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**FILTERING OF NONLINEAR  
STOCHASTIC FEEDBACK SYSTEMS**

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## Abstract

This paper concerns the filtering problem for the class of stochastic nonlinear systems endowed with an output feedback, for which there exists a finite-dimensional filter. It is proven that the optimal filter for the open-loop system remains optimal when the feedback is closed. The result holds whatever the noise distribution may be.

*Key words:* Nonlinear filtering, closed-loop systems, Girsanov theorem, convergence of probability measures



## 1. Introduction

Let us consider the class of stochastic systems described by the equations:

$$dX(t) = f(t, X(t), u(t))dt + dW^1(t), \quad (1.1)$$

$$dY(t) = h(t, X(t), u(t))dt + dW^2(t), \quad (1.2)$$

$$S(t) = F(X(t)), \quad (1.3)$$

where  $X(t) \in \mathbb{R}^d$  is the system state,  $Y(t) \in \mathbb{R}^m$  is the observation process,  $u(t) \in \mathbb{R}^p$  is the input function,  $S(t) \in \mathbb{R}^{d'}$  is the signal to be estimated,  $W^1(t) \in \mathbb{R}^{\nu_1}$ ,  $W^2(t) \in \mathbb{R}^{\nu_2}$  are standard mutually uncorrelated Wiener processes,  $f, h$  are, for any  $t \geq 0$ , suitably dimensioned  $C^\infty$  smooth vector functions and  $F$  is a continuous bounded function. The initial state  $X(0)$  is a random variable independent of  $W^1(t), W^2(t)$   $t \geq 0$ .

Throughout the paper we will use the following *subscript notation*: for a given function  $\xi(t)$ ,  $t \in [0, T]$ , we shall denote with  $\xi_t$  the whole trajectory of  $\xi$  in  $[0, t]$ , that is  $\xi_t(s) = \xi(s)$ ,  $s \in [0, t]$ .

We will consider system (1.1)-(1.3), under the *feedback law*:

$$u_t = \Gamma_t(Y_t) \quad (1.4)$$

where  $\Gamma_t$ , for any  $t \geq 0$ , is a causal  $C^\infty$  map from  $\mathcal{U}_t$  to  $\mathcal{Y}_t$  and  $\mathcal{U}_t, \mathcal{Y}_t$  are suitable metric spaces of measurable functions containing all the trajectories given by the whole feedback system (1.1)-(1.4).

We refer to eq.ns (1.1)-(1.3), when these stand alone, as the *open-loop* system, representing a mathematical model of some plant to be controlled and/or estimated, whereas (1.4), describing the behaviour of some control device, will be called the *controller*. We shall denote by  $X^u, Y^u, S^u$  the solution of the open-loop system, corresponding to the given input function  $u$ . Moreover, for any stochastic process  $\xi$  we shall denote by  $\mathcal{F}_t^\xi$  the  $\sigma$ -algebra generated by  $\{\xi_s, s \leq t\}$ .

We assume that for any given deterministic input  $\bar{u}$  there exists the optimal *open-loop* estimate  $\widehat{S}^{\bar{u}}(t) = E(S^{\bar{u}}(t)/\mathcal{F}_t^{Y^{\bar{u}}})$  in a filter form, that is for any given input function  $\bar{u}$  the estimate is given by the output of a known SDE of the form

$$d\xi^{\bar{u}}(t) = M(t, \xi^{\bar{u}}(t), \bar{u}(t))dt + N(t, \xi^{\bar{u}}(t), \bar{u}(t))dY^{\bar{u}}(t), \quad \xi^{\bar{u}}(0) = \xi_0, \quad (1.5)$$

$$E(S^{\bar{u}}(t)/\mathcal{F}_t^{Y^{\bar{u}}}) = \rho(\xi^{\bar{u}}(t)). \quad (1.6)$$

where  $\xi_0$  is a random variable such that  $E(F(X(0))) = \rho(\xi_0)$ . Let us denote with  $\mathcal{G}(\bar{u}_t, Y_t^{\bar{u}})$  the function associated to equations (1.5) (1.6) that gives the optimal open-loop estimate. So we can write

$$E(S^{\bar{u}}(t)/\mathcal{F}_t^{Y^{\bar{u}}}) = \mathcal{G}(\bar{u}_t, Y_t^{\bar{u}}). \quad (1.7)$$

