3)} = \mathcal{X}'_{t,(1)} \otimes \mathcal{Y}'_{t,(1)} \tag{2.69}$$

and therefore eq. (2.66) rewrites as:

$$d\mathcal{Y}'_{t,(2)} = \left(C^{(1)}(t)(\mathcal{X}'_{t,(1)} \otimes \mathcal{Y}'_{t,(1)}) + st(I_m)\right) dt + U_m^2 (I_m \otimes \mathcal{Y}'_{t,(1)}) dW_t. \tag{2.70}$$

The comparison between the pair of equations (2.12),(2.39) and the pair (2.65),(2.70) shows that, by substituting X_t^0 with $\mathcal{X}'_{t,(1)}$, the following equality is verified:

$$\mathcal{Y}'_{t,(2)} = (\mathcal{Y}'_{t,(1)})^{[2]}. \tag{2.71}$$

Finally, by writing again equation (2.62) in the following way:

$$dX'_{t,(1)} = A(t)\mathcal{X}'_{t,(1)}dt + \sum_{k=1}^{p} B^k(t)\mathcal{X}'_{t,(1)}dW'_{k,t}$$
$$+ H(t)L(t)\mathbf{\Pi}\left(\mathcal{X}'_{t,(1)}\Big/\mathcal{L}_t^n\left(\mathcal{Y}'_{t,(1)}, (\mathcal{Y}'_{t,(1)})^{[2]}\right)\right) dt, \tag{2.72}$$

where $\mathcal{Y}'_{t,(1)}$ is given by (2.65), and by recalling that:

$$dX_t = A(t)X_t dt + \sum_{k=1}^{p} B^k(t)X_t dW'_{k,t}$$
$$+ H(t)L(t)\mathbf{\Pi}\left(X_t\Big/\mathcal{L}_t^n\left(Y_t, (Y_t)^{[2]}\right)\right) dt, \tag{2.73}$$
$$dY_t = C(t)X_t dt + dW_t, \tag{2.74}$$

by invoking the unicity of a strong solution, we have:

$$\mathcal{X}'_{t,(1)} = X_t, \quad \mathcal{Y}'_{t,(1)} = Y_t. \tag{2.75}$$

Since it has been proven that

$$\begin{bmatrix} \mathcal{X}'_{t,(1)} \\ \mathcal{X}'_{t,(2)} \\ \mathcal{X}'_{t,(3)} \end{bmatrix} = \begin{bmatrix} \mathcal{X}'_{t,(1)} \\ (\mathcal{X}'_{t,(1)})^{[2]} \\ \mathcal{X}'_{t,(1)} \otimes \mathcal{Y}'_{t,(1)} \end{bmatrix}, \tag{2.76}$$

$$\begin{bmatrix} \mathcal{Y}'_{t,(1)} \\ \mathcal{Y}'_{t,(2)} \end{bmatrix} = \begin{bmatrix} \mathcal{Y}'_{t,(1)} \\ (\mathcal{Y}'_{t,(1)})^{[2]} \end{bmatrix}, \tag{2.77}$$

equality (2.75) implies that:

$$\begin{bmatrix} \mathcal{X}'_{t,(1)} \\ \mathcal{X}'_{t,(2)} \\ \mathcal{X}'_{t,(3)} \end{bmatrix} = \begin{bmatrix} X_t \\ X_t^{[2]} \\ X_t \otimes Y_t \end{bmatrix}, \tag{2.78}$$

$$\begin{bmatrix} \mathcal{Y}'_{t,(1)} \\ \mathcal{Y}'_{t,(2)} \end{bmatrix} = \begin{bmatrix} Y_t \\ Y_t^{[2]} \end{bmatrix}, \tag{2.79}$$

so it results

$$\mathcal{X}'_t = \mathcal{X}_t, \quad \mathcal{Y}'_t = \mathcal{Y}_t, \tag{2.80}$$

14.

and then

$$\mu_t = \widehat{\mathcal{X}}_t = \mathbf{\Pi}\left(\mathcal{X}_t/\mathcal{L}_t^\alpha(\mathcal{Y})\right). \tag{2.81}$$

