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FORMATIVE PROCESSES WITH
APPLICATIONS TO THE DECISION PROBLEM
IN SET THEORY: I. POWERSET AND
SINGLETON OPERATORS

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Abstract

This paper introduces formative processes, composed by transitive partitions. Given a family $\mathcal{F}$ of sets, a formative process ending in the Venn partition Σ of $\mathcal{F}$ is shown to exist. Sufficient criteria are also singled out for a transitive partition to model (via a function from set variables to unions of sets in the partition) all set-literals modeled by Σ. On the basis of such criteria a procedure is designed that mimics a given formative process by another where sets have finite rank bounded by $C(|\Sigma|)$, with C a specific computable function.

As a by-product, one of the core results on decidability in computable set theory is rediscovered, namely the one that regards the satisfiability of unquantified set-theoretic formulae involving Boolean operators, the singleton-former, and the powerset operator.

The method described can be extended to solve the satisfaction problem for broader fragments of set theory.

Key words: Satisfaction problem, decidability, Zermelo-Fraenkel set theory, verification of set-based specifications.

The author was one of the researchers working on resolution type systems who "made the switch". It was in trying to prove a rather simple theorem in set theory by paramodulation and resolution, where the program was experiencing a great deal of difficulty, that we became convinced that we were on the wrong track. (From [2], p.2.[1])

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[1]The 'rather simple theorem' to which Bledsoe is referring is $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B)$.

1. Introduction

The availability of sets and functions in high-level specification languages [20, 21, 15, 13] is extremely useful, and enhancements to the services offered by such languages critically depend on the development of specialized proof techniques. In particular, decision algorithms for portions of set theories (cf., e.g., [6]) are essential ingredients of a platform that either assists one in verifying the correctness of detailed efficient algorithm designs [17] (possibly through theory-dependent automated deduction [19]), or directly handles set constraints [12, 14] in an advanced declarative programming language.

Some of the decision algorithms for set contexts are, in the very prototypical form in which they were originally conceived, hard to understand, hard to implement, and even harder to extend with the treatment of set-theoretical constructs that were not built into them from the outset. This is why we deem it useful to re-examine in this paper one of them, [4], in sight of generalizations that are along the way (see [8]).

It is quite plausible, to mention one of these envisaged generalizations, that the unionset operator can be added without disrupting decidability to the fragment of set theory that will be treated in this paper, which comprises Boolean operators and the singleton and powerset formers. Nobody, though, dared to fill in the details of this unified decidability result, because the combinatorial difficulties rapidly became unmanageable, curtailing the growth of computable set theory after the big harvest season of the eighties, when the main breakthroughs were attained (cf. [7]).

A consolidation of the known part of computable set theory is essential not only to promote new discoveries on decidability, but also to convert the theoretical results into technological advances in the field of automated reasoning. Even the most basic layer of automated set reasoning, the so-called *multi-level syllogistic*, benefited from being revisited under a tableaux-based approach, which rendered its implementation far more efficient (cf. [9, 10]).

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The Venn partition associated with a family $\mathcal{F}$ of sets (see Sec.2) is the most fundamental model-theoretic aid to approach the decision problem, already in connection with Boolean rings/algebras where sets/classes are conceived of as flat families of individuals. It still plays a most basic role (see Sec.3) when one comes to genuine theories of sets, where sets are nested one inside another. Beyond the treatment of Boolean operators, conceptual tools more sophisticated than Venn partitions and diagrams become necessary: *formative processes*, to be discussed in Sections 4–6, are the next important device.

Formative processes are special sequences whose components are each associated with a partition Σ_μ, transitive in the sense that $\bigcup\bigcup \Sigma_\mu \subseteq \bigcup \Sigma_\mu$. Every family $\mathcal{F}$ of sets has a formative process, whose length is a successor ordinal (usually transfinite) $\xi + 1$ and which ends in a partition Σ_ξ refining the Venn partition of $\mathcal{F}$.

Mimicking the formative process of a family $\mathcal{F}$ of sets (see Sections 7–10) is the key for retaining the essential features of $\mathcal{F}$ while replacing it with a family much easier to describe. In the paradigmatic case to be studied, we will start with an $\mathcal{F}$ of finite cardinality and will end up with a family where sets are hereditarily finite and have a rank bounded by $C(|\mathcal{F}|)$, with C a specific computable function (see Sec.11). This will offer us the key for a satisfaction algorithm for the above-said fragment of the set-theoretic language.

Basic notation and presupposed notions

The reader is assumed to be familiar with usual set-theoretic notation. Among others, we denote by $[t_1, \ldots, t_N]$ an ordered N-tuple; moreover, by R^{-1} and $R \circ S$, where R and S are dyadic relations (i.e., sets or classes formed by ordered pairs), we denote the relations

$$R^{-1} \quad =_{\mathrm{Def}} \quad \{[Y, X] \mid [X, Y] \in R\},$$
$$R \circ S \quad =_{\mathrm{Def}} \quad \{[X, Y] \mid \exists z([X, z] \in R \wedge [z, Y] \in S)\}.$$

Given a set or class S and a function f, ι_S denotes the identity relation on S, $\mathscr{P}(S)$ denotes the collection of all subsets of S, and $f[S]$, $\mathsf{dom}\, f$ denote the multi-image of S under f and the domain of f respectively:

$$\iota_S \quad =_{\mathrm{Def}} \quad \{[Y, Y] \mid Y \in S\}, \qquad\qquad \mathscr{P}(S) \quad =_{\mathrm{Def}} \quad \{Y \mid (\forall z \in Y) z \in S\},$$
$$f[S] \quad =_{\mathrm{Def}} \quad \{Y \mid (\exists z \in S) Y = f(z)\}, \qquad \mathsf{dom}\, f \quad =_{\mathrm{Def}} \quad \{Y \mid \exists z\, [Y, z] \in f\}.$$

To describe a function $f = \{[X, Y_X] \mid X \in Z)\}$, we will interchangeably use the notation $f = \{Y_X\}_{X \in Z}$ and the notation $X \xmapsto{f} Y_X$ $(X \in Z)$; by Y^Z we will denote the collection $\{\{y_x\}_{x \in Z} \mid (\forall x \in Z)(y_x \in Y)\}$ of all functions from Z into Y.

As is customary, we call:

PREORDER on S, a relation $\trianglelefteq \subseteq S \times S$ meeting the conditions $\iota_S \subseteq \trianglelefteq$ and $\trianglelefteq \circ \trianglelefteq \subseteq \triangleleft$;

EQUIVALENCE RELATION on S, a preorder $\sim$ on S such that $\sim^{-1} \subseteq \sim$;

(PARTIAL) ORDERING on S, a preorder $\leq$ on S such that $\leq \cap \leq^{-1} \subseteq \iota_S$;

LINEAR ORDERING on S, a partial ordering $\leq$ on S such that $\leq \cup \leq^{-1} = S \times S$;

WELL-ORDERING on S, a linear ordering $\leq$ on S with respect to which every non-null set $X \subseteq S$ has a minimum: $\qquad (\forall X \subseteq S)(X \neq \emptyset \rightarrow (\exists m \in X)(\forall v \in X) m \leq v).$

To briefly indicate that two sets x, y are *not* disjoint, i.e., $x \cap y \neq \emptyset$, we employ the notation $x \ni\in y$.

A very quick recollection of basic notions on ordinal numbers follows (for a deeper presentation, see [16]).

Definition 1. *A set T is said to be* TRANSITIVE *if $T \subseteq \mathscr{P}(T)$ or, equivalently, if $\bigcup T \subseteq T$.*
A set μ is said to be an ORDINAL *(number) if μ is transitive and is linearly ordered (and hence well-ordered) by the relation $\in \cup \iota_\mu$.* $\qquad\qquad \square$

As is well known, $\in \cup \iota_\mathcal{O}$ behaves as a well-ordering on the class $\mathcal{O}$ of all ordinals, it coincides there with $\subseteq$, and hence is $\bigcup$-*inductive* in the following sense:

> For every non-null set $C \subseteq \mathcal{O}$ which is linearly ordered by $\in \cup \iota_C$ (with $\in$ restricted to C), it turns out that $\bigcup C \in \mathcal{O}$ and $\bigcup C$ is the smallest of all ordinals m for which $(\forall v \in C)(v(\in \cup \iota_{\iota_\mathcal{O}})m)$ holds.[2] In short, $\bigcup C = \sup C$; and, if $\bigcup C \in C$, then $\bigcup C = \max C$.

One reason to be interested in ordinals is the following fundamental theorem:

Theorem 1. *Let $\trianglelefteq$ be a well-ordering on the set x. Then there exist, and are uniquely determined, an ordinal ξ and a function $f \in x^\xi$ such that $f[\xi] = x$ holds and, for any pair $\nu, \mu < \xi$ of ordinals:*
$$f\nu \neq f\mu \text{ holds when } \nu \neq \mu, \text{ and moreover } f\nu \trianglelefteq f\mu \text{ when } \nu \leq \mu.$$
$\qquad\qquad \square$

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[2]It is customary to denote the relation $\in \cup \iota_\mathcal{O}$ between ordinals simply by $\leq$ and membership between ordinals by $<$. We will adhere to such a convention throughout the paper.

By virtue of the axiom of choice, a well-ordering can be imposed on any set. Therefore the following definition makes sense:

Definition 2. *The* CARDINALITY *of a set* x, *to be denoted* $|x|$, *is the least ordinal* ν *such that there exists a function* $f \in x^\nu$ *with* $f[\nu] = x$. *A* CARDINAL *(number) is an ordinal* μ *such that* $\mu = |\mu|$. □

Example 1. *Natural* numbers, intended *à la* von Neumann, which is by the rules
$$0 =_{\mathrm{Def}} \emptyset, \qquad i + 1 =_{\mathrm{Def}} i \cup \{i\},$$
constitute the initial segment of the class of ordinals; their set, $\omega =_{\mathrm{Def}} \{0, 1, 2, 3, \dots\}$, is the first ordinal which exceeds all natural numbers, often denoted $\aleph_0$.

Even for ordinals (such as ω) which are not natural numbers, it is convenient to assign the meaning just indicated to the increment operation '+1': we will hence have, among ordinals, $\omega + 1$, $\omega + 1 + 1$, etc. The ordinals of the form $\mu + 1$ are called *successors*; all others, save 0, are called *limit ordinals*. The latter comprise ω, $\omega + \omega$, $\underbrace{\omega + \cdots + \omega}_{\omega \text{ times}}$, etc.

(we are making an appeal to the intuition of the reader).

All elements of $\omega + 1$ are cardinal numbers; but $\omega + 1$ itself is *not* such a number. □

Definition 3. *By* ξ-SEQUENCE, *where* ξ *is an ordinal, one means a function* $\{Y_\mu\}_{\mu \in \xi}$, *usually denoted* $(Y_\mu)_{\mu < \xi}$, *whose domain is* ξ.

By SEQUENCE *(without indication of* ξ*), one means* ω-*sequence.* □

In the traditional conception of sets developed by Zermelo, Fraenkel, Skolem, and von Neumann, one has that a function rk exists that is univocally defined on all sets through the recursive rule
$$\mathsf{rk}\, X = \bigcup \{\, (\mathsf{rk}\, Y) + 1 \mid Y \in X \,\};$$
this function associates an ordinal to each set X, and is called the RANK function. Thanks to the axiom of choice, a well-ordering $\trianglelefteq$ can be imposed to any given set x so that
$$y \trianglelefteq z \quad \text{when} \quad \mathsf{rk}\, y < \mathsf{rk}\, z \text{ and } y, z \in x.$$

The class $\mathcal{V}_\mu$ of all sets whose rank is smaller than μ is, for every ordinal μ, a *set*, which is easily recognized to be transitive. Among these sets, one has the family $\mathcal{V}_\omega$ of the HEREDITARILY FINITE sets, which are those sets that are *finite* and whose elements, elements of elements, etc., all are finite. Notice that $\mathcal{V}_{\mu+1} = \mathscr{P}(\mathcal{V}_\mu)$, for every ordinal μ.

Example 2. Putting
$$\emptyset^n =_{\mathrm{Def}} \underbrace{\{\{ \dots \{}_{n} \emptyset \underbrace{\} \dots \}\}}_{n},$$
we can briefly describe the initial stages of von Neumann's *cumulative hierarchy* as follows:
$$
\begin{aligned}
\mathcal{V}_0 &= \emptyset = 0\,, \\
\mathcal{V}_1 &= \{\emptyset\} = \emptyset^1 = 1\,, \\
\mathcal{V}_2 &= \{\emptyset, \emptyset^1\} = 2\,, \\
\mathcal{V}_3 &= \{\emptyset, \emptyset^1, \emptyset^2, \mathcal{V}_2\} = 3 \cup \emptyset^3\,, \\
\mathcal{V}_4 &= \{\emptyset, \emptyset^1, \emptyset^2, \emptyset^3, \{\emptyset, \emptyset^1\}, \{\emptyset, \emptyset^2\}, \{\emptyset^1, \emptyset^2\}, \dots, \mathcal{V}_3\} \\
&= \bigcup_{b \in \{0,3\}} \left(\emptyset^{b+1} \cup \{3 \cup \emptyset^b\} \cup \{\{i,j\} \cup \emptyset^b \mid j \in 3 \wedge i \in j + 1\} \right), \\
&\vdots \quad \vdots \quad \vdots
\end{aligned}
$$
On this basis, one observes that
$$
\begin{aligned}
\mathsf{rk}\, \emptyset &= 0, &\text{indeed}& \quad \emptyset \in \mathcal{V}_1 \text{ and } \emptyset \notin \mathcal{V}_0; \\
\mathsf{rk}\, \emptyset^2 &= 2, &\text{indeed}& \quad \emptyset^2 \in \mathcal{V}_3 \text{ and } \emptyset^2 \notin \mathcal{V}_2; \\
\mathsf{rk}\, \{\emptyset, \emptyset^2\} &= 3, &\text{indeed}& \quad \{\emptyset, \emptyset^2\} \in \mathcal{V}_4 \text{ and } \{\emptyset, \emptyset^2\} \notin \mathcal{V}_3.
\end{aligned}
$$
□

2. Transitive partitions and Venn partition of a set

Definition 4. *A family Σ of pairwise disjoint sets, none of which is $\emptyset$, is said to be a* PARTITION *(of $\bigcup \Sigma$); its members are called* BLOCKS *of Σ.*
The set $\varsigma_\Sigma =_{Def} \mathscr{P}(\bigcup \Sigma) \setminus \bigcup \Sigma$ (to be often denoted simply as ς) will occasionally be treated as a block of the partition too; then it is called the OUTER BLOCK *of Σ.* □

As is well known, the function
$$\Sigma \overset{\sim_\Sigma}{\longmapsto} \{[X,Y] \mid (\exists\, b \in \Sigma)(X \in b \wedge Y \in b)\}$$
establishes a one-to-one correspondence between the partitions of a given set S and the equivalence relations on S.

A useful relation $\sqsubseteq$ on $\mathscr{P}(\mathscr{P}(S))$ is defined by putting
$$\mathcal{B} \sqsubseteq \mathcal{A} \text{ iff}_{\text{Def}} \ (\forall\, a \in \mathcal{A})(\exists\, B \subseteq \mathcal{B})\, a = \bigcup B \ .$$
One reads $\mathcal{B} \sqsubseteq \mathcal{A}$ as '$\mathcal{B}$ is *finer* than $\mathcal{A}$', or as '$\mathcal{A}$ is *coarser* than $\mathcal{B}$'; this obviously is a *preorder* relation. When restricted to the set $\varpi(S)$ of all partitions of S, $\sqsubseteq$ becomes a *partial ordering*.

Definition 5. *A partition Σ is said to be* TRANSITIVE *if $\bigcup \Sigma$ is transitive as by Def.1.* □

Remarks 1. (1) It is easy to prove that $\emptyset \in T$ holds for every non-null transitive set T (whereas $\emptyset$ belongs to no partition).[3] Hence no partition but $\emptyset$ is a transitive set, and therefore no confusion can arise from the abuse of terminology of calling a partition Σ transitive when $\bigcup \Sigma$ is transitive.

(2) Notice also that saying that a partition Σ is transitive amounts to the same as saying that its outer block ς fulfills the equality $\mathscr{P}(\bigcup \Sigma) = \varsigma \cup \bigcup \Sigma$. □

The following lemma is easily proved:

Lemma 1. *Every transitive partition Σ fulfills the following conditions:*

1. $\bigcup \Sigma \in \varsigma$; *(hence ς is non-null, $\Sigma \cup \{\varsigma\}$ is a partition, and furthermore)*

2. $\Sigma \cup \{\varsigma\}$ *is a transitive partition;*

3. $\mathsf{rk}\, \sigma < \mathsf{rk}\, \varsigma$ *for every block $\sigma \in \Sigma$;*

4. $\bigcup \Gamma \in \varsigma \cup \bigcup \Sigma$ *holds for every $\Gamma \subseteq \Sigma$; hence*

5. ς *is the only block ϑ for which there is no $\Gamma \subseteq \Sigma \cup \{\varsigma\}$ that meet both $\vartheta \in \Gamma$ and $\bigcup \Gamma \in \varsigma \cup \bigcup \Sigma$.*

Proof.

1: One has $\bigcup \Sigma \notin \bigcup \Sigma$ due to the acyclicity of membership, and, trivially, $\bigcup \Sigma \in \mathscr{P}(\bigcup \Sigma)$. Thus, by the very definition of ς, $\bigcup \Sigma \in \varsigma \neq \emptyset$ and $\sigma \cap \varsigma = \emptyset$ for all $\sigma \in \Sigma$.

2: Assuming that $\bigcup \Sigma \subseteq \mathscr{P}(\bigcup \Sigma)$, we have that $\varsigma \cup \bigcup \Sigma \subseteq \mathscr{P}(\bigcup \Sigma) \subseteq \mathscr{P}(\varsigma \cup \bigcup \Sigma)$ too, thanks to the inclusion $\varsigma \subseteq \mathscr{P}(\bigcup \Sigma)$ entailed by the definition of ς.

---

[3]Let us assume that $T \neq \emptyset$ and $T \subseteq \mathscr{P}(T)$. The first hypothesis yields, by the regularity (or 'foundation') axiom of set theory, that there is a $z \in T$ such that $z \cap T = \emptyset$. Then, thanks to the second hypothesis, we get $z \subseteq T$, i.e. $z \cap T = z$, and hence $z = \emptyset \in T$.

3: Since $\sigma \subseteq \bigcup \Sigma \in \varsigma$, we have $\operatorname{rk} \sigma \leq \operatorname{rk} \bigcup \Sigma < \operatorname{rk} \varsigma$, for all $\sigma \in \Sigma$.

4: Assuming $\Gamma \subseteq \Sigma$, one gets $\bigcup \Gamma \subseteq \bigcup \Sigma$, and hence $\bigcup \Gamma \in \mathscr{P}(\bigcup \Sigma) = \varsigma \cup \bigcup \Sigma$.

5: By contradiction, assume that $\varsigma \in \Gamma \subseteq \Sigma \cup \{\varsigma\}$ and $\bigcup \Gamma \in \varsigma \cup \bigcup \Sigma$, so that $\varsigma \subseteq \bigcup \Gamma \in \sigma$ for some $\sigma \in \Sigma \cup \{\varsigma\}$, and hence $\operatorname{rk} \varsigma \leq \operatorname{rk} \bigcup \Gamma < \operatorname{rk} \sigma$, against **3**. On the other hand, if $\sigma \in \Sigma$ then $\sigma \in \{\sigma\} \subseteq \Sigma \subseteq \Sigma \cup \{\varsigma\}$, and $\bigcup \{\sigma\} \in \varsigma \cup \bigcup \Sigma$ holds by **4**. ∎

As far as the Boolean constructs $\emptyset, \cap, \backslash, \cup, =, \neq, \subseteq, \nsubseteq$ are concerned, all relevant information about a family of sets is conveyed by the following structure:

Definition 6. *Given a family $\mathcal{F}$, we call* Venn partition *of $\mathcal{F}$ the coarsest of all partitions Σ of $\bigcup \mathcal{F}$ which fulfill the condition*
$$(\forall x \in \mathcal{F})(\forall p \in \Sigma)(\, p \ni\in x \ \rightarrow \ p \subseteq x \,).$$
□

Here is perhaps the most straightforward way of determining the Venn partition $\Sigma_{\mathcal{F}}$ of $\mathcal{F}$:
$$\Sigma_{\mathcal{F}} = \big\{\, (\bigcap r) \backslash \bigcup(\mathcal{F} \backslash r) \mid r \in \mathscr{P}(\mathcal{F}) \backslash \{\emptyset\} \,\big\} \backslash \{\emptyset\}.$$
It can be shown that the same task can be solved, when $|\bigcup \mathcal{F}|$ is finite, by an algorithm based on a positive strategy having complexity $\mathcal{O}(|\bigcup \mathcal{F}|)$.

We will be interested not only in Boolean constructs but also in the operations of singleton and powerset formation, and hence we need to extract from a given $\mathcal{F}$ a more informative structure than $\Sigma_{\mathcal{F}}$:

Theorem 2. *For any family $\mathcal{F}$, there is a* transitive *partition Σ such that*

(a) for every $x \in \mathcal{F}$ there is a (unique) $\Gamma_x \subseteq \Sigma$ such that $x = \bigcup \Gamma_x$;

(b) $|\Sigma| \leq 2^{|\mathcal{F}|}$.

Proof. Let $V \supseteq \mathcal{F} \cup \{\emptyset\}$ be a transitive set (e.g., $V = \mathcal{V}_{(\operatorname{rk} \mathcal{F})+1}$; moreover it is easy to see that there exists a smallest possible V w.r.t. $\subseteq$). Then it can easily be seen that the Venn partition Σ of $\mathcal{F} \cup \{V\}$ is $\big\{ (\bigcap r) \backslash \bigcup(\mathcal{F} \backslash r) \mid r \in \mathscr{P}(\mathcal{F}) \backslash \{\emptyset\} \big\} \cup \{V \backslash \bigcup \mathcal{F}\} \backslash \{\emptyset\}$, so that clearly $\Gamma_x = \{\sigma \mid (\exists r \subseteq \mathcal{F}) (x \in r \wedge \sigma = (\bigcap r) \backslash \bigcup(\mathcal{F} \backslash r) \wedge \sigma \neq \emptyset)\}$. Moreover, $|\Sigma| \leq 2^{|\mathcal{F}|}$. ∎

3. Deciding a fragment of set theory by simulating a partition

Let us consider a function $\mathcal{M} \in \{\text{ sets }\}^{\mathcal{X}}$ defined on a collection $\mathcal{X}$ of set variables.

If someone supplied us with the Venn partition Σ of the set $\mathcal{M}[\mathcal{X}]$, while keeping $\mathcal{M}$ hidden, $\mathcal{M}$ would be traceable among the functions $v \mapsto \bigcup \Im(v)$ that biunivocally correspond to the functions $\Im \in \mathscr{P}(\Sigma)^{\mathcal{X}}$.

When $\mathcal{X}$ is finite, Σ is also finite, and hence the host of $\Im$s that may encode $\mathcal{M}$ is finite too. The endless variety of possible values for $\mathcal{M}$ hence narrows down —provided Σ is known— to a finite inventory of possibilities. More exactly, since the cardinality $|\Sigma|$ does not exceed $2^{|\mathcal{X}|} - 1$, the number $2^{|\Sigma| \cdot |\mathcal{X}|}$ of possibilities for $\Im$ cannot exceed the number $2^{(2^{|\mathcal{X}|}-1) \cdot |\mathcal{X}|}$.

It should be apparent that the latter is an overestimate of the number of possible $\mathcal{M}$s; indeed, those $\Im$s for which $\bigcup \Im[\mathcal{X}] \neq \Sigma$ should not be taken into account. The number

can, moreover, drastically decrease when we know one or more set formulae (involving no variables outside $\mathcal{X}$) that hold true in the set-valued assignment $\mathcal{M}$.