We assume that the filter defined by eq.ns (1.5), (1.6) is *smooth*, that is the function  $\mathcal{G}(\cdot, \cdot)$  above defined, is a continuous map from  $\mathcal{U}_t \times \mathcal{Y}_t$  to  $\mathbb{R}^{d'}$ . We refer readers to [10] and references therein reported, for a discussion about the existence of finite-dimensional filters for nonlinear systems.

Let  $\xi^\Gamma$  be the process satisfying the filter equation (1.5) when the feedback process  $u$  given by eq. (1.4) is used, i.e.

$$d\xi^\Gamma(t) = M(t, \xi^\Gamma(t), \Gamma_t(Y_t^\Gamma))dt + N(t, \xi^\Gamma(t), \Gamma_t(Y_t^\Gamma))dY^\Gamma(t), \quad \xi^\Gamma(0) = \xi_0. \quad (1.8)$$

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The following question arises: is the optimal estimate of  $S^\Gamma(t)$  given by the following expression?

$$E(S^\Gamma(t)/\mathcal{F}^{Y_t^\Gamma}) = \rho(\xi^\Gamma(t)). \quad (1.9)$$

Stated in other words: let us apply the *open-loop filter to the closed-loop system*. Then, the question is: does the estimate agree with the optimal state-estimate for the closed-loop system?

In this paper we will show that the answer is affirmative for the considered class of nonlinear systems. We point out that our class includes a subclass of the so called *conditionally linear systems*, (see [5], [6] and references therein included).

The addressed topic is a theoretical one, up to now solved only in a very particular case (i.e. for linear-Gaussian systems under nonlinear feedback [1]), important also from an application point of view. Indeed, it may be simpler to write down the filter equations for open loop systems then considering the whole feedback system; furthermore, in many areas, such as stochastic control or telecommunications, the presence of structural feedbacks gives rise to the question about the usage of these equations when the feedback is closed. In particular, it is crucial to understand the changing in the estimation performance.

## 2. The case of countable feedback law

In the following  $(\Omega, \mathcal{F}, P)$  will denote the underlying probability triple.

From (1.1)-(1.4) it follows that, for any  $t$ , we can define (Borel) functions  $\tilde{\Theta}$ ,  $\tilde{\Phi}$ , such that

$$\begin{aligned} X(t) &= \tilde{\Phi}(t, u_t, W_t^1, X(0)); \\ Y(t) &= \tilde{\Theta}(t, u_t, W_t^1, W_t^2, X(0)); \\ S(t) &= F \circ \tilde{\Phi}(t, u_t, W_t^1, X(0)); \end{aligned}$$

however, in order to shorten the notations, in the following we will use the functions  $\Theta$ ,  $\Phi$ , defined as

$$\begin{aligned} \Phi(t, u_t, \omega) &= \tilde{\Phi}(t, u_t, W_t^1(\omega), X(0)(\omega)), \\ \Theta(t, u_t, \omega) &= \tilde{\Theta}(t, u_t, W_t^1(\omega), W_t^2(\omega), X(0)(\omega)); \\ \Phi_F(t, u_t, \omega) &= F \circ \tilde{\Phi}(t, u_t, W_t^1(\omega), X(0)(\omega)); \end{aligned} \quad (2.1)$$

For a given process  $\{v(t), t \geq 0\}$  with the symbol  $\mathcal{F}_t^v$  we will denote the  $\sigma$ -algebra generated by  $\{v(s), s \leq t\}$ .