∎

Using Theorem 2.4 we have the following equations for $\widehat{\mathcal{X}}_t$:

$$d\widehat{\mathcal{X}}_t = \mathcal{A}(t)\widehat{\mathcal{X}}_t dt + \Lambda(t)\widehat{\mathcal{X}}_t dt + \mathcal{K}(t)d\nu_t, \tag{2.82}$$

$$d\nu_t = d\mathcal{Y}_t - \left(\mathcal{C}(t)\widehat{\mathcal{X}}_t + \mathcal{U}(t)\right)dt, \tag{2.83}$$

where we have substituted $d\nu_t^0$ for $d\nu_t$. Moreover, the $\mathcal{X}_t$, $\mathcal{Y}_t$ processes satisfy the equations:

$$d\mathcal{X}_t = \mathcal{A}(t)\mathcal{X}_t dt + \Lambda(t)\widehat{\mathcal{X}}_t dt + \sum_{k=1}^{2p} \mathcal{B}_k(t)d\widetilde{W}_{k,t}, \tag{2.84}$$

$$d\mathcal{Y}_t = (\mathcal{C}(t)\mathcal{X}_t + \mathcal{U}(t))\,dt + \sum_{k=1}^{2p} \mathcal{D}_k(t)d\widetilde{W}_{k,t}. \tag{2.85}$$

Since $\widehat{\mathcal{X}}_t = \begin{bmatrix} \widehat{X_t} \\ \widehat{X_t^{[2]}} \\ \widehat{X_t \otimes Y_t} \end{bmatrix}$, from (2.82) we can extract the equation for $\widehat{X}_t$:

$$d\widehat{X}_t = A(t)\widehat{X}_t dt + H(t)L(t)\widehat{X}_t dt + \mathcal{E}\mathcal{K}(t)d\nu_t, \tag{2.86}$$

with $\mathcal{E}$ as in (2.10). Now, we can rewrite the index $J(u)$:

$$J(u) = \frac{1}{2}\mathbf{E}\left\{\left(X_{t_f}\;,\;FX_{t_f}\right) + \int_{t_0}^{t_f}\left\{\left(X_t\;,\;Q(t)X_t\right) + \left(u_t\;,\;R(t)u_t\right)\right\}dt\right\}$$

as a function of $\widehat{X}_t$. As a matter of fact, one has

$$\left(X_t\;,\;Q(t)X_t\right) = \left(X_t - \widehat{X}_t\;,\;Q(t)X_t\right) + \left(\widehat{X}_t\;,\;Q(t)X_t\right)$$
$$= \left(X_t - \widehat{X}_t\;,\;Q(t)(X_t - \widehat{X}_t)\right) + \left(X_t - \widehat{X}_t\;,\;Q(t)\widehat{X}_t\right)$$
$$+ \left(\widehat{X}_t\;,\;Q(t)\widehat{X}_t\right) + \left(\widehat{X}_t\;,\;Q(t)(X_t - \widehat{X}_t)\right).$$

Taking the expectations of the latter equality:

$$\mathbf{E}\left\{\left(X_t\;,\;Q(t)X_t\right)\right\} = q(t) + \mathbf{E}\left(\widehat{X}_t\;,\;Q(t)\widehat{X}_t\right), \tag{2.87}$$

where

$$q(t) \doteq \mathbf{E}\left\{\left(X_t - \widehat{X}_t\;,\;Q(t)(X_t - \widehat{X}_t)\right)\right\} \tag{2.88}$$

and we have exploited the orthogonality of $\widehat{X}_t$ and $(X_t - \widehat{X}_t)$. The function $q(t)$ defined in (2.88) is a transformation of the error covariance matrix, and then it depends only on time t (it does not depend on u). In the same way it is proven, but a constant, that:

$$\mathbf{E}\left\{\left(X_{t_f}\;,\;FX_{t_f}\right)\right\} = \mathbf{E}\left(\widehat{X}_{t_f}\;,\;F\widehat{X}_{t_f}\right). \tag{2.89}$$

