ISTITUTO DI ANALISI DEI SISTEMI ED INFORMATICA
CONSIGLIO NAZIONALE DELLE RICERCHE

F. Carravetta, G. Mavelli

A SECOND ORDER ANALYSIS FOR A CLASS OF STOCHASTIC OPTIMAL CONTROL PROBLEMS

R. 523 Marzo 2000

Francesco Carravetta – Istituto di Analisi dei Sistemi ed Informatica del CNR, viale Manzoni 30 - 00185 Rome, Italy. Email: carravetta@iasi.rm.cnr.it.

Gabriella Mavelli – Istituto di Analisi dei Sistemi ed Informatica del CNR, viale Manzoni 30 - 00185 Rome, Italy. Email: mavelli@iasi.rm.cnr.it.

ISSN: 1128–3378

Collana dei Rapporti
dell'Istituto di Analisi dei Sistemi ed Informatica, CNR

viale Manzoni 30, 00185 ROMA, Italy

tel. ++39-06-77161
fax ++39-06-7716461
email: iasi@iasi.rm.cnr.it
URL: http://www.iasi.rm.cnr.it

Abstract

The stochastic optimal control problem for systems described by Ito equations with linear drift and nonlinear diffusion is considered. The optimal feedback control law in a class of linear controllers is found. The optimality criterion is the classical quadratic one for a fixed-interval state-regulation problem. The solution of this problem is presented for both the complete/incomplete information cases.

Key words: stochastic systems, stochastic control, LQG optimal control, Kalman-Bucy filter, nonlinear filtering, separation principle, Brownian motion, Ito formula.

1. Introduction

As well known, the linear-quadratic Gaussian (LQG), finite-horizon, optimal control problem admits a feedback solution resulting in a linear transformation either of the state-process values (for the *complete* information case) or of the state-conditional-expectation with respect to the observations (for the *incomplete* information case). The reader is referred to [1], [2], [3], and references therein, for a complete explanation of the LQG control problem. The optimal solution results to be given in both the complete/incomplete information cases by the *same* matrix time-function (optimal control matrix) performing the linear map from the state or from the state-expectation. Note that the optimal control matrix is also the same matrix of the deterministic case. This result represents a particular case of the *separation theorem* [4]. The optimal control matrix results to be defined by means of a backward Riccati difference/differential equation for the discrete/continuous-time cases respectively, depending only on the system-parameters, and on the weights of the performance-index. When one is faced by the incomplete-information case, the state conditional mean can be obtained by optimal filtering. To this purpose, since the LQG problem concerns a linear Gaussian system, the Kalman filter can be used.

The aim of this paper is to present some results concerning a possible generalization of the LQG control to systems involving non-Gaussian processes. In particular, the system we are faced with in this paper is represented by a pair of stochastic Ito equations. The first one is a stochastic differential equation with linear drift, nonlinear diffusion and additive control, giving the state evolution. The second one represents the observation as an Ito process with additive Gaussian noise. The state process is then a non-Gaussian process and the LQG algorithm is not useful in this case in order to minimize the standard quadratic cost-criterion. In particular the Kalman-Bucy filter (see for instance [5], [6], [7], [8]) is unable to give the conditional state-expectation.

A way to handle the control problem, in this case, can be followed by considering a smaller *admissible set* of controls. Indeed, the solution of the LQG control problem is the minimizing point in a wide class of controls (the set of all the continuous and uniformly Lipschitz functions of the observations). In this paper we focus our attention on the *linear-optimal control* (LOC) problem, that is to say the problem of finding the control that minimizes the standard quadratic cost-criterion in the family of linear maps of the observation process. We highlight that in the LQG control problem the optimal control *results to be* a linear one whereas, when one considers the LOC problem for a nonlinear system, the result is forced to be a linear map.

The main result of this paper consists in finding the linear map solving the LOC problem for the previously mentioned class of nonlinear systems. It will result that the solution is formally the same linear map solving the LQG problem.

The paper is organized as follows. In §2 the precise setting of the problem is given. In §3 and §4 the solution of the LOC problem is presented for the complete and incomplete state information cases respectively.

2. Setting of the LOC problem

First of all we introduce the basic notations and symbols that will be used throughout the paper. $(\Omega, \mathcal{F}, P)$ will denote the basic probability triple. $\mathbf{E}\{\cdot\}$ denotes the expectation operator. $L^2(\mathcal{E})$, with $\mathcal{E}$ linear space, denotes the Hilbert space of all the $\mathcal{E}$-valued square-integrable random variables defined on $(\Omega, \mathcal{F}, P)$. Let $\mathcal{I}$ be a linear space endowed with some inner product, and $\xi, \eta \in \mathcal{I}$. We use the notation $\big(\xi \ , \ \eta\big)$ to denote the inner product between ξ and η. For any matrix M, the notation $M_{i,j}$ will be used to denote its (i,j)-entry. For the identity matrix in $\mathbb{R}^n$ it will be used the symbol I_n.

Let I be a real interval and $\xi : I \to L^2(\mathbb{R}^i)$ an $\mathbb{R}^i$-valued stochastic process; we shall denote with $\mathcal{F}_t^\xi$ the σ-algebra generated by $\{\xi_s;\ s \in I, s \leq t\}$. For a vector-valued process $\{\xi_t\}$, the notation ξ_t^j shall indicate the j-th entry. If $\{\xi_t\}$ and $\{\eta_t\}$ are two second-order scalar stochastic processes, the notation $\{\langle \xi, \eta \rangle_t\}$ will be used to indicate the mutual quadratic variation process. The notation $\langle \xi \rangle_t$ will be also used in place of $\langle \xi, \xi \rangle_t$. When ξ and η are vector-valued, the same notation $\langle \xi, \eta \rangle$ will denote the matrix whose (i,j) entry is given by $\langle \xi^i, \eta^j \rangle$.