Let  $\bar{u}$  be a given deterministic function; we denote with  $Y^{\bar{u}}$  the observation process corresponding to the input function  $\bar{u}$ :

$$Y^{\bar{u}}(t) = \Theta(t, \bar{u}_t, \omega). \quad (2.2)$$

We will need the following Lemma.

**Lemma 2.1.** *Let  $\mathcal{H}$  be a sub  $\sigma$ -algebra of  $\mathcal{F}$  and  $\bar{A} \in \mathcal{H}$ . Let  $\bar{\mathcal{H}}$  be the  $\sigma$ -algebra:  $\bar{\mathcal{H}} = \mathcal{H} \cap \bar{A}$ . Then, for any integrable random variable  $X$ , it results  $P$ -almost-surely in  $\bar{A}$ :*

$$E(X/\mathcal{H}) = E(X/\bar{\mathcal{H}}).$$

*Proof.* By definition of conditional expectation it results

$$\int_A E(X/\mathcal{H})dP = \int_A XdP, \quad \forall A \in \mathcal{H} \quad (2.3)$$

Because  $\bar{\mathcal{H}} \subset \mathcal{H}$ , from (2.3) we have

$$\int_A E(X/\mathcal{H})dP = \int_A XdP, \quad \forall A \in \bar{\mathcal{H}} \quad (2.4)$$

and apply again the definition of conditional expectation:

$$\int_A E(X/\bar{\mathcal{H}})dP = \int_A XdP, \quad \forall A \in \bar{\mathcal{H}}. \quad (2.5)$$

From (2.4), (2.5), it follows that

$$\int_A E(X/\bar{\mathcal{H}})dP = \int_A E(X/\mathcal{H})dP, \quad \forall A \in \bar{\mathcal{H}},$$

from which, taking into account that the restriction to  $\bar{A}$  of  $E(X/\mathcal{H})$  is  $\bar{\mathcal{H}}$ -measurable, the thesis follows.  $\square$

**Definition 2.2.** A causal feedback law  $\Gamma_t : \mathcal{Y}_t \rightarrow \mathcal{U}_t$  is said to be a countable range feedback (CRF) function if there exists a countable set  $\{u^\alpha\} \subset \mathcal{U}_T$ ,  $\alpha \in \mathbb{N}$ , such that

$$\mathcal{R}(\Gamma_t) \subseteq \{u_t^\alpha\},$$

where  $u_t^\alpha$  is the restriction of  $u^\alpha$  in the interval  $[0, t]$ ,  $t \leq T$ .

We are now in a position to write down the following theorem.

**Theorem 2.3.** *If  $\Gamma_t$  is a CRF function, then it results*

$$E(S(t)/\mathcal{F}_t^Y) = \mathcal{G}(\Gamma_t(Y_t), Y_t), \quad (2.6)$$

where  $\mathcal{G}$  is the optimal open-loop estimator defined in (1.7).

*Proof.* Being  $\Gamma_t$  a CRF function by assumption, then for any  $t > 0$  it is

$$P\left(\bigcup_{\alpha=1}^{+\infty} \{\omega; \Gamma_t(Y_t) = u_t^\alpha\}\right) = 1. \quad (2.7)$$

Let  $\hat{S}(t)$  be the optimal signal estimate

$$\hat{S}(t) \triangleq E(S(t)/\mathcal{F}_t^Y).$$

From the mean square optimality of  $\hat{S}(t)$  it follows that

$$\begin{aligned} & \int_{\Omega} \|\hat{S}(t) - S(t)\|^2 dP \\ & \leq \int_{\Omega} \|\tilde{S}(t) - S(t)\|^2 dP, \quad \forall \tilde{S}(t) \mathcal{F}_t^Y - \text{measurable}, \end{aligned} \quad (2.8)$$