Finally, by (2.87) and (2.89), we have that $J(u)$ has, but a constant, the following expression:

$$J(u) = \frac{1}{2}\mathbf{E}\left\{ \left(\widehat{X}_{t_f} \ , \ F\widehat{X}t_f \right) + \int_{t_0}^{t_f} \left\{ \left(\widehat{X}_t \ , \ Q(t)\widehat{X}_t \right) + \left(u_t \ , \ R(t)u_t \right) \right\} dt \right\}, \qquad (2.90)$$

with $u_t = H(t)L(t)\widehat{X}_t$, and then the problem

$$\min_{u_t \in \widehat{\mathcal{U}}} J(u),$$

with the equation (2.86) as differential constraint, has the same solution of the complete information control problem: $u^o = P^o(t)\widehat{X}_t$ (see [10]).

3. Conclusions

This paper presents a new approach to solve the optimal control problem with a quadratic cost criterion for stochastic bilinear systems. The solution is obtained by searching for a quadratic closed loop optimal controller. It results in a linear map of the quadratic-optimal state estimate with the same control matrix function as in the Linear Optimal Control (LOC) problem (which is formally equal to the one solving the classical LQG control problem). The improvement in the controller performance has been proven to be entirely ascribed to the improvement in the suboptimal state-estimate which can be obtained using estimators better than the linear one.

Appendix A. Kronecker algebra

Throughout this work we have widely used Kronecker algebra. For the sake of completeness, in this appendix, we recall some important definitions, properties and rules about this subject.

Definition A.1. Let M and N be matrices of dimension $r \times s$ and $p \times q$ respectively. Then the Kronecker product $M \otimes N$ is defined as the $(r \cdot p) \times (s \cdot q)$ matrix

$$M \otimes N = \begin{bmatrix} m_{11}N & \dots & m_{1s}N \\ \dots & \dots & \dots \\ m_{r1}N & \dots & m_{rs}N \end{bmatrix},$$

where the m_{ij} are the entries of M.

Note that this kind of product is not commutative.

Definition A.2. Let M be the $r \times s$ matrix

$$M = [\, m_1 \quad m_2 \quad \dots \quad m_s \,], \qquad (A.1)$$

where m_i denotes i-th column of M, then the stack of M is the $r \cdot s$ vector

$$st(M) = [\, m_1^T \quad m_2 \quad \dots \quad m_s \,]^T. \qquad (A.2)$$

16.

Observe that a vector as in (A.2) can be reduced to a matrix M as in (A.1) by considering the inverse operation of the stack denoted by st^{-1}. With reference to the Kronecker product and the stack operation, the following properties hold (see also [13]-[15]):

$$
\begin{aligned}
(A + B) \otimes (C + D) =& A \otimes C + A \otimes D \\
&+ B \otimes C + B \otimes D, && \text{(A.3)} \\
A \otimes (B \otimes C) =& (A \otimes B) \otimes C, && \text{(A.4)} \\
(A \cdot C) \otimes (B \cdot D) =& (A \otimes B) \cdot (C \otimes D), && \text{(A.5)} \\
(A \otimes B)^T =& A^T \otimes B^T, && \text{(A.6)} \\
st(A \cdot B \cdot C) =& (C^T \otimes A) \cdot st(B), && \text{(A.7)} \\
u \otimes v =& st(v \cdot u^T), && \text{(A.8)} \\
tr(A \otimes B) =& tr(A) \cdot tr(B), && \text{(A.9)}
\end{aligned}
$$

where A, B, C, D are suitably dimensioned matrices, u, v are vectors and $tr(M)$ denotes the trace of a square matrix M. The Kronecker power of the matrix M is defined as:

$$
\begin{aligned}
M^{[0]} =& 1, \\
M^{[n]} =& M \otimes M^{[n-1]} = M^{[n-1]} \otimes M, \quad n > 0.
\end{aligned}
$$

As an easy consequence of (A.4) and (A.9) it follows

$$
tr(A^{[h]}) = \left(tr(A)\right)^h. \tag{A.10}
$$

Although the Kronecker product is not commutative, the following property holds [15, 16].