Let $\mathcal{S} \subset L^2(\mathcal{E})$ be a linear space and $X \in L^2(\mathcal{E})$; then the symbol $\mathbf{\Pi}\big\{X/\mathcal{S}\big\}$ will denote the orthogonal projection of X onto $\mathcal{S}$. Anytime the underlying space is understood we will use the notation $\widehat{X}$ to denote the orthogonal projection. As well known, the projection $\widehat{X}$ represents the best (in the sense of the error-variance) estimate of X using estimators $\alpha \in \mathcal{S}$, and it is characterized by the following property:

$$\mathbf{E}\big\{(X - \widehat{X}, \alpha)\big\} = 0, \quad \forall \alpha \in \mathcal{S}. \tag{2.1}$$

Let us consider the following stochastic system:

$$dX_t = A(t)X_t dt + H(t)u_t dt + g(X_t)dW_t, \quad X_{t_0} = \overline{X}, \tag{2.2}$$

$$dY_t = C(t)X_t dt + dW_t, \qquad Y_{t_0} = 0, \tag{2.3}$$

where $t \in I$, $I = [t_0, t_f] \subset \mathbb{R}$, $A : I \to \mathbb{R}^{n\times n}$, $H : I \to \mathbb{R}^{n\times p}$, $C : I \to \mathbb{R}^{m\times n}$, are continuous matrix functions; $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is Borel measurable and uniformly Lipschitz on $\mathbb{R}^n$, and W is the standard m-dimensional Brownian motion. The σ-algebra generated by W will be denoted with $\mathcal{F}_t$. The initial point $\overline{X}$ has a finite second moment and is $\mathcal{F}_0$-measurable. The control function $u : I \to \mathbb{R}^p$ is assumed to be adapted to the non-decreasing family $\{\mathcal{F}_t^Y\}_{t\in I}$. We will denote with $\mathcal{L}_t^i(Y)$ the set of $\mathbb{R}^i$-valued linear transformations of $\{Y_s;\ s \in I, s \leq t\}$. One has that $\mathcal{L}_t^i(Y)$ is a closed linear subspace of $L^2(\mathbb{R}^i)$. Let $\widehat{X}_t = \mathbf{\Pi}\big\{X_t/\mathcal{L}_t^n(Y)\big\}$, the *linear-optimal estimate* of the state X solution of (2.2); then we can introduce the subspace $\overline{\mathcal{L}}_t^p(Y) \subset \mathcal{L}_t^p(Y)$ defined as:

$$\overline{\mathcal{L}}_t^p(Y) \doteq \Big\{u_t \in \mathcal{L}_t^p(Y) \ / \ \exists\, T \in \mathbb{R}^{p\times n}, \text{ such that } u_t = T\widehat{X}_t\Big\}.$$

We shall denote with $\mathcal{L}^i(Y)$ the set of functions $\xi : I \times \Omega \to \mathbb{R}^i$ such that $\xi_t \in \mathcal{L}_t^i(Y)$, for all $t \in I$. The space $\overline{\mathcal{L}}^i(Y)$ has a similar definition.

We are now in a position to give a precise definition of the LOC problem:

$$\min_{u \in \mathcal{L}^p(Y)} J(u), \tag{2.4}$$

$$J(u) = \frac{1}{2}\mathbf{E}\left\{\int_{t_0}^{t_f} \Big(\big(X_t \ , \ Q(t)X_t\big) + \big(u_t \ , \ R(t)u_t\big)\Big)\, dt + \big(X_{t_f} \ , \ FX_{t_f}\big)\right\}, \tag{2.5}$$

where $\forall t$, $Q(t) = Q(t)^T \geq 0$, $R(t) = R(t)^T > 0$, and $F = F^T \geq 0$, under the differential constraints represented by system (2.2), (2.3). In this technical report we limit ourselves to give the solution of a simplified version of the LOC problem. Indeed, we will solve the following problem:

$$\min_{u \in \overline{\mathcal{L}}^p(Y)} J(u),$$

under the contraints (2.2), (2.3), where the cost-criterion J is again given by (2.5).

3. The case of complete information on the state

In this section, we consider the particular case in which the state process solution of eq. (2.2) is available. Although this is a situation very difficult to occur in practice, it is very important in view of the general solution. Indeed, as will be shown in the subsequent section, the incomplete information case can be reduced, in some sense, to the present one. In the complete information case, since the state process is available, we are concerned with a different family of controls than $\mathcal{L}^p(Y)$, previously considered for the general case. As a matter of fact, we will consider as admissible control functions the ones resulting as a linear map of X_t. This kind of space will be denoted with $\mathcal{L}_t^p(X)$. Hence, the LOC problem for the complete information case can be stated as follows:

$$\min_{u \in \mathcal{L}^p(X)} J(u), \tag{3.1}$$

under the differential contraint represented by eq. (2.2) alone, where J is given by (2.5).