If we now consider a family $\mathcal{X}$ of set variables, along with a partition Σ and with a function $\Im \in \mathscr{P}(\Sigma)^{\mathcal{X}}$ representing the interpretation $v \mapsto \bigcup \Im(v)$, to what extent will it depend on the specificities of Σ that certain literals of the forms

$$
\begin{array}{llll}
v = w, & v \neq w, & v = \emptyset, & v = u \cup w, \\
v = u \cap w, & v = u \setminus w, & v \subseteq u, & v \not\subseteq u, \\
v \in w, & v \notin w, & v = \mathscr{P}(w), & v = \{w_0, w_1, \ldots, w_H\},
\end{array} \qquad (\dagger)
$$

where u, v, w, w_i are in $\mathcal{X}$, are true in the interpretation[4]

As regards literals of the forms $v = w$, $v \neq w$, $v = \emptyset$, $v = u \star w$ (with $\star$ in $\{\cap, \setminus, \cup\}$), $v \subseteq u$, $v \not\subseteq u$, the only feature of Σ that counts is its cardinality. In the following sense: Any other partition $\widehat{\Sigma}$ for which an injective function β of Σ onto $\widehat{\Sigma}$ exists, will continue to satisfy literals of these forms, under the interpretation $v \mapsto \bigcup \beta[\Im(v)]$.

In order to take also literals of the three forms $v \in w$, $v \notin w$, $v = \mathscr{P}(w)$ into account, we will refer to the following notion:

Definition 7. *A partition $\widehat{\Sigma}$ is said to* SIMULATE *another partition, Σ, when there is a bijection $\beta \in \widehat{\Sigma}^{\Sigma}$ such that, for $X, Y \subseteq \Sigma$,*

- $\bigcup \beta[X] \in \bigcup \beta[Y]$ *if and only if* $\bigcup X \in \bigcup Y$;

- $\bigcup \beta[X] = \mathscr{P}(\bigcup \beta[Y])$ *if* $\bigcup X = \mathscr{P}(\bigcup Y)$. $\qquad \square$

Finally, to take also literals of the form $v = \{w_1, \ldots, w_H\}$ into account, our subsequent study will take advantage of the following notion, where L indicates an upper bound for the value of H:

Definition 8. *A partition $\widehat{\Sigma}$ is said to* L-SIMULATE *another partition, Σ, when there is a bijection $\beta \in \widehat{\Sigma}^{\Sigma}$ such that*

for $X, Y \subseteq \Sigma$:
$$
\begin{array}{ccc}
\bigcup \beta[X] \in \bigcup \beta[Y] & \text{if and only if} & \bigcup X \in \bigcup Y, \\
\bigcup \beta[X] = \mathscr{P}(\bigcup \beta[Y]) & \text{if} & \bigcup X = \mathscr{P}(\bigcup Y);
\end{array}
$$
for $X, Y_1, \ldots, Y_L \subseteq \Sigma$:
$$
\bigcup \beta[X] = \{\bigcup \beta[Y_1], \ldots, \bigcup \beta[Y_L]\} \ \text{if} \ \bigcup X = \{\bigcup Y_1, \ldots, \bigcup Y_L\}.
$$
(The last condition is vacuously satisfied when $L = 0$.) $\qquad \square$

One easily sees that if Σ, $\widehat{\Sigma}$, and β are interrelated as in Def.8, $\mathcal{K}$ is a finite collection of literals of the above-said forms ($\dagger$), and $\Im \in \mathscr{P}(\Sigma)^{\mathcal{X}}$ induces —as explained at the beginning— a set-valued assignment $\mathcal{M}$ making $\mathcal{K}$ true, then $\mathcal{K}$ holds true also in $\Im \circ \overline{\beta}$, where $\overline{\beta} \in \mathscr{P}(\widehat{\Sigma})^{\mathscr{P}(\Sigma)}$ is the function $\Gamma \mapsto \beta[\Gamma]$. Let us postpone a proof of this important, though simple, combinatorial fact till Lemma 21 in Sec.11.

Our strategy to establish whether or not a finite collection $\mathcal{K}$ of literals is satisfiable will be to L-SIMULATE a transitive partition associated (cf. Thm.2) with the family $\mathcal{F}$

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[4]Reasons why we do not feel compelled to treat negative literals that involve set operators will emerge from Sec.11. We will benefit from this syntactical restraint in the statements of Definitions 7, 8, 9, in some of whose conditions we can, thanks to it, use implication instead of bi-implication. Further restraints could be imposed on ($\dagger$); for instance one could do without literals of the forms $v = \emptyset$ and $v = u \cap w$.

of sets assigned to variables in a hypothetical interpretation satisfying $\mathcal{K}$; the simulating partition will have a finite rank, bounded by a computable function in the overall number of variables in $\mathcal{K}$. A direct simulation, however, is hard to perform; it will demonstrate easier to base the simulation on a 'formative process' suitably describing the inner constitution of $\mathcal{F}$. We will see in Sec.4 how such a formative process can be conceived; then, in Sec.7, how to imitate the process in order to obtain the simulating partition.[5]

In sight of these developments, let us for the moment introduce a convenient strengthening of the notion of simulating partition that may surrogate it in practice.

For any set X, we put

$$\mathscr{P}^*(X) =_{\mathrm{Def}} \{ \, Y \mid Y \subseteq \bigcup X \wedge (\forall z \in X)(\, z \ni \in Y \,) \, \} \, .$$

That is, the elements of $\mathscr{P}^*(X)$ are all sets Y obtainable by extracting from each $z \in X$ a non-null $W_z \subseteq z$ and then forming $Y = \bigcup_{z \in X} W_z$.

Here are some useful, and easily verified, properties of $\mathscr{P}^*$:

Lemma 2. *(1) For every set S,   if $\emptyset \in S$, then $\mathscr{P}^*(S) = \emptyset$, else $\bigcup S \in \mathscr{P}^*(S)$.*

(2) $\mathscr{P}^*(\emptyset) = \{\emptyset\}$; $\mathscr{P}^*(\{\sigma\}) = \mathscr{P}(\sigma) \setminus \{\emptyset\}$.

(3) If $g \in \{\, \text{sets} \,\}^X$ is such that $g(z) \subseteq z$ for all $z \in X$,
    then $\mathscr{P}^*(g[X]) \subseteq \mathscr{P}^*(X)$.

(4) For every family $\mathcal{F}$, $\mathscr{P}(\bigcup \mathcal{F}) = \bigcup \mathscr{P}^*[\mathscr{P}(\mathcal{F})]$.

(5) For every partition Σ and for all $\Gamma_0, \Gamma_1 \subseteq \Sigma$,

   (a) $\mathscr{P}^*(\Gamma_0) \neq \emptyset$ and $\bigcup \mathscr{P}^*(\Gamma_0) = \bigcup \Gamma_0$;
   (b) if $\Gamma_0 \neq \Gamma_1$, then $\mathscr{P}^*(\Gamma_0) \cap \mathscr{P}^*(\Gamma_1) = \emptyset$;
   (c) $\mathscr{P}^*(\Gamma_0 \cup \Gamma_1) = \{\, X \cup Y \mid X \in \mathscr{P}^*(\Gamma_0) \wedge Y \in \mathscr{P}^*(\Gamma_1) \,\}$;
   (d) $\mathscr{P}^*(\Sigma)$ determines Σ uniquely.

(6) Let Γ be a partition and let X be any set. If $\bigcup \Gamma \setminus \bigcup (\mathscr{P}^*(\Gamma) \cap X) \neq \emptyset$, then

$$|\mathscr{P}^*(\Gamma) \setminus X| \geq 2^{\sup\{|\sigma| \mid \sigma \in \Gamma\} - 1}.$$

   (The antecedent of the latter implication reads: "Some block in Γ has an element x_0 such that none of the sets in $\mathscr{P}^*(\Gamma)$ which have x_0 as an element belongs to X.")

Proof. We limit ourselves to proving properties (4)–(6) only. A verification of (4) runs as follows: on the one hand, the inclusions $\mathscr{P}^*(\Gamma) \subseteq \mathscr{P}(\bigcup \Gamma) \subseteq \mathscr{P}(\bigcup \mathcal{F})$ hold trivially for all $\Gamma \subseteq \mathcal{F}$; on the other, if $X \subseteq \bigcup \mathcal{F}$, then $X \in \mathscr{P}^*(\{Y \mid Y \in \mathcal{F} \wedge X \ni \in Y\})$.

Concerning (5.a), notice that $\bigcup \Gamma_0 \in \mathscr{P}^*(\Gamma_0)$, by (1), and that every element of $\mathscr{P}^*(\Gamma_0)$ is included in $\bigcup \Gamma_0$. As for (5.b), assume that $\sigma \in \Gamma_{1-b} \setminus \Gamma_b$, with $b \in \{0, 1\}$, so that every set in $\mathscr{P}^*(\Gamma_{1-b})$ intersects σ, whereas, since blocks are pairwise disjoint and $\sigma \notin \Gamma_b$, every set in Γ_b is disjoint from σ; then every set in $\mathscr{P}(\bigcup \Gamma_b) \supseteq \mathscr{P}^*(\Gamma_b)$ is disjoint from σ, and therefore $\mathscr{P}^*(\Gamma_0) \cap \mathscr{P}^*(\Gamma_1) = \emptyset$. Clause (5.c) is easy and is left to the reader.

---

[5]If we simplified Def.8, by requiring simply that for $X, Y \subseteq \Sigma$
$$\bigcup \beta[X] \in \bigcup \beta[Y] \quad \text{if and only if} \quad \bigcup X \in \bigcup Y \, ,$$
$$\bigcup \beta[X] \setminus \{\bigcup \beta[Z] \mid Z \subseteq \Sigma\} = \emptyset \qquad \text{if} \qquad \bigcup X \setminus \{\bigcup Z \mid Z \subseteq \Sigma\} = \emptyset$$
(clearly, with this change we would be leaving out of consideration the literals $v = \mathscr{P}(w)$), then simulation could simply be based on knowing the function $\Gamma \mapsto \sigma^\Gamma$, where $\Gamma \subseteq \Sigma$, $\sigma^\Gamma \in \Sigma \cup \{\varsigma_\Sigma\}$, and $\bigcup \Gamma \in \sigma^\Gamma$.

A verification of (5.d) runs as follows: clearly the maximum w.r.t. inclusion in $\mathscr{P}^*(\Sigma)$ is $\bigcup \Sigma$; moreover, any two distinct elements a, b of $\bigcup \Sigma$ fall into the same block of Σ iff for every element y of $\mathscr{P}^*(\Sigma)$ such that $a \in y$, one has $y \setminus \{b\} \in \mathscr{P}^*(\Sigma)$. Notice that the analogue of (5.d) with an arbitrary family $\mathcal{F}$ in place of Σ does not hold: for example, $\mathcal{F}_0 \neq \mathcal{F}_1$ and $\mathscr{P}^*(\mathcal{F}_0) = \mathscr{P}^*(\mathcal{F}_1) = \{$ infinite subsets of ω $\}$ hold together when $\mathcal{F}_0$ consists of all sets of the form $\omega \setminus Y$, where Y has finite cardinality, and $\mathcal{F}_1$ consists of all sets of the form $\omega \setminus Y$, where the cardinality of $Y \subseteq \omega$ is both finite and even.

Finally, concerning (6), let $x_0 \in \bigcup \Gamma \setminus \bigcup(\mathscr{P}^*(\Gamma) \cap X)$ and let $\sigma \in \Gamma$. If $x_0 \notin \sigma$, then
$$\{ \{x_0\} \cup Y \cup Z \mid Y \in \mathscr{P}^*(\{\sigma\}) \wedge Z \in \mathscr{P}^*(\Gamma \setminus \{\sigma\})\} \subseteq \mathscr{P}^*(\Gamma) \setminus X,$$
otherwise
$$\{ \{x_0\} \cup Y \cup Z \mid Y \in \mathscr{P}(\sigma \setminus \{x_0\}) \wedge Z \in \mathscr{P}^*(\Gamma \setminus \{\sigma\})\} \subseteq \mathscr{P}^*(\Gamma) \setminus X.$$
In the former case we have $|\mathscr{P}^*(\Gamma) \setminus X| \geq 2^{|\sigma|} - 1$, while in the latter we have $|\mathscr{P}^*(\Gamma) \setminus X| \geq 2^{|\sigma|-1}$. Since $\sigma \neq \emptyset$, and hence $2^{|\sigma|} - 1 \geq 2^{|\sigma|-1}$, in either case we get $|\mathscr{P}^*(\Gamma) \setminus X| \geq 2^{|\sigma|-1}$. Since this holds for all $\sigma \in \Gamma$, we conclude that $|\mathscr{P}^*(\Gamma) \setminus X| \geq 2^{\sup\{|\sigma| \mid \sigma \in \Gamma\}-1}$. ∎

From the preceding lemma and from Remark 1(2), we immediately get the following:

Lemma 3. *When its domain gets restricted to a set of the form $\mathscr{P}(\Sigma)$, where Σ is a partition, $\mathscr{P}^*$ is an injective function in $\mathscr{P}^*[\mathscr{P}(\Sigma)]^{\mathscr{P}(\Sigma)}$ whose image is a partition of $\mathscr{P}(\bigcup \Sigma)$.*

If Σ is transitive, then $\mathscr{P}^*[\mathscr{P}(\Sigma)]$ is a partition of $\varsigma \cup \bigcup \Sigma$. ∎

Definition 9. *A partition $\widehat{\Sigma}$ is said to* IMITATE *another partition, Σ, when there is a bijection $\beta \in \widehat{\Sigma}^\Sigma$ such that, for $X \subseteq \Sigma$, $\sigma \in \Sigma$,*

(0) if $\mathscr{P}^*(\beta[X]) \ni \in \beta(\sigma)$, then $\mathscr{P}^*(X) \ni \in \sigma$;

(1) $\bigcup \beta[X] \in \beta(\sigma)$ if and only if $\bigcup X \in \sigma$;

(2) if $\mathscr{P}^*(X) \subseteq \bigcup \Sigma$, then $\mathscr{P}^*(\beta[X]) \subseteq \bigcup \widehat{\Sigma}$.

If in addition to $\widehat{\Sigma}$ imitating Σ one has the condition

(3) $|\beta(\sigma)| = |\sigma|$ when $|\sigma| < \varrho$

fulfilled, where ϱ is a fixed number, then $\widehat{\Sigma}$ is said to ϱ-IMITATE Σ. □

Lemma 4. *If $\Sigma, \widehat{\Sigma}$ are partitions, $\widehat{\Sigma}$ is transitive, and $\widehat{\Sigma}$ imitates Σ, then $\widehat{\Sigma}$ simulates Σ. If, furthermore, $\widehat{\Sigma}$ ϱ-imitates Σ and $L < \varrho$, then $\widehat{\Sigma}$ L-simulates Σ.*

Proof. Let Σ and $\widehat{\Sigma}$ be partitions, let $\widehat{\Sigma}$ be transitive, and assume that $\widehat{\Sigma}$ imitates Σ via the bijection $\beta \in (\widehat{\Sigma})^\Sigma$. Let $X, Y \subseteq \Sigma$. Then we have:

- $\bigcup \beta[X] \in \bigcup \beta[Y]$ iff $(\exists \widehat{\sigma} \in \beta[Y])(\bigcup \beta[X] \in \widehat{\sigma})$ iff $(\exists \sigma \in Y)(\bigcup \beta[X] \in \beta(\sigma))$ iff $(\exists \sigma \in Y)(\bigcup X \in \sigma)$ iff $\bigcup X \in \bigcup Y$.

- Assuming now that $\bigcup X = \mathscr{P}(\bigcup Y)$, let us prove that $\mathscr{P}(\bigcup \beta[Y]) \subseteq \bigcup \beta[X]$. Indeed, suppose $t \subseteq \bigcup \beta[Y]$ and let $\widehat{\Sigma}_t$ be the subset of $\widehat{\Sigma}$ for which $t \in \mathscr{P}^*(\widehat{\Sigma}_t)$ (so that $\widehat{\Sigma}_t \subseteq \beta[Y]$). As $\beta^{-1}[\widehat{\Sigma}_t] \subseteq Y$, it follows that $\mathscr{P}^*(\beta^{-1}[\widehat{\Sigma}_t]) \subseteq \mathscr{P}(\bigcup Y) = \bigcup X \subseteq \bigcup \Sigma$. Therefore, $\mathscr{P}^*(\widehat{\Sigma}_t) \subseteq \bigcup \widehat{\Sigma}$, so that $t \in \bigcup \widehat{\Sigma}$. Let $\widehat{\sigma}_t$ be the block in $\widehat{\Sigma}$ to which t belongs, and let σ_t be the block in Σ for which $\beta(\sigma_t) = \widehat{\sigma}_t$. Then, since $\mathscr{P}^*(\widehat{\Sigma}_t) \ni \in \widehat{\sigma}_t$, we will have that $\mathscr{P}^*(\beta^{-1}[\widehat{\Sigma}_t]) \ni \in \sigma_t$, which yields $\bigcup X = \mathscr{P}(\bigcup Y) \supseteq \mathscr{P}^*(\beta^{-1}[\widehat{\Sigma}_t]) \ni \in \sigma_t$, so that $\bigcup X \ni \in \sigma_t$, $\sigma_t \in X$, and hence $t \in \widehat{\sigma}_t \in \beta[X]$, which in turn yields $t \in \bigcup \beta[X]$.

- Assuming again that $\bigcup X = \mathscr{P}(\bigcup Y)$, let us now prove that $\bigcup \beta[X] \subseteq \mathscr{P}(\bigcup \beta[Y])$. Indeed, for each $t \in \bigcup \beta[X]$ there is a unique $\sigma_t \in X$ such that $t \in \beta(\sigma_t)$; moreover, by the transitivity of $\bigcup \widehat{\Sigma}$, there is a unique $\Gamma \subseteq \Sigma$ for which $t \in \mathscr{P}^*(\beta[\Gamma])$; finally, since $\mathscr{P}^*(\beta[\Gamma]) \ni \in \beta(\sigma_t)$, we also have that $\mathscr{P}^*(\Gamma) \ni \in \sigma_t$. We can hence take a $t' \in \sigma_t \cap \mathscr{P}^*(\Gamma)$, which, since $\sigma_t \subseteq \bigcup X = \mathscr{P}(\bigcup Y)$, is to fulfill $t' \in \mathscr{P}^*(Z)$ for a suitable $Z \subseteq Y$. In conclusion $\Gamma = Z$, and therefore $t \subseteq \bigcup \beta[\Gamma] = \bigcup \beta[Z] \subseteq \bigcup \beta[Y]$.

- To prove the second statement of the lemma, assuming that $\bigcup X = \{\bigcup Y_1, \ldots, \bigcup Y_L\}$ where $L < \varrho$ and $Y_1, \ldots, Y_L$ are distinct, we must check that $\bigcup \beta[X] = \{\bigcup \beta[Y_1], \ldots, \bigcup \beta[Y_L]\}$. Since $\sum_{\sigma \in X} |\sigma| = |\bigcup X| = L < \varrho$, we have $|\sigma| < \varrho$ and therefore $|\beta(\sigma)| = |\sigma|$ for each $\sigma \in X$; this easily yields the desired conclusion, because $\bigcup \beta[Y_i] \in \beta(\sigma)$ iff $\bigcup Y_i \in \sigma$, and because $\beta[Y_1], \ldots, \beta[Y_L]$ —and, accordingly, $\bigcup \beta[Y_1], \ldots, \bigcup \beta[Y_L]$— are pairwise distinct. ∎

4. Formative processes and traces

Definition 10. *Let Σ and Σ' be two partitions and let $\Gamma \subseteq \Sigma$. We say that Σ'* PROLONGATES Σ VIA Γ *when the following conditions are met:*

1. *for all $\sigma \in \Sigma$, there is a $\sigma' \in \Sigma'$ such that $\sigma \subseteq \sigma'$;*

2. $\bigcup \Sigma' \setminus \bigcup \Sigma \subseteq \mathscr{P}^*(\Gamma)$;

3. $\Sigma \neq \Sigma'$.

When condition 1. is met, possibly without 2. or 3. being fulfilled, then we say that Σ' EXTENDS Σ; *if both 1. and 3. are met, then Σ' is said to extend Σ* PROPERLY. □

Remarks 2. Concerning Def.10, notice that:

(1) The relation $'$ defined in 1. is univocally determined and actually it is an injective map from Σ into Σ'.

(2) Condition 2. entails that

- $\sigma' \setminus \sigma \subseteq \mathscr{P}^*(\Gamma) \setminus \bigcup \Sigma$, for all $\sigma \in \Sigma$;
- $\tau \subseteq \mathscr{P}^*(\Gamma) \setminus \bigcup \Sigma$, for all $\tau \in \Sigma' \setminus \{\sigma' \mid \sigma \in \Sigma\}$ (i.e., for all $\tau \in \Sigma'$ such that $\tau \cap \bigcup \Sigma = \emptyset$).

(3) Saying that Σ' prolongates Σ via $\Gamma \subseteq \Sigma$ amounts to the same as saying that there exist a partition Γ^* and a function $\Delta \in (\Gamma^* \cup \{\emptyset\})^\Sigma$ such that

- $\emptyset \neq \bigcup \Gamma^* \subseteq \mathscr{P}^*(\Gamma) \setminus \bigcup \Sigma$;
- $\Delta \sigma_1 \cap \Delta \sigma_2 = \emptyset$, for every $\sigma_1, \sigma_2 \in \Sigma$ such that $\sigma_1 \neq \sigma_2$;
- $\Sigma' = \{\sigma \cup \Delta \sigma \mid \sigma \in \Sigma\} \cup (\Gamma^* \setminus \Delta[\Sigma])$.

Indeed, assuming Σ' to prolongate Σ, let $\Gamma^* = (\{\sigma' \setminus \sigma \mid \sigma \in \Sigma\} \cup (\Sigma' \setminus \{\sigma' \mid \sigma \in \Sigma\})) \setminus \{\emptyset\}$. Then notice that from $\Sigma \neq \Sigma'$ it follows that $\Gamma^* \neq \emptyset$, and since $\bigcup \Gamma^* = \bigcup \Sigma' \setminus \bigcup \Sigma$ and $\emptyset \notin \Gamma^*$, we also have $\bigcup \Sigma' \neq \bigcup \Sigma$.

The converse is obvious. □

Definition 11 (Coherence requirement) *Let* Γ, Σ', *and* Σ'' *be partitions, with* Σ' *extending* Γ *and* Σ'' *extending* Σ'. *Then* Σ'' *is said to extend* Σ' COHERENTLY *with* Γ *if no element of* $\bigcup \Sigma''$ *belongs to* $\mathscr{P}^*(\Gamma) \setminus \bigcup \Sigma'$. $\qquad\square$

Definition 12. *Let* ξ *be an ordinal and let* $\left(\{ q^{(\mu)} \}_{q \in P} \right)_{\mu \leq \xi}$ *be a* $(\xi + 1)$-*sequence of functions all of which are defined on the same domain* P. *Put* $B^{(\mu)} =_{Def} \{ q^{(\mu)} \mid q \in P \}$ *for all* $B \subseteq P$, *and let* $\Sigma_\mu = P^{(\mu)} \setminus \{ \emptyset \}$, *for all* $\mu \leq \xi$.
Assume the following conditions to be fulfilled:

- $q^{(\mu)} \cap p^{(\mu)} = \emptyset$ *when* $p, q \in P$, $p \neq q$, *and* $\mu \leq \xi$;

- $q^{(\nu)} \subseteq q^{(\nu+1)}$ *for all* $q \in P$ *when* $\nu < \xi$;

- $q^{(\lambda)} = \bigcup_{\nu < \lambda} q^{(\nu)}$ *for every* $q \in P$ *and every limit ordinal* $\lambda \leq \xi$;

- Σ_μ *is a partition, for every* $\mu \leq \xi$;

- $\Sigma_0 = \emptyset$; $\qquad \emptyset \notin \Sigma_\xi$.