Theorem A.3. *For any given pair of matrices $A \in \mathbb{R}^{r \times s}$, $B \in \mathbb{R}^{n \times m}$, we have*

$$
B \otimes A = C_{r,n}^T (A \otimes B) C_{s,m}, \tag{A.11}
$$

where the commutation matrix $C_{u,v}$ is the $(u \cdot v) \times (u \cdot v)$ matrix such that its (h, l) entry is given by:

$$
\{C_{u,v}\}_{h,l} = \begin{cases} 1, & \text{if } l = (|h - 1|_v)u + \left(\left[\frac{h-1}{v}\right] + 1\right); \\ 0, & \text{otherwise.} \end{cases} \tag{A.12}
$$

Observe that $C_{1,1} = 1$, hence in the vector case when $a \in \mathbb{R}^r$ and $b \in \mathbb{R}^n$, (A.11) becomes

$$
b \otimes a = C_{r,n}^T (a \otimes b). \tag{A.13}
$$

By applying two times rule (A.5) it is easily obtained the following equality:

$$
\begin{aligned}
(A \otimes v_1) \cdot v_2 &= (A \otimes v_1) \cdot (v_2 \otimes 1) \\
&= (A \cdot v_2) \otimes v_1 \\
&= (A \otimes I) \cdot (v_2 \otimes v_1), \tag{A.14}
\end{aligned}
$$

where matrix A and vectors v_1 and v_2 are suitably dimensioned and I is an identity matrix.

Appendix B. The vector Ito formula in the Kronecker formalism

By using a formalism derived from the Kronecker algebra, it is possible to write a new version of the Ito formula. It has, with respect to the classical formulation, the advantage of being much more compact and allows the calculation, for a given stochastic process ϕ, of the stochastic differential of the process $\phi^{[h]}$, where $[h]$ is any integer Kronecker power.

Let $x \in \mathrm{I\!R}^n$ and F be any C^2 function in $\mathrm{I\!R}^{m \times p}$, we introduce the matrix $(d/dx) \otimes F(x)$, having dimensions $m \times (n \cdot p)$, defined as

$$\frac{d}{dx} \otimes F(x) \triangleq \left[\frac{\partial F(x)}{\partial x_1} \ \cdots \ \frac{\partial F(x)}{\partial x_n} \right], \tag{B.1}$$

where the operator d/dx is given by

$$\frac{d}{dx} \triangleq \left[\frac{\partial}{\partial x_1} \ \cdots \ \frac{\partial}{\partial x_n} \right]. \tag{B.2}$$

Note that in (B.1) the rules defining the Kroneker product between matrices (see definition A.1) are formally satisfied, provided that the "multiplication" between the differential operator $\partial/\partial x_i$ and a matrix function $F(x)$ is conventionally defined as

$$\frac{\partial}{\partial x_i} \cdot F(x) = \frac{\partial F(x)}{\partial x_i}$$

where the right hand side has the usual meaning. Similarly, we can define the operator:

$$\frac{d}{dx} \otimes \frac{d}{dx} \triangleq \left[\frac{\partial^2}{\partial x_1^2} \ \frac{\partial^2}{\partial x_1 \partial x_2} \ \cdots \ \frac{\partial^2}{\partial x_n^2} \right].$$

Also in this case the composition rule of the Kronecker product is satisfied, but the "multiplication" between the differential operators $\partial/\partial x_i$ and $\partial/\partial x_i$ had to be interpreted as resulting in the differential operator $\partial^2/\partial x_i \partial x_j$. In general, we will adopt the convention: the multiplication between a differential operator and a function F results in a function (the derivative of F) whereas the multiplication between two differential operators results in a differential operator (the second order differential operator). Obviously, this convention could be generalized in order to give a precise meaning to the quantity:

$$\frac{d^{[h]}}{dx^{[h]}} \otimes F(x),$$

for any integer $h \geq 0$. However, in this paper we are concerned at most with second order derivatives.