Theorem 3.1. *The solution of the problem* (3.1), *under the differential contraint* (2.2), *is the following:*

$$u_t^o = P^o(t) X_t, \tag{3.2}$$

$$P^o(t) = -R(t)^{-1} H(t)^T G(t), \tag{3.3}$$

$$\begin{aligned} &G(t_f) = F \\ &\dot{G}(t) = -A(t)^T G(t) - G(t)A(t) - Q(t) + G(t)^T H(t) R(t)^{-1} H(t)^T G(t), \end{aligned} \tag{3.4}$$

$$\begin{aligned} J(u^o) = &\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{p} \int_{t_0}^{t_f} G_{i,i}(t) \mathbf{E}\Big\{ \Big(g_{i,j}(X_t) \Big)^2 \Big\} dt \\ &+ \frac{1}{2} \Big(\operatorname{tr}\big\{ G(t_0) \Psi_X(t_0) \big\} + \Big(m_X(t_0) \ , \ G(t_0) m_X(t_0) \Big) \Big), \end{aligned} \tag{3.5}$$

where Ψ_X and m_X denote the covariance matrix and the mean value of X respectively.

Proof. In the following we omit, for short, time dependencies provided that this does not cause confusion. Let us define the functional J_t as

$$J_t(u) = \frac{1}{2} \mathbf{E} \left\{ \int_t^{t_f} \Big(\Big(X_\tau \ , \ Q X_\tau \Big) + \Big(u_\tau \ , \ R u_\tau \Big) \Big) d\tau + \Big(X_{t_f} \ , \ F X_{t_f} \Big) \right\}.$$

Since u is of the form $u = PX$ with $P : [t_0, t_f] \to \mathbb{R}^{p \times n}$ continuous, the index J_t can be rewritten as follows:

$$J_t(P) = \frac{1}{2} \mathbf{E} \left\{ \int_t^{t_f} \Big(X_\tau \ , \ (Q + P^T R P) X_\tau \Big) d\tau + \Big(X_{t_f} \ , \ F X_{t_f} \Big) \right\}.$$

Let $V(t)$ be the (unique) solution of the following equation

$$\dot{V} = -(A+HP)^T V - V(A+HP) - Q - P^T RP, \quad V(t_f) = F. \tag{3.6}$$

Note that $V = V^T$. Let us now define the function $\xi(s,\alpha)$, $s \in [t_0, t_f]$, $\alpha \in \mathbb{R}^n$, as

$$\xi(t, X_t) = \Big(X_t \ , \ V(t)X_t\Big). \tag{3.7}$$

Since

$$\int_t^{t_f} d\xi = \Big(X_{t_f} \ , \ FX_{t_f}\Big) - \Big(X_t \ , \ V(t)X_t\Big),$$

we have

$$J_t(P) = \frac{1}{2}\mathbf{E}\left\{\int_t^{t_f} \Big(X_\tau \ , \ (Q+P^T RP)X_\tau\Big)d\tau + \int_t^{t_f} d\xi + \Big(X_t \ , \ V(t)X_t\Big)\right\}. \tag{3.8}$$

By the Ito formula (see for instance [7]) it results:

$$d\xi(t,X_t) = \frac{\partial \xi}{\partial s}(t,X_t)dt + \sum_{i=1}^n \frac{\partial \xi}{\partial \alpha_i}(t,X_t)dX_t^i + \frac{1}{2}\sum_{i,j} \frac{\partial^2 \xi}{\partial \alpha_i \partial \alpha_j}(t,X_t)d\langle M^i, M^j\rangle_t, \tag{3.9}$$

where we have denoted

$$M_t = \int_{t_0}^t g(X_\tau)dW_\tau. \tag{3.10}$$

Now, from the definition of the process ξ given in (3.7), we have:

$$\frac{\partial \xi}{\partial s}(t,X_t) = \Big(X_t \ , \ \dot{V}(t)X_t\Big), \tag{3.11}$$

$$\sum_{i=1}^n \frac{\partial \xi}{\partial \alpha_i}(t,X_t)dX_t^i = \Big(dX_t \ , \ V(t)X_t\Big) + \Big(X_t \ , \ V(t)dX_t\Big), \tag{3.12}$$

$$\frac{\partial^2 \xi}{\partial \alpha_i \partial \alpha_j}(t,X_t) = 2V_{i,j}(t). \tag{3.13}$$

Substituting (3.11), (3.12), (3.13) in (3.9), and taking into account (2.2), the expression of u, and that $\langle M^i, M^j\rangle_t = 0$, for $i \neq j$, we obtain

$$\begin{aligned} d\xi(t,X) = {} & \Big(X \ , \ (\dot{V} + (A+HP)^T V + V(A+HP))X\Big)dt \\ & + 2\Big(X \ , \ Vg(X)dW\Big) + \sum_{i=1}^n V_{i,i} d\langle M^i\rangle. \end{aligned} \tag{3.14}$$

Since $\mathbf{E}\int(X \ , \ Vg(X)dW) = 0$, taking into account (3.6), the substitution of (3.14) in (3.8) results in

$$J_t(P) = \frac{1}{2}\mathbf{E}\left\{\sum_{i=1}^n \int_t^{t_f} V_{i,i}(\tau)d\langle M^i\rangle_\tau + \Big(X_t \ , \ V(t)X_t\Big)\right\}. \tag{3.15}$$