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this implies that, for any  $\Omega$ -subset  $\Omega^* \in \mathcal{F}_t^Y$  we have

$$\begin{aligned} & \int_{\Omega^*} \|\widehat{S}(t) - S(t)\|^2 dP \\ & \leq \int_{\Omega^*} \|\widetilde{S}(t) - S(t)\|^2 dP, \quad \forall \widetilde{S}(t) \mathcal{F}_t^Y - \text{measurable}. \end{aligned} \quad (2.9)$$

For, suppose that there exist an  $\Omega' \in \mathcal{F}_t^Y$  and an  $S'(t)$   $\mathcal{F}_t^Y$ -measurable such that

$$\begin{aligned} & \int_{\Omega'} \|\widehat{S}(t) - S(t)\|^2 dP \\ & > \int_{\Omega'} \|S'(t) - S(t)\|^2 dP, \end{aligned} \quad (2.10)$$

and let

$$\bar{S}(t) \triangleq S'(t)\chi_{\Omega'} + \widehat{S}(t)\chi_{\Omega'^c} \quad (2.11)$$

where  $\chi_A$ ,  $A^C$  denote the characteristic function and the set-theoretic complement of a set  $A$ . Obviously, it results that  $\bar{S}(t)$  is an  $\mathcal{F}_t^Y$ -measurable function. Now, we have

$$\begin{aligned} E(\|\bar{S}(t) - S(t)\|^2) &= \int_{\Omega} \|\bar{S}(t) - S(t)\|^2 dP \\ &= \int_{\Omega'} \|\bar{S}(t) - S(t)\|^2 dP + \int_{\Omega'^c} \|\bar{S}(t) - S(t)\|^2 dP \\ &= \int_{\Omega'} \|S'(t) - S(t)\|^2 dP + \int_{\Omega'^c} \|\widehat{S}(t) - S(t)\|^2 dP, \end{aligned} \quad (2.12)$$

where we have used (2.11). From (2.12) and recalling (2.10), we obtain

$$\begin{aligned} & E(\|\bar{S}(t) - S(t)\|^2) \\ & < \int_{\Omega'} \|\widehat{S}(t) - S(t)\|^2 dP \\ & \quad + \int_{\Omega'^c} \|\widehat{S}(t) - S(t)\|^2 dP \\ & = E(\|\widehat{S}(t) - S(t)\|^2), \end{aligned}$$

which is against (2.8). Now, for any given deterministic function  $\{u(t), t \geq 0\}$ , and for any integer  $t \geq 0$  let us define the set  $\mathcal{T}(u_t) \subset \Omega$ :

$$\mathcal{T}(u_t) = \{\omega : \Gamma_t(Y_t) = u_t, \} \quad (2.13)$$

and the  $\sigma$ -field, namely  $\mathcal{F}_t^{Y^u}$ , generated by  $\{Y^u(s) = \Theta(s, u_s, \omega), s \leq t\}$ . Obviously it results  $\mathcal{T}(u_t) \in \mathcal{F}_t^Y$ . Moreover, for any Borel set  $B$ , we can consider the identity:

$$\begin{aligned} & \{\omega : \Theta(s, u_s, \omega) \in B, s \leq t\} \cap \{\omega : u_s = \Gamma_s(Y_s), s \leq t\} \\ & = \{\omega : \Theta(s, \Gamma_s(y_s), \omega) \in B, s \leq t\} \cap \{\omega : u_s = \Gamma_s(Y_s), s \leq t\}, \end{aligned}$$

from which it follows that:

$$\mathcal{T}(u_t) \cap \mathcal{F}_t^{Y^u} = \mathcal{F}_t^Y \cap \mathcal{T}(u_t). \quad (2.14)$$