Assume moreover that to each $\nu < \xi$ *there corresponds a* $\Gamma_\nu \subseteq \Sigma_\nu$ *such that*

- $\Sigma_{\nu+1}$ *prolongates* Σ_ν *via* Γ_ν *(cf. Def.10);*

- Σ_ξ *extends* $\Sigma_{\nu+1}$ *coherently with* Γ_ν *(cf. Def.11).*

Then the sequence $\left(\{ q^{(\mu)} \}_{q \in P} \right)_{\mu \leq \xi}$ *is called a* (STRONG) FORMATIVE PROCESS *for* Σ_ξ, *and the* ξ-*sequences* $(A_\nu)_{\nu < \xi}$, $(A_\nu, T_\nu)_{\nu < \xi}$ *such that both of the conditions*

- $A_\nu^{(\nu)} = \Gamma_\nu$,

- $\{ q^{(\nu+1)} \setminus q^{(\nu)} \mid q \in T_\nu \}$ *is a partition of* $\left(\mathscr{P}^*(\Gamma_\nu) \setminus \bigcup \Sigma_\nu \right) \cap \bigcup \Sigma_{\nu+1}$

hold for each ν *are called the* TRACE *of the formative process, and a* HISTORY *of* Σ_ξ, *respectively.*

A WEAK FORMATIVE PROCESS *is like a formative process, save that the coherence requirement is withdrawn from the definition. A* WEAK TRACE *is defined similarly.* $\quad\square$

Remarks 3. (1) An indirection could easily be eliminated from the definition of formative process by requiring that $\Sigma_\xi = P$ and $\{ q^{(\xi)} \}_{q \in P} = \iota_P$. Indeed, characterizing a formative process without P and $q^{(\mu)}$s —directly in terms of the sequence $(\Sigma_\mu)_{\mu \leq \xi}$ of partitions— would lead to a more concise and essential definition; however it does not seem to be particularly convenient to proceed so either on the technical plan, or to convey a better intuitive grasp. (We will come back to this idea only once, namely within the proof of the trace theorem—cf. Corollary 1.)

Assuming $\left(\{ q^{(\mu)} \}_{q \in P} \right)_{\mu \leq \xi}$ to be a formative process, and maintaining the above notation, notice that

(2) If $\nu < \mu \leq \xi$, then $\bigcup \Sigma_\nu \subsetneqq \bigcup \Sigma_\mu$, because clearly Σ_μ extends Σ_ν and does so properly when $\mu = \nu + 1$.

(3) For each $e \in \bigcup \Sigma_\xi$, there is a unique $\nu < \xi$, denoted $\nu(e)$, for which $e \in \mathscr{P}^*(\Gamma_\nu) \setminus \bigcup \Sigma_\nu$, because $\Sigma_0 = \emptyset$ and new elements enter into the $\bigcup \Sigma_\mu$s only through prolongation steps. Clearly, we will have for all $\mu \leq \xi$

$$e \in \bigcup \Sigma_\mu \leftrightarrow \nu(e) < \mu.$$

(4) $P = \bigcup_{\nu < \xi} T_\nu$. In fact, for each $q \in P$, by taking an $e \in q^{(\xi)}$ we will have $\nu(e) < \xi$ and $q \in T_{\nu(e)}$.

(5) $A_\nu \subseteq \bigcup_{\mu < \nu} T_\mu$, and $q^{(\nu)} \neq \emptyset$ for every $q \in A_\nu$, for all $\nu < \xi$. In fact, each set $A_\nu^{(\nu)} = \Gamma_\nu$ is a partition (were it not so, $\mathscr{P}^*(\Gamma_\nu)$ would fail to contribute elements to $\bigcup \Sigma_{\nu+1} \setminus \bigcup \Sigma_\nu$); hence, if $q \in A_\nu$, then by taking $e \in q^{(\nu)}$ we will have $\nu(e) < \nu$ and $q \in T_{\nu(e)}$. $\square$

Lemma 5. *Every constituent Σ_μ of a formative process is a transitive partition.*

Proof. Assuming that $e \in \bigcup \Sigma_\mu$, in view of the above Remarks 3(2),(3), we have $e \in \mathscr{P}^*(\Gamma_{\nu(e)}) \subseteq \mathscr{P}(\bigcup \Gamma_{\nu(e)}) \subseteq \mathscr{P}(\bigcup \Sigma_{\nu(e)}) \subseteq \mathscr{P}(\bigcup \Sigma_\mu)$, and therefore $e \subseteq \bigcup \Sigma_\mu$. ∎

Lemma 6. *Let Σ and Σ'' be partitions such that*

- *Σ'' properly extends Σ (cf. Def.10),*

- *Σ'' is a transitive partition (cf. Def.5).*

Then there are a $\Gamma \subseteq \Sigma$ and a partition Σ' that both prolongates Σ via Γ and is extended by Σ'' coherently with Γ (cf. Def.10 and Def.11).

Proof. Let s be an element of smallest rank in $\bigcup \Sigma'' \setminus \bigcup \Sigma$. Thanks to the transitivity of Σ'', we have $s \subseteq \bigcup \Sigma''$; therefore, every $t \in s$ belongs to $\bigcup \Sigma$. By taking
$$\Gamma = \{ Z \mid Z \in \Sigma \wedge s \ni\in Z \},$$
we will have $s \in \mathscr{P}^*(\Gamma)$ and we must put
$$\Sigma' = \{ (\mathscr{P}^*(\Gamma) \cap \sigma'') \cup \sigma \mid \sigma'' \in \Sigma'' \wedge \sigma \in \Sigma \wedge \sigma \subseteq \sigma'' \}$$
$$\cup \{ \mathscr{P}^*(\Gamma) \cap \sigma'' \mid \sigma'' \in \Sigma'' \wedge \sigma'' \ni\in \mathscr{P}^*(\Gamma) \wedge (\forall \sigma \in \Sigma)(\sigma \not\subseteq \sigma'') \} .$$
Clearly Σ' prolongates Σ via Γ, since $s \in \bigcup \Sigma' \setminus \bigcup \Sigma$; moreover, it is obvious that Σ'' extends Σ' coherently with Γ. ∎

Corollary 1 (Trace theorem) *Every transitive partition Σ has a history $(A_\mu, T_\mu)_{\mu < \xi}$ with the cardinality of ξ not exceeding $|\bigcup \Sigma|$.*

Proof. Given Σ, simply take $P = \Sigma$. We begin with $\Sigma_0 = \emptyset$ and then, for every ordinal μ:

- If μ is a limit ordinal, we put
$$\Sigma_\mu = \bigcup_{\gamma < \mu} \{ \bigcup_{\gamma < \nu < \mu} \sigma^{\langle \nu \rangle} \mid \sigma \in \Sigma_\gamma \},$$
where $\sigma^{\langle \nu \rangle}$ indicates the block τ of Σ_ν for which $\sigma \subseteq \tau$.

- In any case, we define $\Sigma_{\mu+1}$ to be the same as Σ_μ if $\Sigma_\mu = \Sigma$; otherwise we choose a $\Gamma_\mu \subseteq \Sigma_\mu$ and a partition $\Sigma_{\mu+1}$ that both prolongates Σ_μ via Γ_μ and is extended by Σ coherently with Γ_μ, as by the preceding lemma.

- If $\Sigma_{\mu+1} \neq \Sigma_\mu$, we take $A_\mu = \{ q \in P \mid (\exists \sigma \in \Gamma_\mu)(\sigma \subseteq q) \}$ and $T_\mu = \{ q \in P \mid (\exists \sigma \in \Sigma_{\mu+1} \setminus \Sigma_\mu)(\sigma \subseteq q) \}$.

Since the sequence of the $\bigcup \Sigma_\mu$s strictly increases w.r.t. $\subseteq$ until it has reached $\bigcup \Sigma$, and since $\bigcup \Sigma_\mu \subseteq \bigcup \Sigma$ holds for all μ, certainly there will be an ordinal ξ for which $\bigcup \Sigma_\xi = \bigcup \Sigma$ with $|\xi| \leq |\bigcup \Sigma|$; the conclusion that $(A_\mu, T_\mu)_{\mu < \xi}$ is the desired history hence follows. ∎

The simplifying notation to be introduced next will be helpful in the ongoing.

Definition 13. *Let* $(\{q^{(\mu)}\}_{q \in P})_{\mu \leq \xi}$ *be a weak formative process. Then, for* $q \in P$, $B \subseteq P$, *and* $\nu < \xi$, *we put*

$$
q^{(\bullet)} \quad =_{Def} \quad q^{(\xi)}, \qquad B^{(\bullet)} \quad =_{Def} \quad B^{(\xi)},
$$
$$
\Delta^{(\nu)}(q) \quad =_{Def} \quad q^{(\nu+1)} \setminus \bigcup P^{(\nu)}.
$$

$\square$

Example 3. Resuming the notation $\emptyset^n$ of Example 2, let us take $\Sigma = \{0^{(\bullet)}, \ldots, 4^{(\bullet)}\}$, where

$$
\begin{aligned}
0^{(\bullet)} &= \{\emptyset\}, & 1^{(\bullet)} &= \{\emptyset^1\}, & 2^{(\bullet)} &= \{\emptyset^2, \{\emptyset, \emptyset^1\}\}, \\
3^{(\bullet)} &= \{\emptyset^3, \{\{\emptyset, \emptyset^1\}\}\}, & 4^{(\bullet)} &= \{\{\emptyset, \emptyset^2\}\}.
\end{aligned}
$$

One readily sees that $\bigcup_{j=0}^{4} j^{(\bullet)}$ is a transitive set; hence Σ is a transitive partition.

With the elements s of $\bigcup \Sigma$ ordered by non-decreasing ranks, we easily associate with each of them the set A for which $s \in \mathscr{P}^*(\{q^{(\bullet)} \mid q \in A\})$:

s	$\emptyset$	$\emptyset^1$	$\emptyset^2$	$\{\emptyset, \emptyset^1\}$	$\{\{\emptyset, \emptyset^1\}\}$	$\{\emptyset, \emptyset^2\}$	$\emptyset^3$
A	$\emptyset$	$\{0\}$	$\{1\}$	$\{0, 1\}$	$\{2\}$	$\{0, 2\}$	$\{2\}$

In this concrete example, the construction of the trace theorem proceeds according to the following table:

i	s	A_i	T_i	$0^{(i)}$	$1^{(i)}$	$2^{(i)}$	$3^{(i)}$	$4^{(i)}$
0	$\emptyset$	$\emptyset$	$\{0\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	$\emptyset^1$	$\{0\}$	$\{1\}$	$\{\emptyset\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	$\emptyset^2$	$\{1\}$	$\{2\}$	$\{\emptyset\}$	$\{\emptyset^1\}$	$\emptyset$	$\emptyset$	$\emptyset$
3	$\{\emptyset, \emptyset^1\}$	$\{0, 1\}$	$\{2\}$	$\{\emptyset\}$	$\{\emptyset^1\}$	$\{\emptyset^2\}$	$\emptyset$	$\emptyset$
4	$\{\emptyset, \emptyset^2\}$	$\{0, 2\}$	$\{4\}$	$\{\emptyset\}$	$\{\emptyset^1\}$	$\{\emptyset^2, \{\emptyset, \emptyset^1\}\}$	$\emptyset$	$\emptyset$
5	$\emptyset^3$	$\{2\}$	$\{3\}$	$\{\emptyset\}$	$\{\emptyset^1\}$	$\{\emptyset^2, \{\emptyset, \emptyset^1\}\}$	$\emptyset$	$\{\{\emptyset, \emptyset^2\}\}$
6	—	—	—	$\{\emptyset\}$	$\{\emptyset^1\}$	$\{\emptyset^2, \{\emptyset, \emptyset^1\}\}$	$\{\emptyset^3, \{\{\emptyset, \emptyset^1\}\}\}$	$\{\{\emptyset, \emptyset^2\}\}$

The sequences $(A_i)_{i \leq 5}$ and $(A_i, T_i)_{i \leq 5}$ are a *trace* and a *history* of the partition Σ, respectively. $\square$

5. Useful lemmas about formative processes

In sight of the main proofs in this paper, which will constitute Sections 8 and 10, let us review some useful properties of the sets $q^{(\mu)}, B^{(\mu)}, \Delta^{(\nu)}(q), q^{(\bullet)}$, and $B^{(\bullet)}$.

Lemma 7. *Assume that* $P^{(\bullet)} \not\subseteq \{\{\emptyset\}\}$ *in a weak formative process whose (weak) trace is* $(A_\nu)_{\nu < \xi}$. *Then the following conditions are fulfilled, for* $q \in P$, $B \subseteq P$, $\mu \leq \xi$, *and* $\nu < \xi$:

(1) $q^{(0)} = \emptyset$, $\quad \bigcup \Delta^{(0)}[P] = \{\emptyset\}$, $\quad A_\nu = \emptyset$ *iff* $\nu = 0$,
$\bigcup \Delta^{(1)}[P] = \{\{\emptyset\}\}$, $\quad \left| A_1^{(\nu+1)} \right| = 1$ *and* $A_1^{(\nu+1)}$ *extends* $\{\{\emptyset\}\}$;

(2) if $\nu \leq \mu$, *then*

- $\emptyset \notin A_\nu^{(\mu)}$ *(hence* $A_\nu^{(\mu)}$ *is a partition),*
- $q^{(\nu)} \subseteq q^{(\mu)}$, *and* $B^{(\mu)}$ *extends* $B^{(\nu)}$
  (more accurately stated, $B^{(\mu)} \setminus \{\emptyset\}$ *extends* $B^{(\nu)} \setminus \{\emptyset\}$*),*
  and hence $\bigcup B^{(\nu)} \subseteq \bigcup B^{(\mu)}$,

- rk $q^{(\mu)} \le \mu$,
- rk $B^{(\mu)} \le \mu + 1$;

(3) $q^{(\mu)} = q^{(\nu)} \cup \bigcup_{\nu \le \vartheta < \mu} \Delta^{(\vartheta)}(q)$, *where unions have disjoint operands,*
    and therefore $q^{(\mu)} \setminus q^{(\nu)} = \bigcup_{\nu \le \vartheta < \mu} \Delta^{(\vartheta)}(q)$, *if* $\nu < \mu$;

(4) $\Delta^{(\nu)}(q) \cap \mathscr{P}^*(B^{(\mu)}) = \Delta^{(\nu)}(q) \cap \mathscr{P}^*(B^{(\nu)})$ *if* $\nu \le \mu < \xi$;

(5) $q^{(\nu)} \cap \mathscr{P}^*(B^{(\mu)}) = q^{(\nu)} \cap \mathscr{P}^*(B^{(\nu)})$ *if* $\nu < \mu$,

(6) *if* $\bigcup B^{(\bullet)} \in \bigcup P^{(\nu+1)}$, *then* $B^{(\nu)} = B^{(\bullet)}$ *and* $\bigcup_{\nu \le \vartheta < \xi} \bigcup \Delta^{(\vartheta)}[B] = \emptyset$.

When the formative process is strong, *the following further conditions will be met:*

(7) $A_\nu^{(\nu)} \ne A_\mu^{(\mu)}$ *if* $\nu < \mu < \xi$;

(8) $q^{(\bullet)} \cap \mathscr{P}^*(A_\nu^{(\nu)}) = q^{(\mu)} \cap \mathscr{P}^*(A_\nu^{(\nu)})$ *if* $\nu < \mu$.

These laws have straightforward proofs, which are left to the reader. ∎

The next definition and proposition, still concerning a weak formative process of length $\xi + 1$, offer a reasoning tactic to be repeatedly exploited in the verifications of Sec.10.

Definition 14. *Let* $\nu < \mu \le \xi$, *and let* X *be any set. Then*
$$\mathcal{S}(\nu, \mu, q, X) \ =_{Def} \ \{\, \eta \mid \nu < \eta \le \mu \ \wedge \ q^{(\eta)} \setminus q^{(\nu)} \ni\in X \,\}.$$
□

Taking into account that $q^{(\eta)} \setminus q^{(\nu)} = \bigcup_{\nu \le \vartheta < \eta} \Delta^{(\vartheta)}(q)$ Lemma 7(3), we get:

Lemma 8. *Let* $\mathcal{S}(\nu, \mu, q, X) \ne \emptyset$. *Then* $\min \mathcal{S}(\nu, \mu, q, X)$ *is a successor ordinal.*

Proof. Assume that $\mathcal{S}(\nu, \mu, q, X) \ne \emptyset$ and let $\overline{\eta} = \min \mathcal{S}(\nu, \mu, q, X)$, so that in particular $\nu < \overline{\eta} \le \mu$ and $\bigcup_{\nu \le \vartheta < \overline{\eta}} \Delta^{(\vartheta)}(q) \ni\in X$ both hold. One has, moreover, that
$$\forall \eta \left(\nu < \eta < \overline{\eta} \to \bigcup_{\nu \le \vartheta < \eta} \Delta^{(\vartheta)}(q) \cap X = \emptyset \right),$$
and hence
$$\forall \eta \, \forall \vartheta \ \left(\nu \le \vartheta < \eta < \overline{\eta} \to \Delta^{(\vartheta)}(q) \cap X = \emptyset \right),$$
holds.

If $\overline{\eta}$ were a limit ordinal, then $(\forall \vartheta < \overline{\eta})(\exists \eta < \overline{\eta})(\vartheta < \eta)$ would hold, easily yielding
$$\forall \vartheta \ (\nu \le \vartheta < \overline{\eta} \to \Delta^{(\vartheta)}(q) \cap X = \emptyset).$$
This would lead to the absurd conclusion that $\bigcup_{\nu \le \vartheta < \overline{\eta}} \Delta^{(\vartheta)}(q) \cap X = \emptyset$. ∎

Another useful combinatorial lemma is the following:

Lemma 9. *Let* $(A_\nu)_{\nu < \xi}$ *be the trace of a* strong *formative process with a non-empty finite domain* P. *Let moreover* $\varrho \in \omega$ *be such that* $2^{\varrho - 1} > \varrho \cdot |P|$. *If there are an ordinal* $\overline{\mu} < \xi$ *and a* $\overline{q} \in A_{\overline{\mu}}$ *such that* $\left| \overline{q}^{(\overline{\mu})} \right| \ge \varrho$, *and* $\mathscr{P}^*(A_{\overline{\mu}}^{(\bullet)}) \subseteq \bigcup P^{(\bullet)}$, *then* $\left| \bigcup \Delta^{(\overline{\mu})}[P] \right| > \varrho \cdot |P|$, *and hence there must be an* $r \in P$ *such that* $\left| \Delta^{(\overline{\mu})}(r) \right| > \varrho$.

Proof. W.l.o.g., assume $\varrho \ge 3$ (indeed, for $\varrho = 0$, the thesis becomes trivial). Consider the set $H = \{\, \eta \mid \eta < \overline{\mu} \wedge A_\eta = A_{\overline{\mu}} \,\}$. If $H = \emptyset$, then $\left| \bigcup \Delta^{(\overline{\mu})}[P] \right| \ge \left| \mathscr{P}(\overline{q}^{(\overline{\mu})}) \setminus \{\emptyset\} \right| \ge 2^\varrho - 1 \ge 2^{\varrho - 1} > \varrho \cdot |P|$, and we are done. Otherwise we put $\overline{\vartheta} = \sup H$ and treat the cases $\overline{\vartheta} \notin H, \overline{\vartheta} \in H$ separately.

16.

If $\overline{\vartheta} \notin H$ then, plainly, H is infinite and, by the coherence requirement, $\bigcup A_{\overline{\mu}}^{(\overline{\mu})}$ must be infinite. W.l.o.g. assume that $\overline{q}^{(\overline{\mu})}$ is infinite and let $s_0, s_1, \ldots, s_{\varrho \cdot |P|}$ be $\varrho \cdot |P| + 1$ distinct elements in $\overline{q}^{(\overline{\mu})}$. For $i = 0, 1, \ldots, \varrho \cdot |P|$, let $S_i = \bigcup A_{\overline{\mu}}^{(\overline{\mu})} \setminus \{s_i\}$, so that clearly $S_i \in \mathscr{S}^*(A_{\overline{\mu}}^{(\overline{\mu})})$. If it were the case that $S_i \in \bigcup P^{(\overline{\mu})}$, then $S_i \in \bigcup \Delta^{(\eta)}[P] \subseteq \mathscr{S}^*(A_\eta^{(\eta)}) \subseteq \mathscr{S}^*(A_\eta^{(\overline{\mu})})$ for some $\eta < \overline{\mu}$. Therefore we would get from Lemma 2(5.b) that $A_\eta = A_{\overline{\mu}}$, and hence $\eta \in H$, implying, by our current hypothesis that $\overline{\vartheta} \notin H$, the existence of an $\eta' \in H$ such that $\eta < \eta' < \overline{\mu}$. Then the coherence requirement would yield
$$\bigcup A_{\overline{\mu}}^{(\overline{\mu})} \setminus \{s_i\} \subseteq A_{\overline{\mu}}^{(\eta)} \subsetneqq A_{\overline{\mu}}^{(\eta')} \subsetneqq \bigcup A_{\overline{\mu}}^{(\overline{\mu})},$$
which is a contradiction; hence $S_i \notin \bigcup P^{(\overline{\mu})}$. Thus, by the coherence requirement and by the hypothesis $\mathscr{S}^*(A_{\overline{\mu}}^{(\bullet)}) \subseteq \bigcup P^{(\bullet)}$, we get
$$\left| \bigcup \Delta^{(\overline{\mu})}[P] \right| = \left| \mathscr{S}^*(A_{\overline{\mu}}^{(\overline{\mu})}) \setminus \bigcup P^{(\overline{\mu})} \right| > \varrho \cdot |P|.$$

Let us assume now that $\overline{\vartheta} \in H$. After fixing a $y_0 \in \bigcup A_{\overline{\mu}}^{(\overline{\mu})} \setminus \bigcup A_{\overline{\mu}}^{(\overline{\vartheta})}$, we consider the cases $y_0 \notin \overline{q}^{(\overline{\mu})}$, $y_0 \in \overline{q}^{(\overline{\mu})}$ separately. In the former case, by picking the $\overline{r} \in A_{\overline{\mu}}$ for which $y_0 \in \overline{r}$, we have
$$\{\{y_0\} \cup Y \mid Y \in \mathscr{S}^*((A_{\overline{\mu}} \setminus \{\overline{r}\})^{(\overline{\mu})})\} \subseteq \mathscr{S}^*(A_{\overline{\mu}}^{(\overline{\mu})}) \setminus \bigcup P^{(\overline{\mu})},$$
where both $\mathscr{S}^*(A_{\overline{\mu}}^{(\overline{\mu})}) \setminus \bigcup P^{(\overline{\mu})} = \bigcup \Delta^{(\overline{\mu})}[P]$ and
$$\left| \{\{y_0\} \cup Y \mid Y \in \mathscr{S}^*((A_{\overline{\mu}} \setminus \{\overline{r}\})^{(\overline{\mu})})\} \right| \geq 2^\varrho - 1 > \varrho \cdot |P|$$
hold, and hence $\left| \bigcup \Delta^{(\overline{\mu})}[P] \right| > \varrho \cdot |P|$. If $y_0 \in \overline{q}^{(\overline{\mu})}$ then we have

$$\{\{y_0\} \cup Y \mid Y \in \mathscr{S}^*((A_{\overline{\mu}} \setminus \{\overline{q}\})^{(\overline{\mu})} \cup \{\overline{q}^{(\overline{\mu})} \setminus \{y_0\}\}) \cup \mathscr{S}^*((A_{\overline{\mu}} \setminus \{\overline{q}\})^{(\overline{\mu})})\} \subseteq$$
$$\mathscr{S}^*(A_{\overline{\mu}}^{(\overline{\mu})}) \setminus \bigcup P^{(\overline{\mu})} = \bigcup \Delta^{(\overline{\mu})}[P],$$

where

$$\left| \{\{y_0\} \cup Y \mid Y \in \mathscr{S}^*((A_{\overline{\mu}} \setminus \{\overline{q}\})^{(\overline{\mu})} \cup \{\overline{q}^{(\overline{\mu})} \setminus \{y_0\}\}) \cup \mathscr{S}^*((A_{\overline{\mu}} \setminus \{\overline{q}\})^{(\overline{\mu})})\} \right| \geq$$
$$2^{\varrho-1} > \varrho \cdot |P|$$

holds, and hence $\left| \bigcup \Delta^{(\overline{\mu})}[P] \right| > \varrho \cdot |P|$. ∎

We conclude this section by proving the following inequality which will be applied later to estimate the length of certain formative processes.