As well known (see for instance [9]), denoting by V^o the solution of (3.6) for $P = P^o$, it results: $V^o(t) = G(t)$, with G solution of the Riccati equation (3.4). Moreover, for any V solution of (3.6):

$$V(t) - G(t) \geq 0, \ \forall t \in [t_0, t_f], \ \forall P \in \mathbb{R}^{p\times n}. \tag{3.16}$$

The theorem is proven as soon as it is shown that $\forall t \in [t_0, t_f]$, $J_t(P) - J_t(P^o) \geq 0, \ \forall P \in \mathbb{R}^{p\times n}$. To this purpose, using (3.15), it results

$$J_t(P) - J_t(P^o) = \frac{1}{2}\mathbf{E}\Big\{ \sum_{i=1}^{n} \int_t^{t_f} (V_{i,i} - G_{i,i}) d\langle M^i\rangle \Big\} + \frac{1}{2}\mathbf{E}\Big\{ \Big(X \ , \ (V-G)X\Big)\Big\}. \tag{3.17}$$

Let $\Psi_X(t)$ and $m_X(t)$ be the covariance matrix and the mean value of X respectively. Moreover, let us recall that, for any pair of suitably dimensioned vectors v, w, and any matrix Φ, it results the property: $\big(v \ , \ \Phi w\big) = \mathrm{tr}\big\{\Phi w v^T\big\}$. Then we have

$$\mathbf{E}\Big\{\Big(X \ , \ (V-G)X\Big)\Big\} = \mathrm{tr}\big\{(V-G)\Psi_X\big\} + \Big(m_X \ , \ (V-G)m_X\Big).$$

Substituting the latter expression in (3.17) it results

$$\begin{aligned} J_t(P) - J_t(P^o) = {} & \frac{1}{2}\mathbf{E}\Big\{ \sum_{i=1}^{n} \int_t^{t_f} (V_{i,i}(\tau) - G_{i,i}(\tau)) d\langle M^i\rangle_\tau \Big\} \\ & + \frac{1}{2}\Big(\mathrm{tr}\big\{\big(V(t) - G(t)\big)\Psi_X(t)\big\} + \Big(m_X(t) \ , \ (V(t)-G(t))m_X(t)\Big)\Big). \end{aligned}$$

The second and third term of the previous expression are obviously nonnegative because of (3.16). By the definition of M given by (3.10), one has

$$d\langle M\rangle_t = g(X_t)g(X_t)^T dt$$

and hence

$$d\langle M^i\rangle_t = \sum_{j=1}^{n} (g_{i,j}(X_t))^2 dt. \tag{3.18}$$

Therefore, taking into account that $\mathrm{tr}\big\{V(t) - G(t)\big\} \geq 0$, one has almost surely that:

$$\sum_{i=1}^{n} \big(V_{i,i}(t) - G_{i,i}(t)\big) d\langle M^i\rangle_t \ = \sum_{i,j=1}^{n} \big(V_{i,i}(t) - G_{i,i}(t)\big)\big(g_{i,j}(X_t)\big)^2 dt \ \geq 0.$$

Finally, by computing (3.15) in $P^o(t)$ (hence $V(t) = V^o(t) = G(t)$), taking into account (3.18), and recalling that $\mathbf{E}\big\{\big(X \ , \ GX\big)\big\} = \mathrm{tr}\big\{G(\Psi_X + m_X m_X^T)\big\}$, we obtain expression (3.5) for the index $J(u^o)$. ∎

4. The case of incomplete information

As mentioned in the previous section, the case of inaccessible state process (indeed, the most common situation in practice) can be solved by means of manipulations which formally reduce the original problem to a complete information case. This will be done in the present section in an important particular case, that is when $g(X_t)$ is an affine map:

$$g_i(X_t) = B^i X_t + f^i, \quad i = 1, ..., m, \tag{4.1}$$

where we have denoted by g_i the i-th column of the matrix-valued function g.

To this purpose, we state in advance a lemma where an important feature of innovation-type processes, built up by using orthogonal projections, is shown. Let $\{a_t(\omega)\}_{t\in I}$ an $\mathbb{R}^n$-valued stochastic process.

Lemma 4.1. *Let us consider the process ξ given by*

$$d\xi_t = a_t(\omega)dt + dW_t, \tag{4.2}$$

where the Wiener process W_t is such that, $\forall\Delta > 0$, $\delta W_t \doteq W_{t+\Delta} - W_t$ is independent of $\{a_s,\ s \le t\}$.
Then, setting $\hat{a}_t(\omega) \doteq \mathbf{\Pi}\left\{a_t(\omega)/\mathcal{L}^n_t(\xi)\right\}$, and $e_t(\omega) \doteq (a_t(\omega) - \hat{a}_t(\omega))$, it results that the innovation process $\widetilde{\xi}_t$, defined by the equations

$$d\widetilde{\xi}_t = d\xi_t - \hat{a}_t(\omega)dt = e_t(\omega)dt + dW_t, \tag{4.3}$$

is a standard wide-sense Wiener (WSW) process.