Now, let  $u_t^{(\alpha)}$  an admissible value for the feedback function  $\Gamma_t(Y_t)$ ; by exploiting (2.9) for  $\Omega^* = \mathcal{T}(u_t^\alpha)$ , it results, for any random variable  $\tilde{S}(t)$   $\mathcal{F}_t^Y$ -measurable:

$$\int_{\mathcal{T}(u_t^\alpha)} \|\widehat{S}(t) - S(t)\|^2 dP \leq \int_{\mathcal{T}(u_t^\alpha)} \|\tilde{S}(t) - S(t)\|^2 dP. \quad (2.15)$$

On the other hand, by using the function  $\Phi_F$  defined in (2.1), it results

$$\begin{aligned} & \int_{\mathcal{T}(u_t^\alpha)} \|\widehat{S}(t) - S(t)\|^2 dP \\ &= \int_{\mathcal{T}(u_t^\alpha)} \|E\left(\Phi_F(t, \Gamma_t(Y_t), \omega) / \mathcal{F}_t^Y\right) - \Phi_F(t, \Gamma_t(y_t), \omega)\|^2 dP \\ &= \int_{\mathcal{T}(u_t^\alpha)} \|E\left(\Phi_F(t, u_t^\alpha, \omega) / \mathcal{F}_t^Y\right) - \Phi_F(t, u_t^\alpha, \omega)\|^2 dP \\ &= \int_{\mathcal{T}(u_t^\alpha)} \|E\left(\Phi_F(t, u_t^\alpha, \omega) / \mathcal{F}_t^Y \cap \mathcal{T}(u_t^\alpha)\right) - \Phi_F(t, u_t^\alpha, \omega)\|^2 dP \end{aligned} \quad (2.16)$$

where we have used the equality  $u_t^\alpha = \Gamma_t(Y_t(\omega))$  that holds for  $\omega \in \mathcal{T}(u_t^\alpha)$ , and applied Lemma 2.1 in the last step. Now, by exploiting (2.14) we have

$$\begin{aligned} & \int_{\mathcal{T}(u_t^\alpha)} \|E\left(\Phi_F(t, u_t^\alpha, \omega) / \mathcal{F}_t^Y \cap \mathcal{T}(u_t^\alpha)\right) - \Phi_F(t, u_t^\alpha, \omega)\|^2 dP \\ &= \int_{\mathcal{T}(u_t^\alpha)} \|E\left(\Phi_F(t, u_t^\alpha, \omega) / \mathcal{F}_t^{Y^{(u_t^\alpha)}} \cap \mathcal{T}(u_t^\alpha)\right) - \Phi_F(t, u_t^\alpha, \omega)\|^2 dP \end{aligned} \quad (2.17)$$

moreover, Lemma 2.1 implies that

$$\begin{aligned} & \int_{\mathcal{T}(u_t^\alpha)} \|E\left(\Phi_F(t, u_t^\alpha, \omega) / \mathcal{F}_t^{Y^{(u_t^\alpha)}} \cap \mathcal{T}(u_t^\alpha)\right) - \Phi_F(t, u_t^\alpha, \omega)\|^2 dP \\ &= \int_{\mathcal{T}(u_t^\alpha)} \|E\left(\Phi_F(t, u_t^\alpha, \omega) / \mathcal{F}_t^{Y^{(u_t^\alpha)}}\right) - \Phi_F(t, u_t^\alpha, \omega)\|^2 dP \end{aligned} \quad (2.18)$$

substituting (2.18) in (2.17) and then the result in (2.16), recalling (2.15) we infer the following inequality

$$\begin{aligned} & \int_{\mathcal{T}(u_t^\alpha)} \|E\left(\Phi_F(t, u_t^\alpha, \omega) / \mathcal{F}_t^{Y^{(u_t^\alpha)}}\right) - \Phi_F(t, u_t^\alpha, \omega)\|^2 dP \\ & \leq \int_{\mathcal{T}(u_t^\alpha)} \|\tilde{S}(t) - \Phi_F(t, u_t^\alpha, \omega)\|^2 dP \end{aligned} \quad (2.19)$$