Lemma 10. *Let $y \geq 1$ and let $\varrho_y = \lceil \frac{25}{24} \lceil \log y \rceil + 5 \rceil$. Then $2^{\varrho_y - 1} > \varrho_y y$.*

Proof. By elementary calculus, it is immediate to check that
$$2^{\frac{x}{24}+4} > \frac{25}{24}x + 6 \,, \text{ for } x \in \mathbb{R} \,.$$
Therefore
$$2^{\lceil \frac{x}{24}+5 \rceil - 1} \geq 2^{\frac{x}{24}+4} > \frac{25}{24}x + 6 > \lceil \frac{25}{24}x + 5 \rceil \,, \text{ for } x \in \mathbb{R} \,.$$
Let $y \geq 1$ and put $x = \lceil \log y \rceil$, $\varrho_y = \lceil \frac{25}{24} \lceil \log y \rceil + 5 \rceil$. Then we have
$$2^{\varrho_y - 1} = 2^{\lceil \frac{25}{24}x + 5 \rceil - 1} = 2^{\lceil \frac{x}{24}+5 \rceil - 1} 2^x > \lceil \frac{25}{24}x + 5 \rceil y = \varrho_y y \,,$$
which proves the lemma. ∎

6. Mimicking a formative process: an illustration

In this section we will see a procedure which, given a weak formative process, develops a strong formative process, usually shorter, ending in the same partition as the original process. Another procedure of this kind will be seen in Sec.7, where the original formative process will, instead, be assumed to be strong and the aim will be just to *simulate* the ending partition through the new process: the latter will no longer be guaranteed to be strong, but its length will be finite even when the original process is infinite.

The transfinitely recursive procedure shown below is simply meant to offer a paradigm for other more useful, similar procedures. It receives in input the trace of the weak formative process $\mathcal{G}$ to be mimicked, along with the ending function $B \mapsto B^{(\bullet)}$ of $\mathcal{G}$, and supplies in output the trace of the mimicking process $\mathcal{G}'$, with indication of how the partitioning function of the latter evolves. The sets which form the traces, which are subsets of the domain P common to all functions in $\mathcal{G}$ or in $\mathcal{G}'$, are metaphorically called *moves*, of $\mathcal{G}$ and of $\mathcal{G}'$ respectively.

procedure strengthenProcess($(A_\mu)_{\mu<\xi}$, $\{\, B^{(\bullet)} \,:\, B \subseteq P \,\}$);
   γ := $\emptyset$;   -- γ assigns to the ν-th move of $\mathcal{G}'$ the position $\gamma(\nu)$
            --   of the move of $\mathcal{G}$ it mimics
   ∇ := $\emptyset$;   -- ∇ assigns to the ν-th move of $\mathcal{G}'$ its associated partition
   for $q \in P$ **loop** $\widehat{q} := \emptyset$; **end loop**;   -- start with void blocks
   notation: throughout, and for all $B \subseteq P$, $\widehat{B} =_{\mathrm{Def}} \{ \widehat{q} \mid q \in B \}$;
   $T := \{\, [B, \{ q \in P \mid q^{(\bullet)} \ni \in \mathscr{P}^*(A^{(\bullet)}) \}] \mid B \subseteq P \,\}$;
          -- 'targets' for moves: each $T(B)$ comprises those q
          --   for which $\mathscr{P}^*(\widehat{B})$ will ever intersect $\widehat{q}$
   for μ **in** $[0, 1, \ldots, \xi - 1]$ **loop**
      if $\bigcup \widehat{P} = \bigcup P^{(\bullet)}$ **then quit loop**; **end if**;
      A := A_μ;
      S := $\bigcup P^{(\bullet)} \cap \mathscr{P}^*(\widehat{A}) \setminus \bigcup \widehat{P}$;
      if $S \neq \emptyset$ **then**
         ν := $\bigcup_{\alpha \in \mathrm{dom}(\gamma)} (\alpha + 1)$;
         $\gamma(\nu)$:= μ;
         assert
            $\left(\exists \{\Delta(p)\}_{p \in T(A)} \right) \Big($
                 isPartition$(\Delta, S) \wedge$
                 $(\forall q \in P)\big(S \ni \in q^{(\bullet)} \rightarrow S \cap q^{(\bullet)} \subseteq \Delta(q) \big) \Big)$;
         let $\nabla(\nu)$ **be one such** Δ;
         let the ν-th move consist of the set A paired with this function $\nabla(\nu)$;
            for $q \in T(A)$ **loop** $\widehat{q} := \widehat{q} \cup \Delta(q)$; **end loop**;
      end if;
   end loop;
   ν := $\bigcup_{\alpha \in \mathrm{dom}(\gamma)} (\alpha + 1)$;
   return
      $\big(A_{\gamma(\alpha)}, \nabla(\alpha) \big)_{\alpha < \nu}$;   -- sequence of mimicking moves
end   strengthenProcess;

procedure   isPartition(Δ, S);
   claim
      if $(\forall q, r \in \mathrm{dom}(\Delta))\big(q \neq r \rightarrow \Delta(q) \cap \Delta(r) = \emptyset \big)$,
      then $\Delta[\mathrm{dom}(\Delta)] \setminus \{\emptyset\}$ is a partition as by Def.4;
   return
      $\bigcup \Delta[\mathrm{dom}(\Delta)] = S \wedge (\forall q, r \in \mathrm{dom}(\Delta))\big(q \neq r \rightarrow \Delta(q) \cap \Delta(r) = \emptyset \big)$;
end   isPartition.

The proof is left to the reader that the formative process returned by strengthenProcess really meets the purpose stated at the beginning of this section. Thus we have

Lemma 11. *Let $(A_\mu)_{\mu < \xi}$ be the trace of a weak formative process for a transitive partition Σ. Then it is possible to extract a subsequence $(A_{\gamma(\alpha)})_{\alpha < \nu}$ from $(A_\mu)_{\mu < \xi}$, with $\nu \leq \xi$ (so that $\gamma(\alpha) < \gamma(\beta)$ for $\alpha < \beta < \nu$), which is the trace of a strong formative process for Σ.* ■

7. The thinning of a transitive partition through its trace

As argued in Sec.3, the capability of L-simulating a given finite transitive partition Σ by means of another partition $\widehat{\Sigma}$ having finite rank, bounded by a computable function in L and $|\Sigma|$, is crucial in order to solve the decision problem for collections of literals of the form (†).

In this section we describe a non-deterministic procedure, imitate, which carries out this task. More specifically, given a strong formative process $(\{q^{(\mu)}\}_{q \in P})_{\mu \leq \xi}$ for a transitive partition $P^{(\bullet)}$ and given a constant $\varrho > L$ such that $2^{\varrho - 1} > \varrho|P|$, the procedure imitate will compute a (weak) formative process $(\{\widehat{q}_\varrho^{[\mu_i]}\}_{q \in P})_{i \leq \ell_\varrho}$ for a transitive partition $\widehat{P}_\varrho^{[\ell_\varrho]}$, with $\ell_\varrho < \varrho|P| \, 2^{|P|} + 3$, which L-simulates $P^{(\bullet)}$.

Let us put $\varrho_0 = \max(\lceil \frac{25}{24} \lceil \log |P| \rceil + 5 \rceil, L+1)$. Then we have $\varrho_0 > L$ and, by Lemma 10, $2^{\varrho_0 - 1} > \varrho_0|P|$. Hence $\ell_{\varrho_0} < \max(\lceil \frac{25}{24} \lceil \log |P| \rceil + 5 \rceil, L+1) |P| \, 2^{|P|} + 3$.

In view of Lemma 11, the above discussion can be summarized as follows.

Lemma 12. *Let $(\{q^{(\mu)}\}_{q \in P})_{\mu \leq \xi}$ be a strong formative process for a transitive partition $P^{(\bullet)}$ and let $L \geq 0$. Then there exists a strong formative process $(\{\widehat{q}^{[\mu_i]}\}_{q \in P})_{i \leq \ell}$ for a transitive partition $\widehat{P}$ which L-simulates $P^{(\bullet)}$ and such that*

$$\ell < \max\left(\left\lceil \tfrac{25}{24} \lceil \log |P| \rceil + 5 \right\rceil, L+1\right) |P| \, 2^{|P|} + 3.$$

Therefore, by Lemma 7(2), $\mathsf{rk} \, \widehat{P} \leq \max\left(\left\lceil \tfrac{25}{24} \lceil \log |P| \rceil + 5 \right\rceil, L+1\right) |P| \, 2^{|P|} + 3.$ ■

The execution of imitate refers as to an oracle to a strong formative process of $P^{(\bullet)}$. Should this process not be available, an execution could nevertheless be performed, albeit non-deterministically, to take into consideration all possible response sequences from the oracle; then, at the tip of each branch of the non-deterministic execution tree, one could directly establish whether or not the sequence $(\{\widehat{q}^{[\mu_i]}\}_{q \in P})_{i \leq \ell}$ constructed by the procedure imitate L-simulates $P^{(\bullet)}$.

For technical reasons, we will assume that $P^{(\bullet)} \not\subseteq \{\{\emptyset\}\}$. It will easily turn out that such a constraint will not affect the applicability of the procedure imitate to the satisfiability problem we are interested in.

procedure imitate($\varrho, P, (A_\nu)_{\nu < \xi}$);
C_0 : **claim**
    $|P| < \aleph_0 \wedge \varrho > L \wedge 2^{\varrho - 1} > \varrho \cdot |P| \wedge (A_\mu)_{\mu < \xi}$ is the trace associated
    strong formative process $(\{q^{(\mu)}\}_{q \in P})_{\mu \leq \xi}$ such that $P^{(\bullet)} \not\subseteq \{\{\emptyset\}\}$;
    explanation: the formative process $(\{q^{(\mu)}\}_{q \in P})_{\mu \leq \xi}$ will be taken as an oracle
        in what follows, by referring to the blocks $q^{(\mu)}$ with $q \in P$ and to the
        partitions $A_\nu^{(\mu)}$ as if they were available as additional inputs;

$$M_1 := \{\, \mu \mid \mu < \xi \wedge (\forall p \in A_\mu)\,(\,|p^{(\mu)}| < \varrho)\,\};$$

$$M_2 := \Big\{\, \mu \mid \mu < \xi \wedge (\exists\, q \in P)\, \Big(\, \Big(|q^{(\mu)}| < \varrho \vee q^{(\mu)} \cap \mathscr{P}^*(A_\mu^{(\mu)}) = \emptyset\Big) \wedge \Delta^{(\mu)}(q) \neq \emptyset\Big)\,\Big\};$$

$$M_3 := \{\, \mu \mid \mu < \xi \wedge \bigcup A_\mu^{(\mu)} = \bigcup A_\mu^{(\bullet)} \in \bigcup P^{(\bullet)}\,\};$$

$\mathsf{A_0}$: **assert**

$$|M_1 \cup M_2 \cup M_3| < \varrho \cdot |P| \cdot 2^{|P|} + 3;$$

let $\{\mu_0, \mu_1, \ldots, \mu_\ell\} = M_1 \cup M_2 \cup M_3 \cup \{\xi\}$, **with** $\mu_0 < \mu_1 < \cdots < \mu_\ell$;

for $q \in P$ **loop** $\widehat{q} := \emptyset$; $\quad \Delta(q) := \emptyset$; **end loop**;

notation: throughout, and for all $B \subseteq P$, $\widehat{B} =_{\mathrm{Def}} \{\widehat{q} \mid q \in B\}$;

for $i \in [0, \ldots, \ell]$ **loop** -- the main loop begins here

    for $q \in P$ **loop** $\widehat{q} := \widehat{q} \cup \Delta(q)$; **end loop**;

$\mathsf{C_1}$:   **claim**

$$(\forall q \in P)\Big(\, \big(\,(\,|q^{(\mu_i)}| < \varrho \vee |\widehat{q}| < \varrho\,) \rightarrow |\widehat{q}| = |q^{(\mu_i)}|\,\big)$$
$$\wedge\, (\forall B \subseteq P)\Big(\, (\forall p \in B)(|p^{(\mu_i)}| < \varrho)$$
$$\rightarrow |q^{(\mu_i)} \cap \mathscr{P}^*(B^{(\mu_i)})| = \Big|\widehat{q} \cap \mathscr{P}^*(\widehat{B})\Big|\,\Big)$$
$$\wedge\, (\forall B \subseteq P)\Big(\, q^{(\mu_i)} \ni\in \mathscr{P}^*(B^{(\mu_i)}) \leftrightarrow \widehat{q} \ni\in \mathscr{P}^*(\widehat{B})\,\Big)\,\Big);$$

if $i = \ell$ **then quit loop; end if**;

    -- the seemingly useless last iteration calls for a final verification of $\mathsf{C_1}$

Δ := **if** $\mu_i \in M_1$ **then**

      $\mathsf{revise_1}(\mu_i, \widehat{})$

    else if $\mu_i \in M_3$ **then**

      $\mathsf{revise_3}(\mu_i, \widehat{})$

    else -- $\mu_i \in M_2$

      $\mathsf{revise_2}(\mu_i, \widehat{})$

    end if;

$\mathsf{C_2}$:   **claim**

$$\mathsf{subPartitions}(\,\Delta,\, A_{\mu_i}\,) \wedge (\bigcup \widehat{P}) \cap \bigcup \Delta[P] = \emptyset;$$

$\mathsf{C_3}$:   **claim**

$$(\forall q \in P)\big(\,\Delta^{(\mu_i)}(q) = \emptyset \rightarrow \Delta(q) = \emptyset\,\big);$$

  end loop;

$\mathsf{C_4}$: **claim**

    $\widehat{P}$ is a transitive partition ϱ-imitating $P^{(\bullet)}$, hence $\widehat{P}$ L-simulates $P^{(\bullet)}$;

return $\widehat{P}$;

procedure $\mathsf{revise_1}(\,\mu,\, \widehat{}\,)$;

    $A := A_\mu$;

$\mathsf{A_1}$:   **assert**

$$\Big(\exists \{\Delta(r)\}_{r \in P}\Big)\,\Big($$
$$\mathsf{subPartitions}(\,\Delta,\, A\,) \wedge$$
$$(\forall q \in P)\big(|\Delta(q)| = |\Delta^{(\mu)}(q)| \wedge$$
$$\big(\,\bigcup \widehat{A} \in \Delta(q) \leftrightarrow \bigcup A^{(\mu)} \in \Delta^{(\mu)}(q)\,\big)\,\big)\,\Big);$$

    pick one such Δ;   **return** Δ;

  end $\mathsf{revise_1}$;

procedure $\mathsf{revise_2}(\,\mu,\, \widehat{}\,)$;

    $A := A_\mu$;

$\mathsf{A_2}$:   **assert**

$$\Big(\exists \{\Delta(p)\}_{p \in P}\Big)\,\Big(\, \mathsf{subPartitions}(\,\Delta,\, A\,) \wedge \bigcup \widehat{A} \notin \bigcup \Delta[P] \wedge$$
$$\big(\,\forall q \in P\,\big)\,\Big($$
$$\textbf{if}\ |q^{(\mu+1)}| < \varrho \vee \Delta^{(\mu)}(q) = \emptyset\ \textbf{then}$$
$$|\Delta(q)| = |\Delta^{(\mu)}(q)|$$
$$\textbf{else if}\ q^{(\mu)} \cap \mathscr{P}^*(A^{(\mu)}) = \emptyset\ \wedge\ |\widehat{q}| \geq \varrho\ \textbf{then}$$
$$|\Delta(q)| \geq 1$$
$$\textbf{else}$$

$$|\Delta(q)| \geq \varrho - |\widehat{q}|$$
$$\textbf{end if } \Big) \wedge$$
$$\left|\mathscr{S}^*(\widehat{A}) \setminus \bigcup \widehat{P} \setminus \bigcup \Delta[P]\right| \geq 1 +$$
$$+ \sum_{\substack{r \in P \\ |r^{(\bullet)}| < \varrho}} \left|\big(r^{(\bullet)} \setminus r^{(\mu+1)}\big) \cap \mathscr{S}^*(A^{(\bullet)})\right| + \sum_{\substack{r \in P \\ |r^{(\bullet)}| \geq \varrho \\ |r^{(\mu+1)}| < \varrho}} \big(\varrho - |r^{(\mu+1)}|\big)$$
$$+ \left|\{r \in P \mid |r^{(\mu+1)}| \geq \varrho \wedge r^{(\bullet)} \cap \mathscr{S}^*(A^{(\bullet)}) \neq \emptyset \wedge r^{(\mu+1)} \cap \mathscr{S}^*(A^{(\mu)}) = \emptyset\}\right|$$
$$\Big);$$

        pick one such Δ;   **return** Δ;
      end revise$_2$;

      procedure revise$_3$(μ, $\widehat{\ }$);
        $A := A_\mu$;

A$_3$:   **assert**
$$\Big(\exists \{\Delta(p)\}_{p \in P}\Big) \Big(\text{subPartitions}(\Delta, A) \wedge$$
$$\big(\forall q \in P\big) \Big(\big(\bigcup \widehat{A} \in \Delta(q) \leftrightarrow \bigcup A^{(\bullet)} \in q^{(\bullet)} \big) \wedge$$
$$\textbf{if } |q^{(\mu+1)}| < \varrho \vee \Delta^{(\mu)}(q) = \emptyset \textbf{ then}$$
$$|\Delta(q)| = |\Delta^{(\mu)}(q)|$$
$$\textbf{else if } q^{(\mu)} \cap \mathscr{S}^*(A^{(\mu)}) = \emptyset \wedge |\widehat{q}| \geq \varrho \textbf{ then}$$
$$|\Delta(q)| \geq 1$$
$$\textbf{else}$$
$$|\Delta(q)| \geq \varrho - |\widehat{q}|$$
$$\textbf{end if } \Big) \wedge$$
$$\big(\mathscr{S}^*(A^{(\bullet)}) \subseteq \bigcup P^{(\bullet)} \rightarrow \bigcup \Delta[P] = \mathscr{S}^*(\widehat{A}) \setminus \bigcup \widehat{P} \big)$$
$$\Big);$$

        pick one such Δ;   **return** Δ;
      end revise$_3$;

      procedure subPartitions(Δ, B);
        return
$$\emptyset \neq \bigcup \Delta[P] \subseteq \mathscr{S}^*(\widehat{B}) \setminus \bigcup \widehat{P} \wedge \big(\forall q, r \in P\big)\big(q \neq r \rightarrow \Delta(q) \cap \Delta(r) = \emptyset\big);$$
      end subPartitions;

end imitate.

8. Proof of the main claim-statements occurring inside imitate

Checking that all claim- and assert-statements occurring inside the procedure imitate are fulfilled whenever such statements are met during execution, or simply getting a clear overall view of what the procedure does, calls for a detailed and lengthy analysis. This is the main task we are undertaking here.

    As a matter of notation and terminology, we introduce the following:

Definition 15. *For $q \in P$, $B \subseteq P$, and $i \in \{0, \ldots, \ell\}$, let $\widehat{q}^{[i]}$ and $\widehat{B}^{[i]}$ be the values of $\widehat{q}$ and $\widehat{B}$ when the claim* C$_1$ *is encountered during the $(i+1)$-st iteration of the main for-loop of* imitate; *moreover, let $\Delta^{[i]}$ be defined similarly when $i < \ell$, but referring to* C$_2$.*

    We will say that a claim- or assert-statement C is FULFILLED *(or is* met*) for $i = 0, \ldots, k$ to mean that whenever C is encountered during one of the initial $k+1$ iterations of the main for-loop of* imitate, *it will turn out to be true. We will say that it eventually gets* VIOLATED *if the opposite event takes place for some k.*   □

(We will also say that a statement C is fulfilled —or is violated— when $i = i_0$, etc.)

The claims to be proved are C_1–C_4 only: C_0, in fact, expresses conditions which the input parameters are supposed to comply with. The statements A_0–A_3 must be proved too; such statements are preceded by the keyword 'assert' instead of by 'claim' simply to stress that one cannot erase them without disrupting executability. In fact A_1–A_3 claim the existence of partitioning functions which are referred to by subsequent executable statements; as regards A_0, unless $M_1 \cup M_2 \cup M_3$ were finite then the semantics of the let-statement following A_0 would become unclear.

The approach to the intricate proof we will carry out is to start with the absurd hypothesis that some of the statements C_1–C_4, A_0–A_3 will fail to be true at some time. In this case one could, at least in principle, isolate as 'culprit' the statement which fails first, and spot out the latest value of the variable i when this event takes place. However, by induction on i, we will reach a contradiction whichever claim- or assert-statement one might indicate as the culprit.

Potential culprits should be passed in review all within the same inductive proof; however, in order to subdivide the difficulties and to let the reader gain a better grasp of the overall mechanism, we prefer to concentrate on C_2, C_3, and C_4 for the rest of this section, while postponing to subsequent sections the treatment of all other potential culprits (which, momentarily, are supposed here to be 'innocent'). Merging the various parts of the proof into a single proof poses, of course, no conceptual challenge.

The claims we have selected for immediate treatment lie, in a sense, at a higher level, and by discussing them we will unroll a landscape view of the ongoing. Here is the leading idea behind the procedure imitate, as it emerges in the light of C_2, taking it momentarily for granted that the statement A_0 is fulfilled (cf. Lemma 15 below):

Lemma 13. *Let $0 \leq k \leq \ell$. If none of the statements C_1–C_3, A_1–A_3 in imitate ever gets violated for $i = 0, 1, \ldots, k-1$, then the functions $\{\widehat{q}^{[j]}\}_{q \in P}$ ($j = 0, 1, \ldots, k$) make a weak formative process on $\{\, q \in P \mid \widehat{q}^{[k]} \neq \emptyset \,\}$ with trace $A_{\mu_0}, \ldots, A_{\mu_{k-1}}$.*

Proof. Regular termination of the k-th iteration of the main loop of imitate is obviously ensured by the assumption that assert-statements are fulfilled every time they get reached. Then one observes that $\widehat{q}^{[0]} = \emptyset$ and $\widehat{q}^{[i+1]} = \widehat{q}^{[i]} \cup \Delta^{[i]}(q)$, where $\{\, \Delta^{[i]}(q) \mid q \in P \,\} \setminus \{\emptyset\}$ is a partition of some non-null $Q \subseteq \mathscr{P}^*(\widehat{A}^{[i]}_{\mu_i}) \setminus \bigcup \widehat{P}^{[i]})$, by C_2, as inspection of subPartitions reveals. By contrasting all of this with Def.12 (ignoring the condition that regards limit ordinals, since $\ell < \omega$, and taking Remark 2(3) into account), one sees that the thesis holds. ∎

Then C_4, that we are about to discuss, explains what the final situation will be: by stating that $\widehat{P}^{[\ell]}$ must be a partition, it in fact indicates that the functions $\{\widehat{q}^{[j]}\}_{q \in P}$ ($j = 0, 1, \ldots, \ell$) will make a weak formative process on P with trace $A_{\mu_0}, \ldots, A_{\mu_{\ell-1}}$.

The second half of C_4 follows immediately from Lemma 4, by the assumption $\varrho > L$ in C_0. Hence, by Def.9 and by the subinstance $(\forall q \in P)\big(\,\big|q^{(\mu_i)}\big| < \varrho \ \to \ \big|\widehat{q}^{[i]}\big| = \big|q^{(\mu_i)}\big|\,\big)$ of C_1 (which, in the case $i = \ell$ can be written more shortly as $(\forall q \in P)\,\big(\big|q^{(\bullet)}\big| < \varrho \to \big|\widehat{q}^{[\ell]}\big| = \big|q^{(\bullet)}\big|\big)$), checking C_4 reduces to verifying the following conditions:

Lemma 14. *(i) $\widehat{P}^{[\ell]}$ is a transitive partition;*

(ii) if $\mathscr{P}^*(\widehat{B}^{[\ell]}) \ni\in \widehat{q}^{[\ell]}$, then $\mathscr{P}^*(B^{(\bullet)}) \ni\in q^{(\bullet)}$;

(iii) $\bigcup \widehat{B}^{[\ell]} \in \widehat{q}^{[\ell]}$ if and only if $\bigcup B^{(\bullet)} \in q^{(\bullet)}$;

(iv) if $\mathscr{S}^*(B^{(\bullet)}) \subseteq \bigcup P^{(\bullet)}$, then $\mathscr{S}^*(\widehat{B}^{[\ell]}) \subseteq \bigcup \widehat{P}^{[\ell]}$.