Proof. First of all, we prove that $\widetilde{\xi}_t$ is an orthogonal increments process. Indeed, we can prove that, for any $t_1, t_2, t_3, t_4 \in I$, $t_1 \le t_2 \le t_3 \le t_4$, it is verified the equality:

$$\mathbf{E}\left\{\left(\widetilde{\xi}_{t_4} - \widetilde{\xi}_{t_3}\right)\left(\widetilde{\xi}_{t_2} - \widetilde{\xi}_{t_1}\right)^T\right\} = 0. \tag{4.4}$$

As a matter of fact, by (4.3) we can write:

$$\mathbf{E}\left\{\left(\widetilde{\xi}_{t_4} - \widetilde{\xi}_{t_3}\right)\left(\widetilde{\xi}_{t_2} - \widetilde{\xi}_{t_1}\right)^T\right\} = \mathbf{E}\left\{\left(\int_{t_3}^{t_4} e_\tau d\tau + W_{t_4} - W_{t_3}\right)\left(\xi_{t_2} - \xi_{t_1} - \int_{t_1}^{t_2} \hat{a}_\tau d\tau\right)^T\right\}. \tag{4.5}$$

Since we have $\xi_{t_2} - \xi_{t_1} \in \mathcal{L}^n_{t_2}(\xi) \subset \mathcal{L}^n_\tau(\xi)$ for $\tau > t_2$, $\int_{t_1}^{t_2} \hat{a}_\tau d\tau \in \mathcal{L}^n_{t_2}(\xi)$, and $e_\tau \perp \mathcal{L}^n_s(\xi)\ \forall s \le \tau$ (hence, for $i = 1, ..., n$, $e^i_\tau \perp \mathcal{L}^1_s(\xi)\ \forall s \le \tau$) it results

$$\mathbf{E}\left\{\left(\int_{t_3}^{t_4} e_\tau d\tau\right)\left(\xi_{t_2} - \xi_{t_1}\right)^T\right\} = \mathbf{E}\left\{\left(\int_{t_3}^{t_4} e_\tau d\tau\right)\left(\int_{t_1}^{t_2} \hat{a}_s ds\right)^T\right\} = 0. \tag{4.6}$$

Using (4.6), eq. (4.5) becomes

$$\begin{aligned}\mathbf{E}\left\{\left(\widetilde{\xi}_{t_4} - \widetilde{\xi}_{t_3}\right)\left(\widetilde{\xi}_{t_2} - \widetilde{\xi}_{t_1}\right)^T\right\} &= \mathbf{E}\left\{(W_{t_4} - W_{t_3})\left(\widetilde{\xi}_{t_2} - \widetilde{\xi}_{t_1}\right)^T\right\} \\ &= \mathbf{E}\left\{(W_{t_4} - W_{t_3})\left(\int_{t_1}^{t_2} e_\tau d\tau + W_{t_2} - W_{t_1}\right)^T\right\}.\end{aligned} \tag{4.7}$$

Now, by hypothesis, δW_t is independent of $\{a_s,\ s \leq t\}$, and hence δW_t is independent of $\{\xi_s,\ s \leq t\}$. Moreover, since the error e_t is a Borel function of $\{a_t, \xi_s,\ s \leq t\}$ it follows that δW_t is independent of $\{e_s,\ s \leq t\}$. Hence we have:

$$\mathbf{E}\left\{(W_{t_4} - W_{t_3})\left(\int_{t_1}^{t_2} e_\tau d\tau\right)^T\right\} = 0.$$

Using this equality in (4.7) one has

$$\mathbf{E}\left\{\left(\widetilde{\xi}_{t_4} - \widetilde{\xi}_{t_3}\right)\left(\widetilde{\xi}_{t_2} - \widetilde{\xi}_{t_1}\right)^T\right\} = \mathbf{E}\left\{(W_{t_4} - W_{t_3})(W_{t_2} - W_{t_1})^T\right\} = 0,$$

that is (4.4).

In order to complete the proof, we must show that

$$\mathbf{E}\left\{\widetilde{\xi}_t\widetilde{\xi}_s^T\right\} = I_n \cdot s, \quad s \leq t. \tag{4.8}$$

For, by the vector Ito formula given in [8, §5], one has:

$$\begin{aligned} d\widetilde{\xi}_t^{[2]} &= U_n^2(I_n \otimes \widetilde{\xi}_t)d\widetilde{\xi}_t + \frac{1}{2}O_n^2\mathrm{st}\{I_n\}dt \\ &= U_n^2(I_n \otimes \widetilde{\xi}_t)e_t dt + U_n^2(I_n \otimes \widetilde{\xi}_t)dW_t + \frac{1}{2}O_n^2\mathrm{st}\{I_n\}dt \\ &= U_n^2(e_t \otimes \widetilde{\xi}_t)dt + U_n^2(I_n \otimes \widetilde{\xi}_t)dW_t + \frac{1}{2}O_n^2\mathrm{st}\{I_n\}dt, \end{aligned} \tag{4.9}$$

where the reader is referred to [8] for the definition of the matrices $U_\cdot^\cdot$ and $O_\cdot^\cdot$. Taking the expectations of (4.9) it results

$$d\mathbf{E}\left\{\widetilde{\xi}_t^{[2]}\right\} = U_n^2\mathbf{E}\left\{(e_t \otimes \widetilde{\xi}_t)\right\}dt + \frac{1}{2}O_n^2\mathrm{st}\{I_n\}dt. \tag{4.10}$$