It is easy to recognize that, for any matrix, namely M, and for any pair of differentiable matrix functions, namely $V(x)$ and $W(x)$, having suitable dimensions, it results

$$\frac{d}{dx} \otimes (V(x) \otimes W(x)) = \left(\frac{d}{dx} \otimes V(x) \right) \otimes W(x) + V(x) \otimes \left(\frac{d}{dx} \otimes W(x) \right). \tag{B.3}$$

$$\frac{d}{dx} \otimes (MW(x)) = M \left(\frac{d}{dx} \otimes W(x) \right). \tag{B.4}$$

Moreover, the following "associative" property holds:

$$\frac{d}{dx} \otimes \frac{d}{dx} \otimes F(x) = \left(\frac{d}{dx} \otimes \frac{d}{dx} \right) \otimes F(x) = \frac{d}{dx} \otimes \left(\frac{d}{dx} \otimes F(x) \right).$$

We can now enunciate the following Lemma (see [8]), in which are used the above notation, and which is very useful in the paper.

18.

Lemma B.1. *For any integer $h \geq 1$ and $x \in \mathbb{R}^n$, it results that*

$$\frac{d}{dx} \otimes x^{[h]} = U_n^h (I_n \otimes x^{[h-1]}) \tag{B.5}$$

and for any $h > 1$:

$$\frac{d}{dx} \otimes \frac{d}{dx} \otimes x^{[h]} = O_n^h (I_{n^2} \otimes x^{[h-2]}), \tag{B.6}$$

where the matrices $C_{u,v}^T$, $u, v \in \mathbb{N}$, are the commutation matrices defined by Theorem A.3 and

$$U_n^h \triangleq \Big(\sum_{\tau=0}^{h-1} C_{n,n^{h-1-\tau}}^T \otimes I_{n^\tau} \Big), \quad O_n^h \triangleq \sum_{\tau=0}^{h-1} \sum_{s=0}^{h-2} (C_{n,n^{h-1-\tau}}^T \otimes I_{n^\tau})(I_n \otimes C_{n,n^{h-2-s}}^T \otimes I_n).$$

Corollary B.2. *The following equality holds:*

$$O_n^2 st\{I_n\} = 2 \cdot st\{I_n\}. \tag{B.7}$$

Proof. From (B.6) it follows that matrix O_n^2 satisfy the relation:

$$\frac{d}{dx} \otimes \frac{d}{dx} \otimes x^{[2]} = O_n^2.$$

By calculating the derivatives in the left hand side of the previous equation, we obtain the matrix-value of O_n^2 from which (B.7) can be directly verified. ∎

Now, we are in a position to quote the vector valued version of the Ito formula in the Kronecker formalism [8].

Theorem B.3. *Let $(X_t, \mathcal{F}_t)$ be a vector continuous semimartingale in $\mathbb{R}^n$ described by the Ito's stochastic differential:*

$$dX_t = d\beta_t + dM_t, \tag{B.8}$$

where $(\beta_t, \mathcal{F}_t)$ is an a.s. continuous bounded variation process and $(M_t, \mathcal{F}_t)$ is a square integrable martingale. Let

$$F : \mathbb{R}^n \to \mathbb{R}^p,$$

be a continuous function endowed with the first and second derivatives. Then the process $Z_t = F(X_t)$ is a square integrable semimartingale, whose differential is given by

$$dZ_t = \Big(\frac{d}{dx} \otimes F(x) \Big)_{x=X_t} dX_t + \frac{1}{2} \Big(\frac{d}{dx} \otimes \frac{d}{dx} \otimes F(x) \Big)_{x=X_t} (dM_t)^{[2]}, \tag{B.9}$$

with $(dM_t)^{[2]}$ denoting the associate quadratic variation process whose arguments are

$$(dM_t)^{[2]} = \begin{bmatrix} d < M_1, M_1 >_t \\ d < M_1, M_2 >_t \\ \vdots \\ d < M_n, M_n >_t \end{bmatrix}. \tag{B.10}$$

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