Proof. **(i):** Thanks to Lemma 13 and to Lemma 5, we can simplify (i) into $\emptyset \notin \widehat{P}^{[\ell]}$. Our task, accordingly, will be to prove that $\widehat{q}^{[\ell]} \neq \emptyset$ holds for each $q \in P$. Since $q^{(\bullet)} \neq \emptyset$ and $q^{(\bullet)} = \bigcup_{0 \leq \mu < \xi} \Delta^{(\mu)}(q)$ by Lemma 7(3),(1), it makes sense to consider the least ordinal $\overline{\mu}$ for which $\Delta^{(\overline{\mu})}(q) \neq \emptyset$; i.e., $\Delta^{(\overline{\mu})}(q) \neq \emptyset \ \wedge \ (\forall \mu)(0 \leq \mu < \overline{\mu} \rightarrow \Delta^{(\mu)}(q) = \emptyset)$, whence $q^{(\overline{\mu})} = \emptyset$ easily follows.

We immediately notice that $\overline{\mu} \in M_2$, so that $\overline{\mu} = \mu_{i_0}$ for some $i_0 < \ell$; thus, if we manage to prove that $\Delta^{[i_0]}(q) \neq \emptyset$ then we can conclude that $\widehat{q}^{[\ell]} \neq \emptyset$, because $\widehat{q}^{[\ell]} \supseteq \Delta^{[i_0]}(q)$. Notice that $\widehat{q}^{[i_0]} = \emptyset$ ensues from $q^{(\mu_{i_0})} = \emptyset$, thanks to the first conjunct in $\mathsf{C_1}$.

If $\overline{\mu} \in M_1$ then, by the assert-statement in $\mathsf{revise_1}$, $\left|\Delta^{[i_0]}(q)\right| = \left|\Delta^{(\overline{\mu})}(q)\right|$ holds. If $\overline{\mu} \in (M_3 \cup M_2) \setminus M_1$, then inspection of the assert-statements $\mathsf{A_3}$ and $\mathsf{A_2}$ reveals two possibilities only: either $\left|\Delta^{[i_0]}(q)\right| = \left|\Delta^{(\overline{\mu})}(q)\right|$, or $\left|\Delta^{[i_0]}(q)\right| \geq \varrho - \left|\widehat{q}^{[i_0]}\right| = \varrho > 0$. In either case $\Delta^{[i_0]}(q) \neq \emptyset$, and hence our thesis $\widehat{q}^{[\ell]} \neq \emptyset$, holds.

(ii): This follows immediately from the third part of claim $\mathsf{C_1}$.

It is worth noticing that even if claim $\mathsf{C_1}$ did not include this sub-statement, one could nevertheless obtain (ii) by the following plain argument. Assume that $\mathscr{S}^*(\widehat{B}^{[\ell]}) \ni \in \widehat{q}^{[\ell]}$. Since $\widehat{q}^{[\ell]} = \bigcup_{0 \leq i < \ell} \Delta^{[i]}(q)$, let $i_0 < \ell$ be such that $\mathscr{S}^*(\widehat{B}^{[\ell]}) \ni \in \Delta^{[i_0]}(q)$, so that $\mathscr{S}^*(\widehat{B}^{[i_0]}) \ni \in \Delta^{[i_0]}(q)$ holds by Lemma 13 and Lemma 7(4). Put $\overline{\mu} = \mu_{i_0}$. Clearly, we must have $B = A_{\overline{\mu}}$. From $\mathsf{C_3}$, it follows that $\Delta^{(\overline{\mu})}(q) \neq \emptyset$; hence $\Delta^{(\overline{\mu})}(q) \ni \in \mathscr{S}^*(A_{\overline{\mu}}^{(\overline{\mu})}) = \mathscr{S}^*(B^{(\overline{\mu})})$, and therefore $q^{(\bullet)} \ni \in \mathscr{S}^*(B^{(\bullet)})$, which is the desired conclusion.

(iii⇒): Assuming that $\bigcup \widehat{B}^{[\ell]} \in \widehat{q}^{[\ell]}$, there must be an $i_0 < \ell$ such that $\bigcup \widehat{B}^{[\ell]} \in \Delta^{[i_0]}(q)$. Therefore, putting $\overline{\mu} = \mu_{i_0}$, we will have $B = A_{\overline{\mu}}$ and $\bigcup \widehat{A}_{\overline{\mu}}^{[\ell]} = \bigcup \widehat{A}_{\overline{\mu}}^{[i_0]}$. Observe that $\overline{\mu} \notin M_2$, since otherwise, by the statement $\mathsf{A_2}$, $\bigcup \widehat{A}_{\overline{\mu}}^{[i_0]} \notin \bigcup \Delta^{[i_0]}[P]$ should hold. Therefore, only the case $\overline{\mu} \in M_1 \cup M_3$ must be considered.

Assume first that $\overline{\mu} \in M_3 \setminus M_1$. By the statement $\mathsf{A_3}$, it follows readily that $\bigcup B^{(\bullet)} = \bigcup A_{\overline{\mu}}^{(\bullet)} \in q^{(\bullet)}$.

On the other hand, if $\overline{\mu} \in M_1$ then, by the statement $\mathsf{A_1}$, $\bigcup A_{\overline{\mu}}^{(\overline{\mu})} \in \Delta^{(\overline{\mu})}(q) \subseteq q^{(\bullet)}$. Notice that $\left|p^{(\overline{\mu})}\right| < \varrho$ holds for all $p \in A_{\overline{\mu}}$, and hence, in consequence of claim $\mathsf{C_1}$, we have $\left|\widehat{p}^{[i_0]}\right| = \left|p^{(\overline{\mu})}\right|$ for every $p \in A_{\overline{\mu}}$. Moreover, since $\widehat{A}_{\overline{\mu}}^{[\ell]} = \widehat{A}_{\overline{\mu}}^{[i_0]}$, we also have $\left|\widehat{p}^{[\ell]}\right| = \left|\widehat{p}^{[i_0]}\right| < \varrho$ for every $p \in A_{\overline{\mu}}$, so that, again by claim $\mathsf{C_1}$, $\left|p^{(\bullet)}\right| = \left|\widehat{p}^{[\ell]}\right| = \left|\widehat{p}^{[i_0]}\right| = \left|p^{(\overline{\mu})}\right|$, which in turn implies $p^{(\bullet)} = p^{(\overline{\mu})}$, for every $p \in A_{\overline{\mu}}$. Therefore $\bigcup A_{\overline{\mu}}^{(\bullet)} = \bigcup A_{\overline{\mu}}^{(\overline{\mu})} \in q^{(\bullet)}$, and we are done.

(iii⇐): Assuming that $\bigcup B^{(\bullet)} \in q^{(\bullet)}$, there must be a $\overline{\mu}$ such that $\bigcup B^{(\bullet)} \in \Delta^{(\overline{\mu})}(q)$, since $q^{(\bullet)} = \bigcup_{\mu < \xi} \Delta^{(\mu)}(q)$. Therefore $B = A_{\overline{\mu}}$, $\bigcup B^{(\bullet)} = \bigcup B^{(\overline{\mu})}$, and $\overline{\mu} \in M_3$ hold. Let $i_0 < \ell$ be such that $\mu_{i_0} = \overline{\mu}$.

If $\overline{\mu} \in M_1$ then, since $\bigcup A_{\overline{\mu}}^{(\overline{\mu})} \in \Delta^{(\overline{\mu})}(q)$, we have $\bigcup \widehat{A}_{\overline{\mu}}^{[i_0]} \in \Delta^{[i_0]}(q)$ by the statement $\mathsf{A_1}$. Likewise, if $\overline{\mu} \in M_3 \setminus M_1$, then $\bigcup \widehat{A}_{\overline{\mu}}^{[i_0]} \in \Delta^{[i_0]}(q)$, by the statement $\mathsf{A_3}$. Notice also that $\Delta^{(\mu_i)}(p) = \emptyset$ and hence, by $\mathsf{C_3}$, $\Delta^{[i]}(p) = \emptyset$, must hold for all $p \in A_{\overline{\mu}}$ and all $i \in \{i_0, \ldots, \ell - 1\}$. Therefore $\bigcup \widehat{B}^{[\ell]} = \bigcup \widehat{A}_{\overline{\mu}}^{[\ell]} = \bigcup \widehat{A}_{\overline{\mu}}^{[i_0]} \in \Delta^{[i_0]}(q) \subseteq \widehat{q}^{[\ell]}$ holds, as desired.

(iv): Let $\mathscr{S}^*(B^{(\bullet)}) \subseteq \bigcup P^{(\bullet)}$. Then, since $\bigcup B^{(\bullet)} \in \mathscr{S}^*(B^{(\bullet)}) \subseteq \bigcup P^{(\bullet)}$, where $\bigcup P^{(\bullet)} = \bigcup_{p \in P} \bigcup_{\mu < \xi} \Delta^{(\mu)}(p)$ by Lemma 7, we have $\bigcup B^{(\bullet)} \in \Delta^{(\overline{\mu})}(p)$ for suitable $p \in P$ and $\overline{\mu} < \xi$. Hence $B = A_{\overline{\mu}}$ and $\bigcup A_{\overline{\mu}}^{(\overline{\mu})} = \bigcup A_{\overline{\mu}}^{(\bullet)}$, and therefore $\overline{\mu} \in M_3$. Let $i_0 < \ell$ be such that $\overline{\mu} = \mu_{i_0}$.

From $\bigcup A_{\underline{\mu}}^{(\overline{\mu})} = \bigcup A_{\underline{\mu}}^{(\bullet)}$ it follows that $\Delta^{(\mu)}(p) = \emptyset$, for $p \in A_{\overline{\mu}}$ and $\overline{\mu} \le \mu < \xi$, whence, by claim C_3, $\Delta^{[i]}(p) = \emptyset$ ensues for $p \in A_{\overline{\mu}}$ and $i_0 \le i < \ell$. Hence $\widehat{A}_{\underline{\mu}}^{[i_0]} = \widehat{A}_{\underline{\mu}}^{[\ell]}$.

Our goal in what follows is to show that

$$\bigcup \Delta^{[i_0]}[P] = \mathscr{P}^*(\widehat{A}_{\underline{\mu}}^{[i_0]}) \setminus \bigcup \widehat{P}^{[i_0]} \tag{1}$$

holds; this will readily yield that

$$\begin{aligned}
\mathscr{P}^*(\widehat{A}_{\underline{\mu}}^{[\ell]}) \;=\; \mathscr{P}^*(\widehat{A}_{\underline{\mu}}^{[i_0]}) \;&\subseteq\; \bigcup \widehat{P}^{[i_0]} \;\cup\; \bigcup \Delta^{[i_0]}[P] \\
&=\; \bigcup \widehat{P}^{[i_0+1]} \;\subseteq\; \bigcup \widehat{P}^{[\ell]},
\end{aligned}$$

which encompasses our desired conclusion.

If $\overline{\mu} \in M_3 \setminus M_1$, then (1) follows immediately from the statement A_3.

On the other hand, if $\overline{\mu} \in M_1$, then we have

- $\left| \bigcup \Delta^{[j]}[P] \right| = \left| \bigcup \Delta^{(\mu_j)}[P] \right|$ for all j such that $0 \le j \le i_0$ and $A_{\mu_j} = A_{\overline{\mu}}$ (by the statement A_1);

- $\left| \mathscr{P}^*(\widehat{A}_{\underline{\mu}}^{[i_0]}) \right| = \left| \mathscr{P}^*(A_{\underline{\mu}}^{(\overline{\mu})}) \right|$ (by the statement C_1);

- $\left| \mathscr{P}^*(\widehat{A}_{\underline{\mu}}^{[i_0]}) \cap \bigcup \widehat{P}^{[i_0]} \right| = \sum\limits_{\substack{0 \le j < i_0 \\ A_{\mu_j} = A_{\overline{\mu}}}} \left| \bigcup \Delta^{[j]}[P] \right|$

  $= \sum\limits_{\substack{0 \le j < i_0 \\ A_{\mu_j} = A_{\overline{\mu}}}} \left| \bigcup \Delta^{(\mu_j)}[P] \right| = \left| \mathscr{P}^*(A_{\underline{\mu}}^{(\overline{\mu})}) \cap \bigcup P^{(\overline{\mu})} \right|$;

- $\bigcup \Delta^{(\overline{\mu})}[P] = \mathscr{P}^*(A_{\underline{\mu}}^{(\overline{\mu})}) \setminus \bigcup P^{(\overline{\mu})}$

  (by $\mathscr{P}^*(A_{\underline{\mu}}^{(\bullet)}) \subseteq \bigcup P^{(\bullet)}$ and the coherence requirement).

Therefore

$$\begin{aligned}
\left| \bigcup \Delta^{[i_0]}[P] \right| \;&=\; \left| \bigcup \Delta^{(\overline{\mu})}[P] \right| = \left| \mathscr{P}^*(A_{\underline{\mu}}^{(\overline{\mu})}) \setminus \bigcup P^{(\overline{\mu})} \right| \\
&=\; \left| \mathscr{P}^*(A_{\underline{\mu}}^{(\overline{\mu})}) \right| - \left| \mathscr{P}^*(A_{\underline{\mu}}^{(\overline{\mu})}) \cap \bigcup P^{(\overline{\mu})} \right| \\
&=\; \left| \mathscr{P}^*(\widehat{A}_{\underline{\mu}}^{[i_0]}) \right| - \left| \mathscr{P}^*(\widehat{A}_{\underline{\mu}}^{[i_0]}) \cap \bigcup \widehat{P}^{[i_0]} \right| \\
&=\; \left| \mathscr{P}^*(\widehat{A}_{\underline{\mu}}^{[i_0]}) \setminus \bigcup \widehat{P}^{[i_0]} \right|
\end{aligned}$$

holds, which in turn plainly implies (1), concluding our proof. ∎

Inspection of revise_1, revise_2, and revise_3 makes it plain that C_3 holds in consequence of A_1–A_3; the verification that C_2 ensues from A_1–A_3 is equally trivial, as one sees by inspection of the procedure $\mathsf{subPartitions}$.

9. Rough assessment of the complexity of imitate

The following lemma not only shows that the cardinality $|M_1 \cup M_2 \cup M_3|$ is finite (thereby ensuring that the main loop of imitate will be executed finitely many times), but even tightens w.r.t. A_0 the upper bound on this cardinality, setting the ground for the complexity analysis that will be carried out in Sec. 11.

Lemma 15. *Assuming the conditions in claim C_0 to hold, let $n = |P|$. Then*
$$|M_1 \cup M_2 \cup M_3| \le \varrho n 2^{n-1} + 2^{n+1} + (\varrho - 1)n - 1 < \varrho n 2^n + 3.$$

Proof. We begin by first estimating $|M_1|$. Let $B \subseteq P$. Notice that if $A_\mu = A_{\mu'} = B$, with $\mu < \mu'$, $\mu, \mu' \in M_1$, then by the pigeon-hole principle the following inequalities hold:
$|B| \le \left| \bigcup B^{(\mu)} \right| < \left| \bigcup B^{(\mu')} \right| \le (\varrho - 1)|B|$. Hence,

$$|\{\mu \in M_1 \mid A_\mu = B\}| \leq (\varrho - 1)|B| - |B| + 1 = (\varrho - 2)|B| + 1 \; .$$

Therefore

$$
\begin{aligned}
|M_1| \quad \leq \quad & \sum_{B \subseteq P} ((\varrho - 2)|B| + 1) && = \quad \sum_{i=0}^{n} \binom{n}{i} ((\varrho - 2)i + 1) \\
= \quad & (\varrho - 2) \sum_{i=0}^{n} i \binom{n}{i} + 2^n && = \quad (\varrho - 2) \sum_{i=1}^{n} n \binom{n-1}{i-1} + 2^n \\
= \quad & (\varrho - 2) n \sum_{i=0}^{n-1} \binom{n-1}{i} + 2^n && = \quad (\varrho - 2) n 2^{n-1} + 2^n \; .
\end{aligned}
$$

In order to make an estimate of $|M_2 \setminus M_1|$, let us put

$$M_2' = \left\{ \mu \mid \mu < \xi \wedge (\exists q \in P) \left(\left| q^{(\mu)} \right| < \varrho \wedge \Delta^{(\mu)}(q) \neq \emptyset \right) \right\}$$

and

$$M_2'' = \left\{ \mu \mid \mu < \xi \wedge (\exists q \in P) \left(q^{(\mu)} \cap \mathscr{P}^*(A_\mu^{(\mu)}) = \emptyset \wedge \Delta^{(\mu)}(q) \neq \emptyset \right) \right\} \; .$$

Plainly, $M_2 = M_2' \cup M_2''$. We first estimate $|M_2' \setminus M_1|$. Thus, let
$$\Phi(\mu) = \sum_{|q^{(\mu)}| < \varrho} (\varrho - |q^{(\mu)}|)$$
and observe that

- $\Phi(\mu) > \Phi(\mu + 1)$, for all $\mu \in M_2'$;

- $\Phi(\mu) \geq \Phi(\mu + 1)$, for all $\mu < \xi$;

- $0 \leq \Phi(\mu) \leq \varrho n$, for all $\mu < \xi$;

- $0 \in M_1 \cap M_2'$.

From these we immediately get $|M_2' \setminus M_1| \leq \varrho n$.

Next, we estimate $|M_2'' \setminus M_1|$. Let us put
$$M_2''(B, q) =_{\mathrm{Def}} \{\mu \in M_2'' \setminus M_1 \mid A_\mu = B \wedge q^{(\mu)} \cap \mathscr{P}^*(B^{(\mu)}) = \emptyset \wedge \Delta^{(\mu)}(q) \neq \emptyset\},$$
for $B \subseteq P$ and $q \in P$, and observe that

- $M_2''(\emptyset, q) = \emptyset$, for all $q \in P$, since $\emptyset = A_\mu$ iff $\mu = 0$ and moreover $0 \in M_1$;

- $|M_2''(B, q)| \leq 1$, for all $B \subseteq P$ and $q \in P$;

- $M_2'' \setminus M_1 \subseteq \bigcup_{\substack{B \subseteq P \\ q \in P}} M_2''(B, q)$.

From these we immediately obtain $|M_2'' \setminus M_1| \leq n(2^n - 1)$. Therefore
$$|M_2 \setminus M_1| \leq |M_2' \setminus M_1| + |M_2'' \setminus M_1| \leq n(2^n - 1) + \varrho n.$$

Finally we estimate $|M_3 \setminus M_1|$. Since $0 \in M_1 \cap M_3$ and $|\{\mu \in M_3 \mid A_\mu = B\}| \leq 1$, for all $B \subseteq P$, we obtain at once $|M_3 \setminus M_1| \leq 2^n - 1$.

Summing up, we have
$$|M_1 \cup M_2 \cup M_3| \leq \varrho n 2^{n-1} + 2^{n+1} + (\varrho - 1)n - 1 \; .$$

Since $n \geq 1$ and $\varrho \geq 3$ (the latter follows from the assumptions $\varrho > L$ and $2^{\varrho - 1} > \varrho n$ in claim C_0), an easy inductive argument shows that $\varrho n 2^{n-1} + 2^{n+1} + (\varrho - 1)n - 1 < \varrho n 2^n + 3$, thus completing the proof of the lemma. ∎

To end this section, it will be instructive to examine the pattern of calls of the **revise** procedures whose actual parameter μ is associated with some fixed set $B = A_\mu$. It is largely unpredictable how, globally, **revise**-calls are interleaved; but when one focuses on a single B the following regularity emerges:

Lemma 16. *Let* $\mathsf{revise}_{i_1}, \ldots, \mathsf{revise}_{i_m}$ *be the names of* **revise**-*procedures sequenced in the order in which consecutive calls referring to the same subset B of P take place. Then the list $i_1, \ldots, i_m$ of subscripts consists of 0, 1, or more consecutive $\mathbf{1}$s, followed by 0, 1, or more consecutive $\mathbf{2}$s, possibly followed by an occurrence of $\mathbf{3}$.*

Proof. Associate with the given B the set $\mathrm{M}_B = \{\, \mu < \xi \mid A_\mu = B \,\}$. In order to prove the thesis, it will suffice to verify that

(a) if there is a $\overline{\nu} \in \mathrm{M}_B \cap M_3$, then it will meet the condition $\overline{\nu} = \max \mathrm{M}_B$;

(b) if $\mu \in \mathrm{M}_B \cap M_1$, $\nu \in \mathrm{M}_B$, and $\nu < \mu$, then $\nu \in M_1$.

Indeed, (a) —which incidentally entails that $|\mathrm{M}_B \cap M_3| \leq 1$— will restrict the *a priori* possible subscript pattern $(\mathbf{1} \mid \mathbf{2} \mid \mathbf{3})^*$ into the pattern $(\mathbf{1} \mid \mathbf{2})^* \, [\mathbf{3}]$, and (b) will further restrict it into $\mathbf{1}^* \, \mathbf{2}^* \, [\mathbf{3}]$.

To prove (a), notice that if $\overline{\nu} \in \mathrm{M}_B \cap M_3$ then, for every $\mu \in \mathrm{M}_B$, one has $\bigcup A_\mu^{(\mu)} = \bigcup B^{(\mu)} \subseteq \bigcup B^{(\bullet)} = \bigcup A_{\overline{\nu}}^{(\overline{\nu})}$ and hence $\mu \leq \overline{\nu}$, by Lemma 7(2),(7).

To prove (b), assume that $\mu \in \mathrm{M}_B \cap M_1$ and $\nu \in \mathrm{M}_B$, $\nu < \mu$. Then, for any $p \in B = A_\mu$, the former assumption yields $\left| p^{(\mu)} \right| < \varrho$. Since $p^{(\nu)} \subseteq p^{(\mu)}$, we get $\left| p^{(\nu)} \right| < \varrho$. In conclusion, for every $p \in B = A_\nu$ one has $\left| p^{(\nu)} \right| < \varrho$ and hence $\nu \in M_1$. ∎

10. Proof of secondary claim- and assert-statements occurring inside **imitate**

Under the absurd hypothesis that one of $\mathsf{C}_1, \mathsf{A}_1, \mathsf{A}_2, \mathsf{A}_3$ can be violated during the execution of **imitate**, assume i_0 to be the value of i corresponding to such violation and let C be the claim- or assert-statement where this event takes place. Take it for granted here that C_2, and C_3 never get violated, as explained in Sec.8.

It is easy to check that i_0 cannot be 0. Indeed, when $i = 0$, then $\mu_i = 0$ and hence C_1 is met, because $\left| \widehat{q}^{[i]} \right| = \left| q^{(\mu_i)} \right| = 0 < \varrho$ for all q; moreover $\mu_i \in M_1$ and $A_{\mu_i} = \emptyset$ in this case, and hence the statement A_1 in **revise$_1$** is fulfilled (only) by the function $\Delta^{[i]}$ sending to $\emptyset$ every $q \in P$ save the one, $\overline{q}$, for which $\emptyset \in \overline{q}^{(\bullet)}$, which has $\Delta^{[i]}(\overline{q}) = \Delta^{(\mu_i)}(\overline{q}) = \{\emptyset\} = \{\bigcup \widehat{\emptyset}^{[i]}\} = \{\bigcup \emptyset^{(\mu_i)}\}$.

Our goal, with the following series of lemmas, is to show that even by assuming $i_0 \neq 0$ one is led to contradiction whatever C may be. A typical way of reaching contradiction will be by showing that some ordinal μ with $\mu_{i_0-1} < \mu < \mu_{i_0}$ must belong to $M_1 \cup M_2 \cup M_3$. Another way, when $i_0 < \ell$, will be by showing that $A_{\mu_{i_0}}^{(\mu_{i_0})} = A_{\mu_j}^{(\mu_j)}$ for some $j < i_0$: this would in fact contradict the coherence of the formative process (cf. Lemma 7(7)).