Taking into account that: $e_t^i \perp \mathcal{L}_t^1(\xi)$, for $i = 1, ..., n$ and $\hat{a}_\tau \in \mathcal{L}_\tau^n(\xi) \subset \mathcal{L}_t^n(\xi)$, it results $\mathbf{E}\{e_t \otimes \xi_t\} = 0$, $\mathbf{E}\{e_t \otimes \hat{a}_\tau\} = 0$, $0 \leq \tau \leq t$. Then, one has

$$\mathbf{E}\left\{e_t \otimes \widetilde{\xi}_t\right\} = \mathbf{E}\left\{e_t \otimes \left(\xi_t - \int_0^t \hat{a}_\tau d\tau\right)\right\} = \mathbf{E}\left\{e_t \otimes \xi_t\right\} - \int_0^t \mathbf{E}\left\{e_t \otimes \hat{a}_\tau\right\}d\tau = 0. \tag{4.11}$$

It is possible to prove that $O_n^2\mathrm{st}\{I_n\} = 2 \cdot \mathrm{st}\{I_n\}$. Using this one and substituting (4.11) in (4.10), we have

$$d(\mathbf{E}\left\{\widetilde{\xi}_t^{[2]}\right\}) = \mathrm{st}\{I_n\}dt.$$

Taking the inverse-stack in both sides of previous equation results in: $\mathbf{E}\left\{\widetilde{\xi}_t\widetilde{\xi}_t^T\right\} = I_n \cdot t$. From this, and taking into account (4.4), it follows that:

$$\mathbf{E}\left\{\widetilde{\xi}_t\widetilde{\xi}_s^T\right\} = \mathbf{E}\left\{\left(\widetilde{\xi}_t - \widetilde{\xi}_s\right)\widetilde{\xi}_s^T\right\} + \mathbf{E}\left\{\widetilde{\xi}_s\widetilde{\xi}_s^T\right\} = I_n \cdot s.$$

∎

Theorem 4.2. *The solution of the linear-optimal control problem:*

$$\min_{u_t \in \overline{\mathcal{L}}_t^p(Y)} J(u), \qquad t \in [t_0, t_f] \tag{4.12}$$

$$J(u) = \frac{1}{2}\mathbf{E}\left\{ \Big(X_{t_f}\ ,\ FXt_f\Big) + \int_{t_0}^{t_f} \left\{ \Big(X_t\ ,\ Q(t)X_t\Big) + \Big(u_t\ ,\ R(t)u_t\Big) \right\} dt \right\} \tag{4.13}$$

$$dX_t = A(t)X_t dt + H(t)u_t dt + g(X_t)dW_t, \quad X_{t_0} = \overline{X} \tag{4.14}$$

where $g(X_t)$ is of the form (4.1), *is given by:*

$$u_t^o = P^o(t)\widehat{X}_t \tag{4.15}$$

with $P^o(t)$ given by (3.3), *provided that the state-covariance $\Psi_X(t)$ is nonsingular for any $t \in I$.*

Proof. First of all, note that, as shown in [8, Theorem 4.1], eq. (4.14) is equivalent to

$$dX_t = A(t)X_t dt + H(t)u_t dt + \widetilde{F}(t)d\widetilde{W}_t, \tag{4.16}$$

where $\widetilde{F}$ is the following block-matrix:

$$\widetilde{F}(t) = \Big[\widetilde{F}^1(t)\ \vdots\ \dots\ \vdots\ \widetilde{F}^m(t)\ \vdots\ \widetilde{f}^1(t)\ \vdots\ \dots\ \vdots\ \widetilde{f}^m(t) \Big], \tag{4.17}$$

$\widetilde{F}^k(t) \in \mathrm{IR}^{n\times\rho_k}$, $\widetilde{f}^k(t) \in \mathrm{IR}^n$, for $k = 1, ..., m$ where

$$\rho_k \doteq \mathrm{rank}\left\{ B^k\Psi_X(t)B^{k^T} \right\},$$

$$\widetilde{F}^k(t) \doteq \left(B^k\Psi_X(t)B^{k^T} \right)^{(\frac{1}{2})}, \quad \widetilde{f}^k(t) \doteq B^k\mathbf{E}\{X_t\} + f^k.$$

$\widetilde{W}_t$ is a WSW process given by $\widetilde{W}_t{}^T = [\widetilde{W}_t^{1T}\ \dots\ \widetilde{W}_t^{mT}\ W_t{}^T]$, where $\widetilde{W}_t^k \in \mathrm{IR}^{\rho_k}$, $k = 1, ..., m$ are mutually uncorrelated standard WSW processes.

Let us consider the *uncontrolled* system:

$$dX_t^0 = A(t)X_t^0 dt + \widetilde{F}(t)d\widetilde{W}_t, \quad X_{t_0}^0 = \overline{X} \tag{4.18}$$

$$dY_t^0 = C(t)X_t^0 dt + dW_t, \qquad Y_{t_0}^0 = 0. \tag{4.19}$$

Denoting with $\widehat{X}_t^0 = \mathbf{\Pi}\left\{X_t^0/\mathcal{L}_t^n(Y^0)\right\}$ the linear-optimal estimate, we have:

$$d\widehat{X}_t^0 = A(t)\widehat{X}_t^0 dt + K(t)d\nu_t^0, \quad \widehat{X}_0^0 = \mathbf{E}\{\overline{X}\} \tag{4.20}$$

where $d\nu_t^0 = dY_t^0 - C(t)\widehat{X}_t^0 dt$. Let $L(t)$ be a suitably dimensioned matrix function, and m_t be the solution of the equation:

$$dm_t = A(t)m_t dt + H(t)L(t)m_t dt + K(t)d\nu_t^0, \quad m_0 = \mathbf{E}\{\overline{X}\} \tag{4.21}$$

and consider the following system forced by m_t:

$$dX_t = A(t)X_t dt + H(t)L(t)m_t dt + \widetilde{F}(t)d\widetilde{W}_t, \quad X_{t_0} = \overline{X} \tag{4.22}$$

$$dY_t = C(t)X_t dt + dW_t, \qquad Y_{t_0} = 0. \tag{4.23}$$

We will show that $m_t = \mathbf{\Pi}\left\{X_t/\mathcal{L}_t^n(Y)\right\}$ (i.e. $m_t \equiv \widehat{X}_t$). To this purpose, first of all let us prove that

$$\mathcal{L}_t^n(Y) = \mathcal{L}_t^n(Y^0). \tag{4.24}$$

From (4.19), (4.23), (4.22) and (4.18), it results:

$$Y_t - Y_t^0 = \int_{t_0}^{t} C(\tau)(X_\tau - X_\tau^0)d\tau, \tag{4.25}$$

$$X_\tau = \Phi(\tau,t_0)\overline{X} + \int_{t_0}^{\tau} \Phi(\tau,s)H(s)L(s)m_s ds + \int_{t_0}^{\tau} \Phi(\tau,s)\widetilde{F}(s)d\widetilde{W}_s, \tag{4.26}$$

$$X_\tau^0 = \Phi(\tau,t_0)\overline{X} + \int_{t_0}^{\tau} \Phi(\tau,s)\widetilde{F}(s)d\widetilde{W}_s, \tag{4.27}$$

hence

$$Y_t - Y_t^0 = \int_{t_0}^{t} C(\tau)\int_{t_0}^{\tau} \Phi(\tau,s)H(s)L(s)m_s ds d\tau. \tag{4.28}$$

Since, by (4.21), $m_s \in \mathcal{L}_s^n(\nu^0) = \mathcal{L}_s^n(Y^0)$ one has

$$Y_t \in \mathcal{L}_t^m(Y^0). \tag{4.29}$$

Counterwise, since from Lemma 4.1, $d\nu_t = dY_t - C(t)\widehat{X}_t dt$ is a standard WSW process, we can replace ν^0 in (4.21):

$$dm_t = A(t)m_t dt + H(t)L(t)m_t dt + K(t)d\nu_t,$$

from which we see that $m_t \in \mathcal{L}_t^n(\nu) = \mathcal{L}_t^n(Y)$. Then, again from (4.28), it follows that $Y_t^0 \in \mathcal{L}_t^m(Y)$. The latter, together with (4.29) imply (4.24). We now show that $m_t = \mathbf{\Pi}\left\{X_t/\mathcal{L}_t^n(Y^0)\right\}$. As a matter of fact, from (4.21) and (4.20), we have:

$$m_t = \Phi(t,t_0)\mathbf{E}\{\overline{X}\} + \int_{t_0}^{t} H(\tau)L(\tau)m_\tau d\tau + \int_{t_0}^{t} K(\tau)d\nu_\tau^0 \tag{4.30}$$

$$\widehat{X}_t^0 = \Phi(t,t_0)\mathbf{E}\{\overline{X}\} + \int_{t_0}^{t} K(\tau)d\nu_\tau^0, \tag{4.31}$$

hence, defining $U_t = \int_{t_0}^t H(\tau)L(\tau)m_\tau d\tau$, it results

$$m_t = \widehat{X}_t^0 + U_t. \tag{4.32}$$

Similarly, from (4.18) and (4.22), it results:

$$X_t = X_t^0 + U_t. \tag{4.33}$$

Using (4.32) and (4.33) one has, for any linear map $\Lambda_t : \mathbf{C}([t_0,t];\mathbb{R}^m) \to \mathbb{R}^n$ (where $\mathbf{C}([t_0,t];\mathbb{R}^m)$ denote the space of $\mathbb{R}^m$-valued continuous functions on the interval $[t_0,t]$):

$$\mathbf{E}\left\{\Big(X_t - m_t\ ,\ \Lambda_t(Y^0)\Big)\right\} = \mathbf{E}\left\{\Big(X_t^0 - \widehat{X}_t^0\ ,\ \Lambda_t(Y^0)\Big)\right\} = 0, \tag{4.34}$$

where the orthogonality principle has been exploited. Relation (4.34) implies $m_t = \mathbf{\Pi}\left\{X_t/\mathcal{L}_t^n(Y^0)\right\}$ and then, from (4.24), it results:

$$m_t = \mathbf{\Pi}\left\{X_t/\mathcal{L}_t^n(Y)\right\} = \widehat{X}_t. \tag{4.35}$$

From (4.35), we can rewrite (4.21), (4.22), (4.23) as:

$$\begin{aligned} dX_t &= A(t)X_t dt + H(t)L(t)\widehat{X}_t dt + \widetilde{F}(t)d\widetilde{W}_t, & X_{t_0} &= \overline{X} & (4.36)\\ dY_t &= C(t)X_t dt + dW_t, & Y_{t_0} &= 0 & (4.37)\\ d\widehat{X}_t &= A(t)\widehat{X}_t dt + H(t)L(t)\widehat{X}_t dt + K(t)d\nu_t, & \widehat{X}_{t_0} &= \mathbf{E}\{\overline{X}\}. & (4.38)\end{aligned}$$