Lemma 17. *The 'culprit' statement C cannot be the statement A_1.*

Proof. Assuming μ_{i_0} to be in $M_1 \setminus \{0\}$ and C_1 to be fulfilled for $i = 0, \ldots, i_0$, we are to prove that A_1 is met when $i = i_0$. Let $\overline{\mu} = \mu_{i_0}$. Notice that C_1 implies

(a) $\left| \bigcup A_{\mu_i}^{(\mu_i)} \right| = \left| \bigcup \widehat{A}_{\mu_i}^{[i]} \right|$ and, more specifically, $\left| p^{(\mu_i)} \right| = \left| \widehat{p}^{[i]} \right|$ for all $p \in A_{\mu_i}$ and $i = 0, \ldots, i_0$; and moreover

(b) $\left| q^{(\overline{\mu})} \cap \mathscr{P}^*(A_{\overline{\mu}}^{(\overline{\mu})}) \right| = \left| \widehat{q}^{[i_0]} \cap \mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i_0]}) \right|$, for all $q \in P$.

From (a) and (b) we get
$$
\begin{aligned}
\left| \mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i_0]}) \setminus \bigcup \widehat{P}^{[i_0]} \right| &= \left| \mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i_0]}) \right| - \sum_{q \in P} \left| \widehat{q}^{[i_0]} \cap \mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i_0]}) \right| \\
&= \left| \mathscr{P}^*(A_{\overline{\mu}}^{(\overline{\mu})}) \right| - \sum_{q \in P} \left| q^{(\overline{\mu})} \cap \mathscr{P}^*(A_{\overline{\mu}}^{(\overline{\mu})}) \right| \\
&= \left| \mathscr{P}^*(A_{\overline{\mu}}^{(\overline{\mu})}) \setminus \bigcup P^{(\overline{\mu})} \right| .
\end{aligned}
$$

Since $\sum_{q \in P} \left| \Delta^{(\overline{\mu})}(q) \right| \le \left| \mathscr{P}^*(A_{\overline{\mu}}^{(\overline{\mu})}) \setminus \bigcup P^{(\overline{\mu})} \right|$, it is obvious that a function $\{\Delta^{[i_0]}(q)\}_{q \in P}$ exists such that

- $\Delta^{[i_0]}[P] \setminus \{\emptyset\}$ is partition of a subset of $\mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i_0]}) \setminus \bigcup \widehat{P}^{[i_0]}$;
- $\left| \Delta^{[i_0]}(q) \right| = \left| \Delta^{(\overline{\mu})}(q) \right|$, for all $q \in P$.

Notice, incidentally, that the latter condition plainly implies $\left| \bigcup \Delta^{[i_0]}[P] \right| = \left| \bigcup \Delta^{(\overline{\mu})}[P] \right| > 0$.

To ensure that also the rest of $\mathsf{A_1}$ can be met by $\Delta^{[i_0]}$, it will suffice to verify that neither $\bigcup A_{\overline{\mu}}^{(\overline{\mu})} \in \bigcup P^{(\overline{\mu})}$ nor $\bigcup \widehat{A}_{\overline{\mu}}^{[i_0]} \in \bigcup \widehat{P}^{[i_0]}$ holds.

On the one hand, in fact, if we make the absurd hypothesis that $\bigcup A_{\overline{\mu}}^{(\overline{\mu})} \in q^{(\overline{\mu})}$ for some q, then $\bigcup A_{\overline{\mu}}^{(\overline{\mu})} \in \Delta^{(\mu_j)}(q)$ must hold for some $j < i_0$, and hence $\bigcup A_{\overline{\mu}}^{(\overline{\mu})} = \bigcup A_{\mu_j}^{(\mu_j)}$, i.e. $A_{\overline{\mu}}^{(\overline{\mu})} = A_{\mu_j}^{(\mu_j)}$, against Lemma 7(7).

On the other hand, if we make the absurd hypothesis that $\bigcup \widehat{A}_{\overline{\mu}}^{[i_0]} \in \bigcup \widehat{P}^{[i_0]}$, then $\bigcup \widehat{A}_{\overline{\mu}}^{[i_0]} \in \Delta^{[j]}(q)$ for suitable $q \in P$, $j < i_0$, with $A_{\overline{\mu}} = A_{\mu_j}$. Hence $\bigcup \widehat{A}_{\overline{\mu}}^{[i_0]} = \bigcup \widehat{A}_{\mu_j}^{[j]}$, whence $\widehat{A}_{\overline{\mu}}^{[i_0]} = \widehat{A}_{\mu_j}^{[j]}$. However $A_{\overline{\mu}}^{(\overline{\mu})} \ne A_{\mu_j}^{(\mu_j)}$, and therefore $\widehat{A}_{\overline{\mu}}^{[i_0]} \ne \widehat{A}_{\mu_j}^{[j]}$ should hold by (a), which leads to a contradiction. ∎

Lemma 18. *The 'culprit' statement C cannot be the statement* $\mathsf{A_3}$.

Proof. Assuming μ_{i_0} to be in $M_3 \setminus M_1$, all claim- and assert-statements to be fulfilled for $i = 0, \dots, i_0 - 1$, and $\mathsf{C_1}$ to hold when $i = i_0$, we are to prove that $\mathsf{A_3}$ is met when $i = i_0$. Let $\overline{\mu} = \mu_{i_0}$. Recall from Lemma 9 that if $\mathscr{P}^*(A_{\overline{\mu}}^{(\bullet)}) \subseteq \bigcup P^{(\bullet)}$ then there is an $r \in P$ such that $\left| \Delta^{(\overline{\mu})}(r) \right| \ge \varrho$. Accordingly, we may ignore the last requirement of $\mathsf{A_3}$ in what follows: in fact, after checking that a Δ exists fulfilling all other requirements in $\mathsf{A_3}$, if $\mathscr{P}^*(A_{\overline{\mu}}^{(\bullet)}) \subseteq \bigcup P^{(\bullet)}$ then we can put
$$
X = \mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i_0]}) \setminus \bigcup \widehat{P}^{[i_0]} \setminus \bigcup \Delta[P]
$$
and define
$$
\Delta^{[i_0]}(p) = \textbf{if } p \ne r \textbf{ then } \Delta(p) \textbf{ else } \Delta(p) \cup X \textbf{ end if},
$$
so that $\bigcup \Delta^{[i_0]}[P] = \mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i_0]}) \setminus \bigcup \widehat{P}^{[i_0]}$ and even the last requirement of $\mathsf{A_3}$ will be met.

Case A: $A_\nu \ne A_{\overline{\mu}}$ holds for all $\nu \in M_2 \setminus M_1 \setminus M_3$ such that $\nu < \overline{\mu}$. Then

$$
\left| \mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i_0]}) \setminus \bigcup \widehat{P}^{[i_0]} \right| \ge 2^{\varrho - 1} > \varrho \cdot |P| . \tag{2}
$$

Indeed, since $\max\{ \left| \widehat{q}^{[i_0]} \right| \mid q \in A_{\overline{\mu}} \} \ge \varrho$, in order to verify (2) it will suffice by Lemma 2(6) to show that

$$
\bigcup \widehat{A}_{\overline{\mu}}^{[i_0]} \setminus \bigcup \left(\mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i_0]}) \cap \bigcup \widehat{P}^{[i_0]} \right) \ne \emptyset . \tag{3}
$$

Notice that we have

$$\bigcup\left(\mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i_0]})\cap\bigcup\widehat{P}^{[i_0]}\right)$$

$$= \bigcup\left(\bigcup_{\substack{i<i_0\\A_{\mu_i}=A_{\overline{\mu}}}}\mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i]})\cap\bigcup\widehat{P}^{[i_0]}\right)$$

$$\subseteq \bigcup\bigcup_{\substack{i<i_0\\A_{\mu_i}=A_{\overline{\mu}}}}\mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i]}) \;=\; \bigcup_{\substack{i<i_0\\A_{\mu_i}=A_{\overline{\mu}}}}\bigcup\mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i]}) \;=\; \bigcup_{\substack{i<i_0\\A_{\mu_i}=A_{\overline{\mu}}}}\bigcup\widehat{A}_{\overline{\mu}}^{[i]}$$

$$= \begin{cases} \emptyset & \text{if } (\forall\, i<i_0)(A_{\mu_i}\neq A_{\overline{\mu}})\,, \\ \bigcup\widehat{A}_{\overline{\mu}}^{[i_1]} & \text{otherwise, with } i_1=\max\{i\mid i<i_0\wedge A_{\mu_i}=A_{\overline{\mu}}\}\,. \end{cases}$$

If $A_{\mu_i}\neq A_{\overline{\mu}}$ for all $i<i_0$, then (3) plainly holds. On the other hand, if $A_{\mu_i}=A_{\overline{\mu}}$ holds for some $i<i_0$, then $\bigcup\widehat{A}_{\overline{\mu}}^{[i_0]}\setminus\bigcup(\mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i_0]})\cap\bigcup\widehat{P}^{[i_0]})\supseteq\bigcup\widehat{A}_{\overline{\mu}}^{[i_0]}\setminus\bigcup\widehat{A}_{\overline{\mu}}^{[i_1]}$. To see that $\bigcup\widehat{A}_{\overline{\mu}}^{[i_0]}\setminus\bigcup\widehat{A}_{\overline{\mu}}^{[i_1]}\neq\emptyset$, we notice that since $\overline{\mu}\notin M_1$, there exists a $p\in A_{\overline{\mu}}$ such that $|p^{(\overline{\mu})}|\geq\varrho$, and therefore, by the first part of claim C_1, $|\widehat{p}^{[i_0]}|\geq\varrho$. Since $\mu_{i_1}\in M_1$, we get $|p^{(\mu_{i_1})}|<\varrho$, and therefore, again by the first part of claim C_1, $|\widehat{p}^{[i_1]}|<\varrho$, so that $\emptyset\neq\widehat{p}^{[i_0]}\setminus\widehat{p}^{[i_1]}\subseteq\bigcup\widehat{A}_{\overline{\mu}}^{[i_0]}\setminus\bigcup\widehat{A}_{\overline{\mu}}^{[i_1]}$, which in turn implies (3).

From (2), a Δ meeting all requirements in A_3 but the last obviously exists. Notice, marginally, that the requirement in $\mathsf{subPartitions}$ that $\bigcup\Delta[P]$ differ from $\emptyset$ can be fulfilled: indeed, since $\bigcup\Delta^{(\overline{\mu})}[P]\neq\emptyset$, one cannot be forced to put $\Delta(q)=\emptyset$ for all $q\in P$.

Case B: $A_\nu=A_{\overline{\mu}}$ holds for some $\nu\in M_2\setminus M_1\setminus M_3$ with $\nu<\overline{\mu}$. In order to see that a Δ exists meeting all requirements in A_3 but the last, we reason as follows.

Let $\overline{\eta}=\mu_{h_0}=\max\{\eta\in M_2\setminus M_1\setminus M_3\mid\eta<\overline{\mu}\wedge A_\eta=A_{\overline{\mu}}\}$. Observe that the statement A_2 is fulfilled for all $i<i_0$ such that $\mu_i=\overline{\mu}$ and $\mu_i\in M_2\setminus M_1\setminus M_3$. In consequence of this we have $\bigcup\widehat{A}_{\overline{\mu}}^{[i_0]}\notin\bigcup\widehat{P}^{[i_0]}$ and

$$\left|\mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i_0]})\setminus\bigcup\widehat{P}^{[i_0]}\right| \geq \left|\mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[h_0]})\setminus\bigcup\widehat{P}^{[h_0]}\setminus\bigcup\Delta^{[h_0]}[P]\right|$$

$$\geq 1 + \sum_{|q^{(\bullet)}|<\varrho}\left|(q^{(\bullet)}\setminus q^{(\overline{\eta}+1)})\cap\mathscr{P}^*(A_{\overline{\mu}}^{(\bullet)})\right|$$

$$+ \sum_{\substack{|q^{(\bullet)}|\geq\varrho\\|q^{(\overline{\eta}+1)}|<\varrho}}(\varrho-|q^{(\overline{\eta}+1)}|)+|\mathrm{P}(\overline{\eta}+1,\overline{\eta})|,$$

where we are using the notation

$$\mathrm{P}(\mu,\nu)=\{q\in P\mid|q^{(\mu)}|\geq\varrho\wedge q^{(\bullet)}\cap\mathscr{P}^*(A_\nu^{(\bullet)})\neq\emptyset\wedge q^{(\mu)}\cap\mathscr{P}^*(A_\nu^{(\nu)})=\emptyset\},$$

for $\mu,\nu<\xi$ (the same notation will also be used in the next lemma).

Therefore a $\{\partial(q)\}_{q\in P}$ exists meeting the following conditions:

- $\bigcup\partial[P]\subseteq\mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i_0]})\setminus\bigcup\widehat{P}^{[i_0]}$;
- if $p\neq q$ then $\partial(p)\cap\partial(q)=\emptyset$, for $p,q\in P$;
- $\bigcup\widehat{A}_{\overline{\mu}}^{[i_0]}\in\partial(q)\leftrightarrow\bigcup A_{\overline{\mu}}^{(\overline{\mu})}\in\Delta^{(\overline{\mu})}(q)$;
- if $|q^{(\bullet)}|<\varrho$, then
$$|\partial(q)| = \left|(q^{(\bullet)}\setminus q^{(\overline{\eta}+1)})\cap\mathscr{P}^*(A_{\overline{\mu}}^{(\bullet)})\right| \geq \left|(q^{(\bullet)}\setminus q^{(\overline{\mu})})\cap\mathscr{P}^*(A_{\overline{\mu}}^{(\bullet)})\right| = \left|\Delta^{(\overline{\mu})}(q)\right|;$$
- if $|q^{(\bullet)}|\geq\varrho$ and $|q^{(\overline{\eta}+1)}|<\varrho$, then $|\partial(q)|\geq\varrho-|q^{(\overline{\eta}+1)}|\geq\varrho-|q^{(\overline{\mu})}|$;
- if $|q^{(\overline{\eta}+1)}|\geq\varrho$, $q^{(\bullet)}\cap\mathscr{P}^*(A_{\overline{\mu}}^{(\bullet)})\neq\emptyset$, and $q^{(\overline{\eta}+1)}\cap\mathscr{P}^*(A_{\overline{\mu}}^{(\overline{\eta})})=\emptyset$, then $|\partial(q)|\geq1$.

Notice that if $|q^{(\overline{\mu}+1)}|<\varrho$ then $|\partial(q)|\geq|\Delta^{(\overline{\mu})}(q)|$. Indeed, if $|q^{(\bullet)}|<\varrho$ then this ensues immediately from the above; if $|q^{(\bullet)}|\geq\varrho$ holds instead, then, since $|\Delta^{(\overline{\mu})}(q)|=|q^{(\overline{\mu}+1)})|-|q^{(\overline{\mu})}|$, we have $|\partial(q)|\geq\varrho-|q^{(\overline{\mu})}|=\varrho-|q^{(\overline{\mu}+1)}|+|\Delta^{(\overline{\mu})}(q)|>|\Delta^{(\overline{\mu})}(q)|$.

In addition, if $|q^{(\overline{\mu}+1)}|\geq\varrho$, $\Delta^{(\overline{\mu})}(q)\neq\emptyset$, and $q^{(\overline{\mu})}\cap\mathscr{P}^*(A_{\overline{\mu}}^{(\overline{\mu})})=\emptyset$, then $q^{(\bullet)}\cap\mathscr{P}^*(A_{\overline{\mu}}^{(\bullet)})\neq\emptyset$ and $q^{(\overline{\eta}+1)}\cap\mathscr{P}^*(A_{\overline{\mu}}^{(\overline{\eta})})=\emptyset$. Thus, by the above conditions on $\{\partial(q)\}_{q\in P}$, if $|q^{(\overline{\eta}+1)}|<\varrho$ then $|\partial(q)|\geq\varrho-|q^{(\overline{\eta}+1)}|\geq1$, else $|\partial(q)|\geq1$.

Finally, if $\left|q^{(\overline{\mu}+1)}\right| \geq \varrho$ and $\left|\widehat{q}^{[i_0]}\right| < \varrho$, then $\left|q^{(\bullet)}\right| \geq \varrho$ and $\left|q^{(\overline{\eta}+1)}\right| \leq \left|q^{(\overline{\mu})}\right| = \left|\widehat{q}^{[i_0]}\right| < \varrho$, so that $|\partial(q)| \geq \varrho - \left|q^{(\overline{\mu})}\right| = \varrho - \left|\widehat{q}^{[i_0]}\right|$.

In conclusion, a partitioning function $\{\Delta(q)\}_{q \in P}$ exists such that for all $q \in P$:

- $\Delta(q) \subseteq \partial(q)$;
- if $\left|q^{(\overline{\mu}+1)}\right| < \varrho \vee \Delta^{(\overline{\mu})}(q) = \emptyset$, then $|\Delta(q)| = \left|\Delta^{(\overline{\mu})}(q)\right|$, else if $q^{(\overline{\mu})} \cap \mathscr{P}^*(A_{\overline{\mu}}^{(\overline{\mu})}) = \emptyset \wedge \left|\widehat{q}^{[i_0]}\right| \geq \varrho$, then $|\Delta(q)| \geq 1$, else $|\Delta(q)| \geq \varrho - \left|\widehat{q}^{[i_0]}\right|$;
- $\bigcup \widehat{A}_{\overline{\mu}}^{[i_0]} \in \Delta(q) \leftrightarrow \bigcup A_{\overline{\mu}}^{(\bullet)} \in q^{(\bullet)}$;

in agreement with what claimed by the first part of the statement A_3.

As in case A, choosing this Δ so that $\bigcup \Delta[P] \neq \emptyset$ is unproblematic. ∎

Lemma 19. *The 'culprit' statement C cannot be the statement A_2.*

Proof. Assuming μ_{i_0} to be in $M_2 \setminus M_1 \setminus M_3$, all claim- and assert-statements to be fulfilled for $i = 0, \ldots, i_0 - 1$, and C_1 to hold when $i = i_0$, we are to prove that A_2 is met when $i = i_0$. Let $\overline{\mu} = \mu_{i_0}$. The following inequality will demonstrate crucial to our goal:

$$\left|\mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i_0]}) \setminus \bigcup \widehat{P}^{[i_0]}\right| \geq 1 + \sum_{\left|q^{(\bullet)}\right| < \varrho} \left|\left(q^{(\bullet)} \setminus q^{(\overline{\mu})}\right) \cap \mathscr{P}^*(A_{\overline{\mu}}^{(\bullet)})\right|$$
$$+ \sum_{\substack{\left|q^{(\bullet)}\right| \geq \varrho \\ \left|q^{(\overline{\mu})}\right| < \varrho}} \left(\varrho - \left|q^{(\overline{\mu})}\right|\right) + |\mathrm{P}(\overline{\mu}, \overline{\mu})|, \qquad (4)$$

where, as before,

$$\mathrm{P}(\mu, \nu) = \{q \in P \mid \left|q^{(\mu)}\right| \geq \varrho \wedge q^{(\bullet)} \cap \mathscr{P}^*(A_\nu^{(\bullet)}) \neq \emptyset \wedge q^{(\mu)} \cap \mathscr{P}^*(A_\nu^{(\nu)}) = \emptyset\},$$

for $\mu, \nu < \xi$.

We need to analyze the following two cases separately.

Case A: $A_\nu \neq A_{\overline{\mu}}$ holds for all $\nu \in M_2 \setminus M_1 \setminus M_3$ such that $\nu < \overline{\mu}$. By reasoning as in Case A of Lemma 18, we obtain at once $\left|\mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i_0]}) \setminus \bigcup \widehat{P}^{[i_0]}\right| > \varrho \cdot |P|$. Since each $q \in P$ contributes at most ρ units to the right-hand side of inequality (4), we plainly have

$$\varrho \cdot |P| \geq \sum_{\left|q^{(\bullet)}\right| < \varrho} \left|\left(q^{(\bullet)} \setminus q^{(\overline{\mu})}\right) \cap \mathscr{P}^*(A_{\overline{\mu}}^{(\bullet)})\right| + \sum_{\substack{\left|q^{(\bullet)}\right| \geq \varrho \\ \left|q^{(\overline{\mu})}\right| < \varrho}} \left(\varrho - \left|q^{(\overline{\mu})}\right|\right) + |\mathrm{P}(\overline{\mu}, \overline{\mu})|$$

which, together with the previous inequality, trivially yields (4).

Case B: $A_\nu = A_{\overline{\mu}}$ holds for some $\nu \in M_2 \setminus M_1 \setminus M_3$ with $\nu < \overline{\mu}$. Let $\overline{\eta} = \mu_{h_0} = \max\{\eta \in M_2 \setminus M_1 \setminus M_3 \mid \eta < \overline{\mu} \wedge A_\eta = A_{\overline{\mu}}\}$. Observe that by the third conjunct of the statement A_2 instantiated at step h_0, we have

$$\left|\mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i_0]}) \setminus \bigcup \widehat{P}^{[i_0]}\right| \geq \left|\mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[h_0]}) \setminus \bigcup \widehat{P}^{[h_0]} \setminus \bigcup \Delta^{[h_0]}[P]\right|$$
$$\geq 1 + \sum_{\left|q^{(\bullet)}\right| < \varrho} \left|\left(q^{(\bullet)} \setminus q^{(\overline{\eta}+1)}\right) \cap \mathscr{P}^*(A_{\overline{\mu}}^{(\bullet)})\right|$$
$$+ \sum_{\substack{\left|q^{(\bullet)}\right| \geq \varrho \\ \left|q^{(\overline{\eta}+1)}\right| < \varrho}} \left(\varrho - \left|q^{(\overline{\eta}+1)}\right|\right) + |\mathrm{P}(\overline{\eta}+1, \overline{\mu})|$$
$$\geq 1 + \sum_{\left|q^{(\bullet)}\right| < \varrho} \left|\left(q^{(\bullet)} \setminus q^{(\overline{\mu})}\right) \cap \mathscr{P}^*(A_{\overline{\mu}}^{(\bullet)})\right|$$
$$+ \sum_{\substack{\left|q^{(\bullet)}\right| \geq \varrho \\ \left|q^{(\overline{\mu})}\right| < \varrho}} \left(\varrho - \left|q^{(\overline{\mu})}\right|\right) + |\mathrm{P}(\overline{\mu}, \overline{\mu})|,$$

as desired.

Thanks to (4), it will be possible to effect a partitioning $\{\partial(q)\}_{q \in P}$ of a subset of $\mathscr{P}^*(\widehat{A}_{\underline{\mu}}^{[i_0]}) \setminus \bigcup \widehat{P}^{[i_0]} \setminus \{\bigcup \widehat{A}_{\underline{\mu}}^{[i_0]}\}$ so that, for all $q \in P$:

- if $\left|q^{(\bullet)}\right| < \varrho$, then $|\partial(q)| = \left|(q^{(\bullet)} \setminus q^{(\overline{\mu})}) \cap \mathscr{P}^*(A_{\underline{\mu}}^{(\bullet)})\right|$,
- if $\left|q^{(\bullet)}\right| \geq \varrho$ and $\left|q^{(\overline{\mu})}\right| < \varrho$, then $|\partial(q)| \geq \varrho - \left|q^{(\overline{\mu})}\right|$,
- if $\left|q^{(\bullet)}\right| \geq \varrho$, $q^{(\bullet)} \cap \mathscr{P}^*(A_{\underline{\mu}}^{(\bullet)}) \neq \emptyset$, and $q^{(\overline{\mu})} \cap \mathscr{P}^*(A_{\underline{\mu}}^{(\overline{\mu})}) = \emptyset$, then $|\partial(q)| \geq 1$.

Consider now an arbitrary $q \in P$. If $\left|q^{(\overline{\mu}+1)}\right| < \varrho$, then $|\partial(q)| \geq \left|\Delta^{(\overline{\mu})}(q)\right|$. Indeed, if $\left|q^{(\bullet)}\right| < \varrho$ then $|\partial(q)| = \left|(q^{(\bullet)} \setminus q^{(\overline{\mu})}) \cap \mathscr{P}^*(A_{\underline{\mu}}^{(\bullet)})\right| \geq \left|\Delta^{(\overline{\mu})}(q)\right|$. On the other hand, if $\left|q^{(\bullet)}\right| \geq \varrho$, then $|\partial(q)| \geq \varrho - \left|q^{(\overline{\mu})}\right| > \left|\Delta^{(\overline{\mu})}(q)\right|$.