Now, we can rewrite the index

$$J(u) = \frac{1}{2}\mathbf{E}\left\{\left(X_{t_f}\ ,\ FXt_f\right) + \int_{t_0}^{t_f}\left\{\left(X_t\ ,\ Q(t)X_t\right) + \left(u_t\ ,\ R(t)u_t\right)\right\}dt\right\}$$

as a function of $\widehat{X}$. As a matter of fact, one has

$$\begin{aligned}\left(X_t\ ,\ Q(t)X_t\right) &= \left(X_t - \widehat{X}_t\ ,\ Q(t)X_t\right) + \left(\widehat{X}_t\ ,\ Q(t)X_t\right)\\ &= \left(X_t - \widehat{X}_t\ ,\ Q(t)(X_t - \widehat{X}_t)\right) + \left(X_t - \widehat{X}_t\ ,\ Q(t)\widehat{X}_t\right)\\ &+ \left(\widehat{X}_t\ ,\ Q(t)\widehat{X}_t\right) + \left(\widehat{X}_t\ ,\ Q(t)(X_t - \widehat{X}_t)\right).\end{aligned}$$

Taking the expectations of the latter equality:

$$\mathbf{E}\left\{\left(X_t\ ,\ Q(t)X_t\right)\right\} = q(t) + \left(\widehat{X}_t\ ,\ Q(t)\widehat{X}_t\right), \tag{4.39}$$

where

$$q(t) \doteq \mathbf{E}\left\{\left(X_t - \widehat{X}_t\ ,\ Q(t)(X_t - \widehat{X}_t)\right)\right\} \tag{4.40}$$

and we have exploited the orthogonality of $\widehat{X}_t$ and $(X_t - \widehat{X}_t)$.
The function $q(t)$ defined in (4.40) is a transformation of the error covariance matrix, and then it depends only on time t (it does not depend on u). In the same way it is proven, but a constant, that:

$$\mathbf{E}\left\{\left(X_{t_f}\ ,\ FX_{t_f}\right)\right\} = \left(\widehat{X}_{t_f}\ ,\ F\widehat{X}_{t_f}\right). \tag{4.41}$$

Finally, by (4.39) and (4.41), we have that $J(u)$ has, but a constant, the following expression:

$$J(u) = \frac{1}{2}\mathbf{E}\left\{\left(\widehat{X}_{t_f}\ ,\ F\widehat{X}t_f\right) + \int_{t_0}^{t_f}\left\{\left(\widehat{X}_t\ ,\ Q(t)\widehat{X}_t\right) + \left(u_t\ ,\ R(t)u_t\right)\right\}dt\right\}. \tag{4.42}$$

Taking into account that (4.14) is equivalent to (4.16), and that $\widehat{X}$ is solution of (4.38), where $d\nu_t$ is a standard WSW process, the incomplete information control problem (4.12)-(4.14) is reduced to a complete information control problem by using (4.42) as cost criterion and (4.38) as differential constraint. The minimizing matrix function $L(t)$ is then given by the formula (3.3) of the complete information case. ∎

5. Conclusions

The LOC problem has been defined and solved for the class of systems with a bilinear state equation and linear measurement equation. Theorem 3.1 gives the solution of the LOC problem in the complete information case that is when the state-process is accessible. Theorem 3.1 holds for the larger class of stochastic systems described by the eqs. (2.2), (2.3), where a generic nonlinear (uniformly Lipschitz) diffusion coefficient is considered. It results that the equations solving the LOC problem are formally equal to the ones solving the classical LQG control problem. The most important case of incomplete information is treated in §4. In Lemma 4.1 a result concerning innovation-like representations that use orthogonal projections instead of conditional-means is proven. It represents the basic tool in proving the main Theorem (Theorem 4.2) that gives the solution of the LOC problem in the incomplete information case, for a bilinear diffusion. Similarly to the LQG problem, we have that the algorithm solving the LOC problem for incomplete state-information is formally the same of the complete information case, where the state is replaced by its optimal linear estimate. It means that a separation-like principle holds even in the present suboptimal control problem.

REFERENCES

[1] D.P. Bertsekas. "Dynamic Programming and Stochastic Control", vol. 125 Academic Press, New York, 1976.

[2] H.J. Kushner. "Stochastic Stability and Control", vol. 33, Academic Press, New York, 1967.

[3] W. Fleming, R. Rishel. "Deterministic and Stochastic Optimal Control", Springer Verlag, Berlin, 1975.

[4] W. M. Wonham, "On the Separation Theorem of Stochastic Control", *SIAM J. Control*, vol. 6, no.2, pp. 312-326, 1968.

[5] R. E. Kalman and R. S. Bucy, "New Results in Linear Filtering and Prediction Theory", *TRANS. ASME Ser. D. J. Basic Eng.*, vol. 83, pp. 95-108, 1961.

[6] R.S. Liptser, A.N. Shiryayev. "Statistics of Random Processes", vols 1,2, Springer Verlag, New York, 1978.

[7] G. Kallianpur. "Stochastic Filtering Theory", Springer Verlag, New York, 1980.

[8] F. Carravetta, A. Germani, M.K. Shuakayev, "A New Suboptimal Approach to the Filtering Problem for Bilinear Stochastic Differential Systems", *SIAM J. Control Optim.*, vol. 38, no.4, pp. 1171-1203, April 2000.

[9] C. Bruni, D. Iacoviello, "Filtraggio e Controllo Ottimi, Problemi LQG", disp. Corso di Perf. in "Metodi Matematici per l'Analisi, il Controllo e l'Ottimizzazione dei Sistemi", Università degli Studi "La Sapienza", Roma, 1998.