In addition, if $\left|q^{(\overline{\mu}+1)}\right| \geq \varrho$, then $|\partial(q)| \geq \varrho - \left|\widehat{q}^{[i_0]}\right|$. Indeed, if $\left|\widehat{q}^{[i_0]}\right| \geq \varrho$, then $|\partial(q)| \geq \varrho - \left|\widehat{q}^{[i_0]}\right|$ holds trivially. On the other hand, if $\left|\widehat{q}^{[i_0]}\right| < \varrho$ then, by the first part of claim C_1, $\left|\widehat{q}^{[i_0]}\right| = \left|q^{(\overline{\mu})}\right|$ holds, so that $|\partial(q)| \geq \varrho - \left|q^{(\overline{\mu})}\right| = \varrho - \left|\widehat{q}^{[i_0]}\right|$.

Therefore it will be possible to effect a partitioning $\{\Delta^{[i_0]}(r)\}_{r \in P}$ so that for all $q \in P$
- $\Delta^{[i_0]}(r) \subseteq \partial(r)$,
- if $\left|q^{(\overline{\mu}+1)}\right| < \varrho \vee \Delta^{(\overline{\mu})}(q) = \emptyset$, then $\left|\Delta^{[i_0]}(q)\right| = \left|\Delta^{(\overline{\mu})}(q)\right|$, else if $q^{(\overline{\mu})} \cap \mathscr{P}^*(A_{\underline{\mu}}^{(\overline{\mu})}) = \emptyset$ $\wedge \left|\widehat{q}^{[i_0]}\right| \geq \varrho$, then $\left|\Delta^{[i_0]}(q)\right| \geq 1$, else $\left|\Delta^{[i_0]}(q)\right| \geq \varrho - \left|\widehat{q}^{[i_0]}\right|$.

Notice that from the latter it follows also that $\bigcup \Delta^{[i_0]}[P] \neq \emptyset$. Indeed, since $\overline{\mu} \in M_2$, a $p \in P$ exists such that $\Delta^{(\overline{\mu})}(p) \neq \emptyset$ and either $\left|p^{(\overline{\mu})}\right| < \varrho$ or $p^{(\overline{\mu})} \cap \mathscr{P}^*(A_{\underline{\mu}}^{(\overline{\mu})}) = \emptyset$ holds. In the former case, if $\left|p^{(\overline{\mu}+1)}\right| < \varrho$, then $\left|\Delta^{[i_0]}(p)\right| = \left|\Delta^{(\overline{\mu})}(p)\right| > 0$; otherwise $\left|\Delta^{[i_0]}(p)\right| \geq \varrho - \left|\widehat{p}^{[i_0]}\right| = \varrho - \left|p^{(\overline{\mu})}\right| > 0$. In the latter case, namely if $p^{(\overline{\mu})} \cap \mathscr{P}^*(A_{\underline{\mu}}^{(\overline{\mu})}) = \emptyset$, then either $\left|\Delta^{[i_0]}(p)\right| = \left|\Delta^{(\overline{\mu})}(p)\right| > 0$ or $\left|\Delta^{[i_0]}(p)\right| \geq 1$ holds.

Thus the first two conjuncts of the statement A_2 are satisfied by $\{\Delta(q)\}_{q \in P}$.

To see that even the third conjunct of the statement A_2 is satisfied, we proceed as follows. Since $\left|\mathscr{P}^*(\widehat{A}^{[i_0]}) \setminus \bigcup \widehat{P}^{[i_0]}\right| \geq 1 + \sum_{q \in P} |\partial(q)|$ and $\left|\bigcup \Delta^{[i_0]}[P]\right| = \sum_{q \in P} \left|\Delta^{[i_0]}(q)\right|$, we get

$$\left|\mathscr{P}^*(\widehat{A}_{\underline{\mu}}^{[i_0]}) \setminus \bigcup \widehat{P}^{[i_0]} \setminus \bigcup \Delta^{[i_0]}[P]\right| \geq 1 + \sum_{q \in P} \left(|\partial(q)| - \left|\Delta^{[i_0]}(q)\right|\right). \qquad (5)$$

Notice that

- if $\left|q^{(\bullet)}\right| < \varrho$ then

$$\begin{aligned}
|\partial(q)| - \left|\Delta^{[i_0]}(q)\right| &= \left|\left(q^{(\bullet)} \setminus q^{(\overline{\mu})}\right) \cap \mathscr{P}^*(A_{\underline{\mu}}^{(\bullet)})\right| - \left|\Delta^{[i_0]}(q)\right| \\
&= \left|\left(q^{(\bullet)} \setminus q^{(\overline{\mu})}\right) \cap \mathscr{P}^*(A_{\underline{\mu}}^{(\bullet)})\right| - \left|\Delta^{(\overline{\mu})}(q)\right| \\
&= \left|\left(q^{(\bullet)} \setminus q^{(\overline{\mu}+1)}\right) \cap \mathscr{P}^*(A_{\underline{\mu}}^{(\bullet)})\right|;
\end{aligned}$$

- if $\left|q^{(\bullet)}\right| \geq \varrho$ and $\left|q^{(\overline{\mu}+1)}\right| < \varrho$ then

$$\begin{aligned}
|\partial(q)| - \left|\Delta^{[i_0]}(q)\right| &= |\partial(q)| - \left|\Delta^{(\overline{\mu})}(q)\right| \\
&\geq \varrho - \left|q^{(\overline{\mu})}\right| - \left|\Delta^{(\overline{\mu})}(q)\right| \\
&= \varrho - \left|q^{(\overline{\mu}+1)}\right|;
\end{aligned}$$

- if $\left|q^{(\overline{\mu}+1)}\right| \geq \varrho \ \wedge \ q^{(\bullet)} \cap \mathscr{P}^*(A_{\overline{\mu}}^{(\bullet)}) \neq \emptyset \ \wedge \ q^{(\overline{\mu}+1)} \cap \mathscr{P}^*(A_{\overline{\mu}}^{(\overline{\mu})}) = \emptyset$ then

$$|\partial(q)| - \left|\Delta^{[i_0]}(q)\right| \geq 1.$$

Indeed, from $q^{(\overline{\mu}+1)} \cap \mathscr{P}^*(A_{\overline{\mu}}^{(\overline{\mu})}) = \emptyset$ it follows $\Delta^{(\overline{\mu})}(q) = \emptyset$, so that $\Delta^{[i_0]}(q) = \emptyset$. Thus it suffices to show that $|\partial(q)| \geq 1$ holds. The latter inequality follows at once by the third property of $\{\partial(q)\}_{q\in P}$, if $\left|q^{(\overline{\mu})}\right| \geq \varrho$. On the other hand, if $\left|q^{(\overline{\mu})}\right| < \varrho$, then by the second property of $\{\partial(q)\}_{q\in P}$ we have $|\partial(q)| \geq \varrho - \left|q^{(\overline{\mu})}\right| > 0$.

Therefore, by (5) and by the above considerations, we have

$$\begin{aligned}
\left|\mathscr{P}^*(\widehat{A}_{\overline{\mu}}^{[i_0]}) \setminus \bigcup \widehat{P}^{[i_0]} \setminus \bigcup \Delta[P]\right| \geq \quad & 1 \quad + \sum_{|q^{(\bullet)}| < \varrho} \left|(q^{(\bullet)} \setminus q^{(\overline{\mu}+1)}) \cap \mathscr{P}^*(A_{\overline{\mu}}^{(\bullet)})\right| \\
& + \sum_{\substack{|q^{(\bullet)}| \geq \varrho \\ |q^{(\overline{\eta}+1)}| < \varrho}} (\varrho - |q^{(\overline{\mu}+1)}|) + \mathrm{P}(\overline{\mu}+1, \overline{\mu}),
\end{aligned}$$

which completes the verification of the statement A_2 in Case A. ∎

Lemma 20. *The 'culprit' statement C cannot be the claim C_1.*

Proof. Let $0 < i_0 \leq \ell$. Assuming all claim- and assert-statements to be fulfilled for $i = 0, \ldots, i_0 - 1$, we are to prove that C_1 is met when $i = i_0$. Let $\overline{\mu} = \mu_{i_0}$ and let $\overline{\nu} = \mu_{i_0-1}$.

First, assuming that $\left|q^{(\overline{\mu})}\right| < \varrho$, we are to show that $\left|\widehat{q}^{[i_0]}\right| = \left|q^{(\overline{\mu})}\right|$. Indeed, since both facts $\eta \notin M_2$ and $q^{(\eta)} \subseteq q^{(\overline{\mu})}$ hold for every η such that $\overline{\nu} < \eta < \overline{\mu}$, we get $\Delta^{(\eta)}(q) = \emptyset$ for all such η. Hence, recalling that $q^{(\overline{\mu})} = q^{(\overline{\nu})} \cup \bigcup_{\overline{\nu} \leq \eta < \overline{\mu}} \Delta^{(\eta)}(q)$ by Lemma 7(3), we get $q^{(\overline{\mu})} = q^{(\overline{\nu})} \cup \Delta^{(\overline{\nu})}(q)$, where the union is disjoint. Since C_1 holds when $i = i_0 - 1$, we have $\left|\widehat{q}^{[i_0-1]}\right| = \left|q^{(\overline{\nu})}\right|$, and thanks to whichever of A_1–A_3 is encountered when $i = i_0 - 1$, we have $\left|\Delta^{[i_0-1]}(q)\right| = \left|\Delta^{(\overline{\nu})}(q)\right|$. Therefore

$$\left|\widehat{q}^{[i_0]}\right| = \left|\widehat{q}^{[i_0-1]} \cup \Delta^{[i_0-1]}(q)\right| = \left|\widehat{q}^{[i_0-1]}\right| + \left|\Delta^{[i_0-1]}(q)\right| = \left|q^{(\overline{\nu})}\right| + \left|\Delta^{(\overline{\nu})}(q)\right| = \left|q^{(\overline{\mu})}\right|.$$

Secondly, assuming that $\left|\widehat{q}^{[i_0]}\right| < \varrho$, we want to prove that $\left|\widehat{q}^{[i_0]}\right| = \left|q^{(\overline{\mu})}\right|$. In view of what just seen, it is enough to show that $\left|q^{(\overline{\mu})}\right| < \varrho$. Starting with the absurd hypothesis that $\left|q^{(\overline{\mu})}\right| \geq \varrho$, we aim at a contradiction. Since $\left|\widehat{q}^{[i_0-1]}\right| < \varrho$, and C_1 holds when $i = i_0 - 1$, we know that $\left|q^{(\overline{\nu})}\right| = \left|\widehat{q}^{[i_0-1]}\right| < \varrho$, and hence the set

$$\mathcal{S}\left(\overline{\nu}, \overline{\mu}, q, \bigcup P^{(\bullet)}\right) = \{\, \eta \mid \overline{\nu} < \eta \leq \overline{\mu} \ \wedge \ q^{(\eta)} \neq q^{(\overline{\nu})} \,\}$$

is not empty. Lemma 8 then enables us to define $\overline{\eta}$ by the condition $\overline{\eta} + 1 = \min \mathcal{S}\left(\overline{\nu}, \overline{\mu}, q, \bigcup P^{(\bullet)}\right)$, and we obviously have $\overline{\nu} \leq \overline{\eta} < \overline{\mu}$, $q^{(\overline{\eta})} = q^{(\overline{\nu})}$, and $\Delta^{(\overline{\eta})}(q) \neq \emptyset$, so that $\overline{\eta} \in M_2$, and hence $\overline{\eta} = \overline{\nu}$.

Let us show, by a similar argument, that $\left|q^{(\overline{\eta}+1)}\right| \geq \varrho$. Were it not so, then the set

$$\mathcal{S}\left(\overline{\eta}+1, \overline{\mu}, q, \bigcup P^{(\bullet)}\right) = \{\, \vartheta \mid \overline{\eta}+1 < \vartheta \leq \overline{\mu} \ \wedge \ q^{(\vartheta)} \neq q^{(\overline{\eta}+1)} \,\}$$

would have a successor minimum element $\overline{\vartheta} + 1$, so that the conditions $\overline{\nu} < \overline{\eta} + 1 \leq \overline{\vartheta} < \overline{\mu}$, $q^{(\overline{\vartheta})} = q^{(\overline{\eta}+1)} < \varrho$, and $\Delta^{(\overline{\vartheta})}(q) \neq \emptyset$ would be fulfilled, whence the absurdity $\vartheta \in M_2$ would follow.

By inspection of revise_1, revise_2, and revise_3 one sees that only two possibilities exist: either $\left|\Delta^{[i_0-1]}(q)\right| = \left|\Delta^{(\overline{\eta})}(q)\right|$, or $\left|\Delta^{[i_0-1]}(q)\right| \geq \varrho - \left|\widehat{q}^{[i_0-1]}\right|$. However, they both lead to the inequality $\left|\widehat{q}^{[i_0]}\right| \geq \varrho$, conflicting with one of the current hypotheses. The former alternative, in fact, would yield

$$\left|\widehat{q}^{[i_0]}\right| = \left|\widehat{q}^{[i_0-1]}\right| + \left|\Delta^{[i_0-1]}(q)\right| = \left|q^{(\overline{\eta})}\right| + \left|\Delta^{(\overline{\eta})}(q)\right| = \left|q^{(\overline{\eta}+1)}\right| \geq \varrho;$$

while the latter would yield $\left|\widehat{q}^{[i_0]}\right| = \left|\widehat{q}^{[i_0-1]}\right| + \left|\Delta^{[i_0-1]}(q)\right| \geq \varrho.$

In the third place, we are to derive a contradiction from the absurd hypothesis that there are a $q \in P$ and a $B \subseteq P$ such that both $(\forall p \in B)(|p^{(\overline{\mu})}| < \varrho)$ and $|q^{(\overline{\mu})} \cap \mathscr{S}^*(B^{(\overline{\mu})})| \neq |\widehat{q}^{[i_0]} \cap \mathscr{S}^*(\widehat{B}^{[i_0]})|$ hold. On the one hand, since clearly $(\forall p \in B)(|p^{(\overline{\nu})}| < \varrho)$ holds too, and since C_1 is fulfilled when $i = i_0 - 1$, we have that $|q^{(\overline{\nu})} \cap \mathscr{S}^*(B^{(\overline{\nu})})| = |\widehat{q}^{[i_0-1]} \cap \mathscr{S}^*(\widehat{B}^{[i_0-1]})|$; moreover, the equalities

$$
\begin{aligned}
\left|q^{(\overline{\mu})} \cap \mathscr{S}^*(B^{(\overline{\mu})})\right| &= \left|((q^{(\overline{\mu})} \setminus q^{(\overline{\nu})}) \cup q^{(\overline{\nu})}) \cap \mathscr{S}^*(B^{(\overline{\mu})})\right| \\
&= \left|q^{(\overline{\nu})} \cap \mathscr{S}^*(B^{(\overline{\mu})})\right| + \left|(q^{(\overline{\mu})} \setminus q^{(\overline{\nu})}) \cap \mathscr{S}^*(B^{(\overline{\mu})})\right| \\
&= \left|q^{(\overline{\nu})} \cap \mathscr{S}^*(B^{(\overline{\nu})})\right| + \left|(q^{(\overline{\mu})} \setminus q^{(\overline{\nu})}) \cap \mathscr{S}^*(B^{(\overline{\mu})})\right|,
\end{aligned}
\tag{6}
$$

obviously hold (cf. Lemma 7(5)). On the other hand we have

$$
\begin{aligned}
\left|\widehat{q}^{[i_0]} \cap \mathscr{S}^*(\widehat{B}^{[i_0]})\right| &= \left|\widehat{q}^{[i_0-1]} \cap \mathscr{S}^*(\widehat{B}^{[i_0]})\right| + \left|\Delta^{[i_0-1]}(q) \cap \mathscr{S}^*(\widehat{B}^{[i_0]})\right| \\
&= \left|\widehat{q}^{[i_0-1]} \cap \mathscr{S}^*(\widehat{B}^{[i_0-1]})\right| + \left|\Delta^{[i_0-1]}(q) \cap \mathscr{S}^*(\widehat{B}^{[i_0-1]})\right|,
\end{aligned}
\tag{7}
$$

(cf. Lemma 13 and Lemma 7(4),(5)).

Observe now that $|(q^{(\overline{\mu})} \setminus q^{(\overline{\nu})}) \cap \mathscr{S}^*(B^{(\overline{\mu})})| \neq 0$. Assuming the contrary, in fact, we would get $\Delta^{(\overline{\nu})}(q) \cap \mathscr{S}^*(B^{(\overline{\mu})}) = \emptyset$ and hence $\Delta^{(\overline{\nu})}(q) \cap \mathscr{S}^*(B^{(\overline{\nu})}) = \emptyset$ (by Lemma 7(4)). This would yield either $B \neq A_{\overline{\nu}}$ or $\Delta^{(\overline{\nu})}(q) = \emptyset$. In either case, this implies $\Delta^{[i_0-1]}(q) \cap \mathscr{S}^*(\widehat{B}^{[i_0-1]}) = \emptyset$ (in particular $\Delta^{[i_0-1]}(q) = \emptyset$ in the latter case thanks to C_3). Summing up, through (6) and (7) we would easily get $|q^{(\overline{\mu})} \cap \mathscr{S}^*(B^{(\overline{\mu})})| = |\widehat{q}^{[i_0]} \cap \mathscr{S}^*(\widehat{B}^{[i_0]})|$, conflicting with one of the current hypotheses.

It therefore makes sense to consider the smallest ordinal in
$$\mathcal{S}\left(\overline{\nu}, \overline{\mu}, q, \mathscr{S}^*(B^{(\overline{\mu})})\right) = \{\eta \mid \overline{\nu} < \eta \leq \overline{\mu} \wedge q^{(\eta)} \setminus q^{(\overline{\nu})} \ni \in \mathscr{S}^*(B^{(\overline{\mu})})\},$$
which we also know from Lemma 8 to be a successor ordinal $\overline{\eta} + 1$. Accordingly, we will have $\overline{\nu} \leq \overline{\eta} < \overline{\mu}$ and $\Delta^{(\overline{\eta})}(q) \ni \in \mathscr{S}^*(B^{(\overline{\mu})})$, and hence $\Delta^{(\overline{\eta})}(q) \neq \emptyset$ follows. Clearly, $B = A_{\overline{\eta}}$.

It turns out that $\overline{\eta} = \overline{\nu}$, because $\overline{\eta} \in M_1$. Indeed, from $(\forall p \in B)(|p^{(\overline{\mu})}| < \varrho)$, it follows trivially that $(\forall p \in A_{\overline{\eta}})(|p^{(\overline{\eta})}| < \varrho)$, and hence $\overline{\eta} \in M_1$.

Observe also that $|q^{(\overline{\mu})} \cap \mathscr{S}^*(B^{(\overline{\mu})})| = |q^{(\overline{\eta}+1)} \cap \mathscr{S}^*(B^{(\overline{\mu})})|$. Assuming the contrary, in fact, there would be a minimum ordinal, $\overline{\vartheta} + 1$, in
$$\mathcal{S}\left(\overline{\eta} + 1, \overline{\mu}, q, \mathscr{S}^*(B^{(\overline{\mu})})\right) = \{\vartheta \mid \overline{\eta} + 1 < \vartheta \leq \overline{\mu} \wedge q^{(\vartheta)} \setminus q^{(\overline{\eta}+1)} \ni \in \mathscr{S}^*(B^{(\overline{\mu})})\},$$
so that $\overline{\nu} < \overline{\eta} + 1 \leq \overline{\vartheta} < \overline{\mu}$ would hold, and proceeding as above we would reach the absurd conclusion that $\overline{\vartheta} \in M_1$.

Therefore $(q^{(\overline{\mu})} \setminus q^{(\overline{\nu})}) \cap \mathscr{S}^*(B^{(\overline{\mu})}) = \Delta^{(\overline{\nu})}(q) \cap \mathscr{S}^*(B^{(\overline{\mu})})$ holds, and hence, by (6) and by the assumption that all assert- and claim-statements have been fulfilled till now, we get
$$\left|q^{(\overline{\mu})} \cap \mathscr{S}^*(B^{(\overline{\mu})})\right| = \left|\widehat{q}^{[i_0-1]} \cap \mathscr{S}^*(\widehat{B}^{[i_0-1]})\right| + \left|\Delta^{(\overline{\nu})}(q) \cap \mathscr{S}^*(B^{(\overline{\nu})})\right|.$$

Since $B = A_{\overline{\nu}}$ and $(\forall p \in B)(|p^{(\overline{\nu})}| < \varrho)$, we plainly have $\overline{\nu} \in M_1$, so that $|\Delta^{[i_0-1]}(q)| = |\Delta^{(\overline{\nu})}(q)|$. Therefore $|\Delta^{(\overline{\nu})}(q) \cap \mathscr{S}^*(B^{(\overline{\nu})})| = |\Delta^{[i_0-1]}(q) \cap \mathscr{S}^*(\widehat{B}^{[i_0-1]})|$, and hence $|q^{(\overline{\mu})} \cap \mathscr{S}^*(B^{(\overline{\mu})})| = |\widehat{q}^{[i_0]} \cap \mathscr{S}^*(\widehat{B}^{[i_0]})|$, which leads to a contradiction.

In the fourth (and last) place, we show that
$$(\forall q \in P)(\forall B \subseteq P)\left(q^{(\overline{\mu})} \ni \in \mathscr{S}^*(B^{(\overline{\mu})}) \leftrightarrow \widehat{q}^{[i_0]} \ni \in \mathscr{S}^*(\widehat{B}^{[i_0]})\right).$$

Thus, let $q \in P$ and $B \subseteq P$ be such that $q^{(\overline{\mu})} \ni \in \mathscr{P}^*(B^{(\overline{\mu})})$. Then the set
$$\mathcal{S}\left(0, \overline{\mu}, q, B^{(\overline{\mu})}\right) = \{ \eta \mid 0 < \eta \leq \overline{\mu} \wedge q^{(\eta)} \ni \in \mathscr{P}^*(B^{(\eta)}) \}$$
is non-empty and therefore, by Lemma 8, it has a minimum of the form $\overline{\eta} + 1$, with $0 \leq \overline{\eta} < \overline{\mu}$. By Lemma 7(4),(5), we have $q^{(\overline{\eta})} \cap \mathscr{P}^*(B^{(\overline{\eta})}) = \emptyset$ and $q^{(\overline{\eta}+1)} \cap \mathscr{P}^*(B^{(\overline{\eta}+1)}) \neq \emptyset$. Therefore, it plainly follows that $\Delta^{(\overline{\eta})}(q) \neq \emptyset$, so that $\overline{\eta} \in M_2$ and $B = A_{\overline{\eta}}$ hold. Let $0 \leq h_0 < i_0$ be such that $\overline{\eta} = \mu_{h_0}$. By inspection of the procedures revise_1, revise_2, and revise_3, it follows that the only possibilities for $\Delta^{[h_0]}(q)$ are the following:

- $\left|\Delta^{[h_0]}(q)\right| = \left|\Delta^{(\overline{\eta})}(q)\right|$;

- $\left|\Delta^{[h_0]}(q)\right| \geq 1$;

- $\left|\Delta^{[h_0]}(q)\right| \geq \varrho - \left|\widehat{q}^{[h_0]}\right|$, provided that $\left|\widehat{q}^{[h_0]}\right| < \varrho$.

In any case we have $\Delta^{[h_0]}(q) \neq \emptyset$, whence $\widehat{q}^{[h_0+1]} \ni \in \mathscr{P}^*(\widehat{B}^{[h_0]})$, and therefore $\widehat{q}^{[i_0]} \ni \in \mathscr{P}^*(\widehat{B}^{[i_0]})$.

Conversely, let us assume that $\widehat{q}^{[i_0]} \ni \in \mathscr{P}^*(\widehat{B}^{[i_0]})$, for $q \in P$ and $B \subseteq P$, and let h_0 be the smallest integer h such that $\widehat{q}^{[h]} \cap \mathscr{P}^*(\widehat{B}^{[i_0]}) = \emptyset$ and $\widehat{q}^{[h+1]} \cap \mathscr{P}^*(\widehat{B}^{[i_0]}) \neq \emptyset$ both hold. Plainly, $0 \leq h_0 < i_0$, $\Delta^{[h_0]}(q) \neq \emptyset$, and $B = A_{\mu_{h_0}}$ must hold too. Thus, from claim C_3, we get immediately that $\Delta^{(\mu_{h_0})} \neq \emptyset$, so that $q^{(\mu_{h_0}+1)} \ni \in \mathscr{P}^*(B^{(\mu_{h_0})})$, which in turn implies $q^{(\overline{\mu})} \ni \in \mathscr{P}^*(B^{(\overline{\mu})})$. ∎

Remark 4. Notice that the third conjunct of claim C_1, which is

$$(\forall q \in P)(\forall B \subseteq P)\left(q^{(\mu_i)} \ni \in \mathscr{P}^*(B^{(\mu_i)}) \leftrightarrow \widehat{q} \ni \in \mathscr{P}^*(\widehat{B}) \right), \tag{8}$$

has only been exploited once, namely in the proof of Lemma 14(ii); and, in fact, we could have spared even that single utilization, as already discussed within that very proof. We chose to force (8), at the price of some complications in the procedures $\mathsf{imitate}$, revise_2, and revise_3; our gain is that we have set the ground for further applications —postponed to papers to come— of the formative process technique to the set-theoretic satisfiability problem.

In addition to the properties of $\mathsf{imitate}$ discussed in Sections 7, 8, and 10, the fulfillment of two new conditions (cf. (9) and (10) below) turns out to be of basic importance in order to address the decidability problem for extensions of the fragment of set theory described in Sec.3 with literals of the forms $\mathsf{Finite}(v)$, $\neg\mathsf{Finite}(v)$, and $v = \bigcup w$.

Before we can state the new conditions, we need the following concepts. With any given formative process $\mathcal{Q}_{P,\xi} = (\{q^{(\mu)}\}_{q \in P})_{\mu \leq \xi}$, we can associate a sequence $\mathcal{G}_{\mathcal{Q}_{P,\xi}} = (G^{(\mu)})_{\mu \leq \xi}$ of labelled directed graphs defined in the following way:
$$\mathrm{nodes}(G^{(\mu)}) = \mathscr{P}(P),$$
$$\mathrm{edges}(G^{(\mu)}) = \{[A, p, B] \mid A, B \subseteq P \wedge p \in B \wedge p^{(\mu)} \ni \in \mathscr{P}^*(A^{(\mu)})\},$$
for $\mu \leq \xi$ (notice that $[A, p, B]$ denotes the edge $[A, B]$ with label p).

Observe that the sequence $\mathcal{G}_{\mathcal{Q}_{P,\xi}}$ is monotone non-decreasing. Let $skel(\mathcal{G}_{\mathcal{Q}_{P,\xi}})$ denote the longest monotone increasing subsequence of $\mathcal{G}_{\mathcal{Q}_{P,\xi}}$.

Next, let $\widehat{\mathcal{Q}}_{P,\ell} = (\{\widehat{q}^{[j]}\}_{q \in P})_{j \leq \ell}$ be a weak formative process generated by the procedure $\mathsf{imitate}$ in correspondence of an input formative process $(\{q^{(\mu)}\}_{q \in P})_{\mu \leq \xi}$ and let $\mathcal{G}_{\widehat{\mathcal{Q}}_{P,\ell}} = (\widehat{G}^{[j]})_{j \leq \ell}$ be its associated graph sequence.

Then it can be shown that thanks to (8) the following two conditions are met:

$$skel(\mathcal{G}_{\mathcal{Q}_{P,\xi}}) = skel(\mathcal{G}_{\widehat{\mathcal{Q}}_{P,\ell}}), \tag{9}$$

$$\widehat{G}^{[j]} = G^{(\mu_j)}, \qquad \text{for } 0 \leq j \leq \ell. \tag{10}$$

```
    procedure   revise'₂( μ, ^ );
       A := A_μ;
A₂ :  assert
          ( ∃ {Δ(p)}_{p∈P} ) ( subPartitions( Δ, A ) ∧ ⋃ Â ∉ ⋃ Δ[P] ∧
                ( ∀ q ∈ P ) (
                   if |q^{(μ+1)}| < ϱ ∨ Δ^{(μ)}(q) = ∅ then
                        |Δ(q)| = |Δ^{(μ)}(q)|
                   else
                        |Δ(q)| ≥ ϱ − |q̂|
                   end if ) ∧
                   |𝒫*(Â) \ ⋃ P̂ \ ⋃ Δ[P]| ≥ 1 +
                   + ∑_{r ∈ P, |r^{(•)}| < ϱ} |(r^{(•)} \ r^{(μ+1)}) ∩ 𝒫*(A^{(•)})| + ∑_{r ∈ P, |r^{(•)}| ≥ ϱ, |r^{(μ+1)}| < ϱ} (ϱ − |r^{(μ+1)}|)
                );
       pick one such Δ;    return Δ;
    end   revise'₂;

    procedure   revise'₃( μ, ^ );
       A := A_μ;
A₃ :  assert
          ( ∃ {Δ(p)}_{p∈P} ) ( subPartitions( Δ, A ) ∧
                ( ∀ q ∈ P ) ( ( ⋃ Â ∈ Δ(q) ↔ ⋃ A^{(•)} ∈ q^{(•)} ) ∧
                   if |q^{(μ+1)}| < ϱ ∨ Δ^{(μ)}(q) = ∅ then
                        |Δ(q)| = |Δ^{(μ)}(q)|
                   else
                        |Δ(q)| ≥ ϱ − |q̂|
                   end if ) ∧
                   ( 𝒫*(A^{(•)}) ⊆ ⋃ P^{(•)} → ⋃ Δ[P] = 𝒫*(Â) \ ⋃ P̂ )
                );
       pick one such Δ;    return Δ;
    end   revise'₃;
```

Table 1: Simplified variants of revise₂ and revise₃

If one is not interested in properties (9) and (10) above, as is the case when one wants to take into account only the application to the decision problem for Boolean combinations of literals of type (†), cf. Sec.3, then the following simplifications can be made to the procedure imitate:

- define M_2 as the set $\{ \mu \mid \mu < \xi \wedge (\exists q \in P) (|q^{(\mu)}| < \varrho \wedge \Delta^{(\mu)}(q) \neq \emptyset) \}$;

- eliminate the conjunct (8) from claim C_1;

- replace revise₂ and revise₃ by the procedures revise'₂ and revise'₃ shown in Table 1.

$\square$

11. The set satisfiability decision problem again

We address here the *satisfiability problem* for unquantified set formulae φ involving the unary operator $\mathscr{P}$, the binary operators $\cap, \backslash, \cup$, and finite enumerations $\{\_, \ldots, \_\}$, with

their usual meaning. By way of first approximation, what we want to determine for any given formula φ in the said constructs is whether φ is satisfiable or not. More demandingly, we want an algorithm that given φ either finds a set-valued assignment making φ true or establishes that no such assignment exists. By means of simple normalization techniques described in [7], pp.96–99, this problem can be reduced to the satisfaction problem for finite collections $\mathcal{K}$ of literals of the forms (†) listed in Sec.3.

Example 4. In order to establish that the formulae

1. $X \subseteq Y \to \mathscr{P}(X) \subseteq \mathscr{P}(Y)$,

2. $X \in \mathscr{P}(X)$,

3. $\mathscr{P}(X \cap Y) \subseteq \mathscr{P}(X) \cap \mathscr{P}(Y)$,

4. $X \in \mathscr{P}(Y) \wedge V \in X \to V \in Y$

are valid, we deny each of them obtaining

1′. $X \subseteq Y \wedge \mathscr{P}(X) \nsubseteq \mathscr{P}(Y)$,

2′. $X \notin \mathscr{P}(X)$,

3′. $\mathscr{P}(X \cap Y) \nsubseteq \mathscr{P}(X) \cap \mathscr{P}(Y)$,

4′. $X \in \mathscr{P}(Y) \wedge V \in X \wedge V \notin Y$;

then we normalize each of them, obtaining

1″. $X \subseteq Y \wedge V = \mathscr{P}(X) \wedge W = \mathscr{P}(Y) \wedge V \nsubseteq W$,

2″. $V = \mathscr{P}(X) \wedge X \notin V$,

3″. $Z = X \cap Y \wedge T = \mathscr{P}(Z) \wedge V = \mathscr{P}(X) \wedge W = \mathscr{P}(Y) \wedge S = V \cap W \wedge T \nsubseteq S$,

4″. $W = \mathscr{P}(Y) \wedge X \in W \wedge V \in X \wedge V \notin Y$.

By the decision algorithm we are about to introduce, one can show that none of 1″, 2″, 3″, and 4″ is satisfiable, thereby proving that $1, 2, 3$ and 4 are indeed valid. $\qquad\square$

The following reflection lemma, announced in Sec.3 after Def.8, clarifies the importance of L-simulations with respect to the satisfiability problem for the fragment of set theory we are interested in.

Lemma 21. *Consider a set-valued assignment $\mathcal{M} \in \{\text{ sets }\}^{\mathcal{X}}$ defined on a collection $\mathcal{X}$ of variables, together with the Venn partition Σ of the set $\mathcal{M}[\mathcal{X}]$.*

Moreover, let $\widehat{\Sigma}$ and β be a partition and a bijection such that $\widehat{\Sigma}$ L-simulates Σ via β, and let $\mathcal{M}'(v) = \bigcup \beta[\Im(v)]$, where $\Im$ is the function $\Im \in \mathscr{P}(\Sigma)^{\mathcal{X}}$ such that $\mathcal{M}(v) = \bigcup \Im(v)$ holds for every v in $\mathcal{X}$.

Then, for every literal which has one of the forms (†) listed in Sec.3 but is not of the form $v = \{w_1, \ldots, w_H\}$ with $H > L$, and whose variables are drawn from $\mathcal{X}$, the following conditions are fulfilled:

- *if the literal is satisfied by $\mathcal{M}$, then it is satisfied by $\mathcal{M}'$ too;*
- *if the literal is satisfied by $\mathcal{M}'$, and does not involve $\mathscr{P}$ or the construct $\{\_, \ldots, \_\}$, then it is satisfied by $\mathcal{M}$ too.*

Proof. Our thesis can be recast as follows. For u, v, w, and w_is in $\mathcal{X}$, the following conditions hold:

(1) $\bigcup \Im(v) \; \Re \; \bigcup \Im(w)$ iff $\bigcup \beta[\Im(v)] \; \Re \; \bigcup \beta[\Im(w)]$, for $\Re$ in $\{ =, \in, \subseteq \}$;

(2) $\bigcup \Im(v) = \bigcup \Im(u) \star \bigcup \Im(w)$ iff $\bigcup \beta[\Im(v)] = \bigcup \beta[\Im(u)] \star \bigcup \beta[\Im(w)]$, for $\star$ in $\{ \cap, \setminus, \cup \}$; $\qquad \bigcup \Im(v) = \emptyset$ iff $\bigcup \beta[\Im(v)] = \emptyset$;

(3) if $\bigcup \Im(v) = \mathscr{P}(\bigcup \Im(w))$, then $\bigcup \beta[\Im(v)] = \mathscr{P}(\bigcup \beta[\Im(w)])$;

(4) if $0 < H \leq L$ and $\bigcup \Im(v) = \{ \bigcup \Im(w_1), \ldots, \bigcup \Im(w_H) \}$,
then $\bigcup \beta[\Im(v)] = \{ \bigcup \beta[\Im(w_1)], \ldots, \bigcup \beta[\Im(w_H)] \}$.

While $(3), (4)$, and $(1)_\in$ readily follow from Def.8, the proofs of remaining bi-implications rest on the remarks that

- the function $\Gamma \mapsto \beta[\Gamma]$ from $\mathscr{P}(\Sigma)$ to $\mathscr{P}(\widehat{\Sigma})$ is a bijection fulfilling $\beta[\emptyset] = \emptyset$ and $\beta[\Gamma_1 \star \Gamma_2] = \beta[\Gamma_1] \star \beta[\Gamma_2]$, for all $\Gamma_1, \Gamma_2 \subseteq \Sigma$ and $\star$ in $\{\cap, \setminus, \cup\}$, as a consequence of $\sigma \mapsto \beta(\sigma)$ being a bijection from Σ to $\widehat{\Sigma}$;

- all subsets $\Gamma, \Gamma_1, \Gamma_2$ of a partition fulfill the following bi-implications, for each $\star$ in $\{\cap, \setminus, \cup\}$: $\Gamma = \Gamma_1 \star \Gamma_2$ iff $\bigcup \Gamma = \bigcup \Gamma_1 \star \bigcup \Gamma_2$, $\Gamma_1 \subseteq \Gamma_2$ iff $\bigcup \Gamma_1 \subseteq \bigcup \Gamma_2$, $\Gamma = \emptyset$ iff $\bigcup \Gamma = \emptyset$. Thus, to make an example, $\mathcal{M}(v) = \mathcal{M}(u) \setminus \mathcal{M}(w)$ iff $\bigcup \Im(v) = \bigcup \Im(u) \setminus \bigcup \Im(w)$ iff $\Im(v) = \Im(u) \setminus \Im(w)$ iff $\beta[\Im(v)] = \beta[\Im(u) \setminus \Im(w)]$ iff $\beta[\Im(v)] = \beta[\Im(u)] \setminus \beta[\Im(w)]$ iff $\bigcup \beta[\Im(v)] = \bigcup \beta[\Im(u)] \setminus \bigcup \beta[\Im(w)]$ iff $\mathcal{M}'(v) = \mathcal{M}'(u) \setminus \mathcal{M}'(w)$. Or, to make another, $\mathcal{M}(v) \subseteq \mathcal{M}(w)$ iff $\bigcup \Im(v) \subseteq \bigcup \Im(w)$ iff $\Im(v) \setminus \Im(w) = \emptyset$ iff $\beta[\Im(v) \setminus \Im(w)] = \emptyset$ iff $\beta[\Im(v)] \setminus \beta[\Im(w)] = \emptyset$ iff $\beta[\Im(v)] \subseteq \beta[\Im(w)]$ iff $\mathcal{M}'(v) \subseteq \mathcal{M}'(w)$. $\blacksquare$

Let $\mathcal{K}$ be a non-empty finite collection of literals of the form (†) and let

$$H_\mathcal{K} = \max(\{H \mid \mathcal{K} \text{ contains a literal of the form } v = \{w_0, w_1, \ldots, w_H\} \} \cup \{0\}) \, .$$

Let $\mathcal{X}_\mathcal{K}$ be the collection of variables occurring in $\mathcal{K}$ and let $m = |\mathcal{X}_\mathcal{K}|$. Let us assume that $\mathcal{K}$ is satisfiable in a model of Zermelo-Fraenkel set theory and let $\mathcal{M} \in \{ \text{sets} \}^{\mathcal{X}_\mathcal{K}}$ be a set-valued assignment defined on $\mathcal{X}_\mathcal{K}$ and satisfying all literals in $\mathcal{K}$.

Notice that $\mathcal{K}$ cannot contain any literal of type $v = \{w_0, w_1, \ldots, w_H\}$, with v appearing in the list of variables $w_0, w_1, \ldots, w_H$, since by the foundation axiom all such literals are unsatisfiable. Additionally, notice that a literal of type $v = \{w_0, w_1, \ldots, w_H\}$ is satisfied by $\mathcal{M}$ if and only if the corresponding literal $v = \{w_{i_0}, w_{i_1}, \ldots, w_{i_{H'}}\}$, where the list $w_{i_0}, w_{i_1}, \ldots, w_{i_{H'}}$ is obtained from $w_0, w_1, \ldots, w_H$ by dropping multiple occurrences of variables, is satisfied by $\mathcal{M}$. Therefore, under the hypothesis that $\mathcal{M}$ satisfies $\mathcal{K}$, it is not restrictive to assume that for any literal of type $v = \{w_0, w_1, \ldots, w_H\}$ in $\mathcal{K}$

- v does not occur among $w_0, w_1, \ldots, w_H$,

- no variable has multiple occurrences in the list $w_0, w_1, \ldots, w_H$,

so that we may assume that $H_\mathcal{K} < |\mathcal{X}_\mathcal{K}| = m$.

By Thm.2, there exists a transitive partition $\Sigma_\mathcal{K}$ such that

- for every $x \in \mathcal{M}[\mathcal{X}_\mathcal{K}]$ there is a $\Gamma_x \subseteq \Sigma_\mathcal{K}$ such that $x = \bigcup \Gamma_x$;

- $|\Sigma_\mathcal{K}| \leq 2^{|\mathcal{M}[\mathcal{X}_\mathcal{K}]|} \leq 2^{|\mathcal{X}_\mathcal{K}|} = 2^m$.

Let P be any set of cardinality $|\Sigma_{\mathcal{K}}|$ and let $\left(\{q^{(\mu)}\}_{q \in P}\right)_{\mu \leq \xi}$ be a formative process of $\Sigma_{\mathcal{K}}$, whose existence is ensured by Corollary 1.

By Lemma 12, there exists a strong formative process $\left(\{\widehat{q}^{[\mu_i]}\}_{q \in P}\right)_{i \leq \ell}$ for a transitive partition $\widehat{\Sigma}$ which $H_{\mathcal{K}}$-simulates $\Sigma_{\mathcal{K}}$ via a suitable β and is such that

$$
\begin{aligned}
\mathsf{rk}\,\widehat{\Sigma} &\leq \max\left(\left\lceil \tfrac{25}{24}\lceil \log |P| \rceil + 5 \right\rceil, H_{\mathcal{K}} + 1\right) |P|\, 2^{|P|} + 3 \\
&\leq \left\lceil \tfrac{25}{24}m + 5 \right\rceil 2^{2^m + m} + 3,
\end{aligned}
$$

since $|P| \leq 2^m$ and $H_{\mathcal{K}} < m$.

Hence, by putting $\widehat{\mathcal{M}}(v) = \bigcup \beta[\Im(v)]$, where $\Im$ is the function $\Im \in \mathscr{P}(\Sigma_{\mathcal{K}})^{\mathcal{X}_{\mathcal{K}}}$ such that $\mathcal{M}(v) = \bigcup \Im(v)$ holds for every v in $\mathcal{X}_{\mathcal{K}}$, we get from Lemma 21 that $\widehat{\mathcal{M}}$ satisfies $\mathcal{K}$. From the definition of $\widehat{\mathcal{M}}$, we also get $\widehat{\mathcal{M}}(v) \subseteq \bigcup \widehat{\Sigma}$, for every v in $\mathcal{X}_{\mathcal{K}}$. Therefore, since $\widehat{\Sigma}$ is finite, we obtain

$$
\mathsf{rk}\,\widehat{\mathcal{M}}[\mathcal{X}_{\mathcal{K}}] \leq \mathsf{rk}\,\widehat{\Sigma} \leq \left\lceil \frac{25}{24}m + 5 \right\rceil 2^{2^m + m} + 3.
$$

The preceding discussion plainly entails the following result.

Theorem 3 ([4]) *Let $\mathcal{K}$ be a non-empty finite collection of literals of type (†) and let $\mathcal{X}_{\mathcal{K}}$ be the collection of variables occurring in $\mathcal{K}$. If $\mathcal{K}$ is satisfiable, then $\mathcal{K}$ is satisfied by a set-valued assignment $\widehat{\mathcal{M}}$ such that*

$$
\mathsf{rk}\,\widehat{\mathcal{M}}[\mathcal{X}_{\mathcal{K}}] \leq \left\lceil \frac{25}{24}m + 5 \right\rceil 2^{2^m + m} + 3,
$$

where $m = |\mathcal{X}_{\mathcal{K}}|$.

Hence, the satisfiability problem for the fragment of set theory consisting of propositional combinations of literals of the form (†) is decidable. ∎

12. Conclusions

We believe we have made simpler with the new approach, and hence easier to broaden, the result on the decidability of multilevel syllogistic extended with powerset and singleton operators. On the basis of our current research, we are confident that in forthcoming papers we can enrich the decidable fragment of set theory discussed above with either

- literals of the forms $\mathsf{Finite}(v)$, $\neg\mathsf{Finite}(v)$, stating that the cardinality of v is finite and infinite respectively; or
- unionset-literals, of the form $v = \bigcup w$.

Our expectation that the Finite predicate and the operator $\bigcup$ can be treated *together* (along with all other set operators discussed in this paper), although reasonably high, does not rest on any deep investigation so far. The treatment of literals $v = u \times w$ referring to Cartesian product still seems to lie beyond the current techniques (and in fact, as far as we know, it might lead to undecidable fragments, cf. [5]); anyhow, we expect that the study of Cartesian product will benefit from the systematic study of formative processes undertaken in this paper.

We do not have, as yet, a presentation of the satisfaction algorithms for the powerset operator in terms of semantic tableaux, and view this as a necessary future step in sight of any sensible implementation.

Another goal we have in mind is to adapt the decidability results on the powerset operator $\mathscr{P}$ to a theory of sets where membership is not assumed to be well-founded,

or perhaps is subject to Aczel's anti-foundation axiom [18]. A decision algorithm of this kind would find applications in the automation of modal logics, along the lines indicated in [11, 1].

Notice that approaching the satisfiability decision problem in semantic terms, as we have done in this paper for $\mathscr{P}$, presupposes a clear knowledge of the universe $\mathcal{V}$ of all sets. It is likely that such knowledge cannot be based on intuition alone, but needs the support of an axiomatic view of sets. The whole book [7] on decidable fragments of set theory was developed in apparently naive semantic terms, without any explicit mention of the axioms involved, but this does not conflict with the axiomatic approach: for each collection of formulae whose decision problem was solved positively, one could in fact single out a certain number of the Zermelo-Fraenkel axioms that fully account for the correctness of the decision algorithm. Here we are in debt with the reader of an analytic account of the axioms involved in the satisfaction algorithm concerning $\mathscr{P}$; the following two examples, due to C. Piazza,[6] show that this kind of study may lead to surprises.

Examples 5. (1) To prove that $X \neq \mathscr{P}(X)$, one may simply observe that $X \in \mathscr{P}(X)$ whereas $X \notin X$ by the well-foundedness of $\in$. The same conclusion can be reached without recourse to foundation, anyhow, through the separation axiom. The latter enables us to build the set $S = \{ v \in X \mid v \notin v \}$, which plainly fulfills $S \in \mathscr{P}(X)$, and $(\forall v \in S)(v \notin v)$—and hence $S \notin S$. Moreover we have that
$$(\neg \exists\, T)\,(S \in \mathscr{P}(T) \wedge S \neq T \wedge T \in \mathscr{P}(X) \wedge (\forall v \in T)(v \notin v))\ ,$$
by the extensionality axiom. If we make the absurd hypothesis that $X = \mathscr{P}(X)$ and put $T = S \cup \{S\}$, then clearly $S \in \mathscr{P}(T)$; moreover, since $S \notin S$, we have $S \neq T$; in addition, $T \in \mathscr{P}(X)$ holds, because $S \in \mathscr{P}(X)$ and $S \in X$ follows from $S \in \mathscr{P}(X) = X$; finally, $(\forall v \in T)(v \notin v)$ holds, which leads to a contradiction.

(2) Proving that $X \neq \mathscr{P}(X \setminus \{X\})$ would again be a trivial task under the foundation axiom, which yields $X \setminus \{X\} = X$. In the absence of foundation, two cases must be considered. Case $X \in X$: assuming $X = \mathscr{P}(X \setminus \{X\})$, we get $(\forall v \in X)(v \in X \setminus \{X\})$ and in particular $X \in X \setminus \{X\}$, whence the contradiction $X \neq X$. Case $X \notin X$: assuming $X = \mathscr{P}(X \setminus \{X\})$, we get $X \notin \mathscr{P}(X \setminus \{X\})$, hence $(\exists\, v \in X)(v \notin X \vee v = X)$, and therefore $(\exists\, v \in X)(v \notin X)$, which is a contradiction. $\square$

Acknowledgments

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[6]Actually, the former example can be found in [22].

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