which holds for any random variable  $\tilde{S}(t)$ ,  $\mathcal{F}_t^Y$ -measurable. Taking into account the definition of the set  $\mathcal{T}(u_t^\alpha)$ , inequality (2.19) can be rewritten as

$$\begin{aligned} & \int_{\mathcal{T}(u_t^\alpha)} \|E\left(\Phi_F(t, u_t^\alpha, \omega) / \mathcal{F}_t^{Y^{(u_t^\alpha)}}\right) - \Phi_F(t, \Gamma_t(Y_t), \omega)\|^2 dP \\ & \leq \int_{\mathcal{T}(u_t^\alpha)} \|\tilde{S}(t) - \Phi_F(k, \Gamma_t(y_t), \omega)\|^2 dP, \quad \forall \tilde{S}(t) \mathcal{F}_t^Y - \text{meas.} \end{aligned} \quad (2.20)$$

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moreover, by Lemma 2.1, the restriction of  $\widehat{S}(t)$  to  $\mathcal{T}(u_t^\alpha)$  is  $\mathcal{F}_t^Y \cap \mathcal{T}(u_t^\alpha)$ -measurable and hence, by (2.14),  $\mathcal{F}_t^{Y(u^\alpha)} \cap \mathcal{T}(u_t^\alpha)$ -measurable. Hence, (2.15) and (2.20) imply:

$$\widehat{S}(t) \Big|_{\mathcal{T}(u_t^\alpha)} = E\left(\Phi_F(t, u_t^\alpha, \omega) / \mathcal{F}_t^{Y(u^\alpha)}\right) \Big|_{\mathcal{T}(u_t^\alpha)}. \quad (2.21)$$

Taking into account that, by the hypotheses,

$$\bigcup_{\alpha=1}^{+\infty} \mathcal{T}(u_t^\alpha) = \Omega \setminus \mathcal{N},$$

where  $\mathcal{N} \in \mathcal{F}$  is a  $P$ -null set, we have P-a.s.:

$$\widehat{S}(t) = \sum_{\alpha=1}^{+\infty} \widehat{S}(t) \Big|_{\mathcal{T}(u_t^\alpha)} \chi_{\mathcal{T}(u_t^\alpha)}(\omega)$$

and using (2.21) it results

$$\widehat{S}(t) = \sum_{\alpha=1}^{+\infty} E\left(\Phi_F(t, u_t^\alpha, \omega) / \mathcal{F}_t^{Y(u^\alpha)}\right) \chi_{\mathcal{T}(u_t^\alpha)}(\omega)$$

from which, it follows that:

$$E(S(t) / Y_t) = \mathcal{G}(\Gamma_t(Y_t), Y_t), \quad (2.22)$$

Which proves the theorem.  $\square$

In the next section will be presented an extension of Theorem 2.3 to the class of continuous and static feedback laws, which uses the results of this section.

### 3. The extension theorem

Let us consider the stochastic feedback system described by the equations (1.1), (1.2), under the static feedback law:

$$u(t) = \Gamma(Y(t)), \quad \Gamma(0) = 0. \quad (3.23)$$

We assume that the functions  $f, h, \Gamma$  are uniformly Lipschitz:

$$\begin{aligned} \|f(x', u') - f(x'', u'')\| &\leq \gamma_f (\|x' - x''\|^2 + \|u' - u''\|^2)^{1/2}, \\ \|h(x', u') - h(x'', u'')\| &\leq \gamma_h (\|x' - x''\|^2 + \|u' - u''\|^2)^{1/2}, \\ \|\Gamma(y') - \Gamma(y'')\| &\leq \gamma_\Gamma \|y' - y''\|, \end{aligned} \quad (3.24)$$

for any choice of  $x', x'' \in \mathbb{R}^n$ ,  $u', u'' \in \mathbb{R}^p$  and  $y', y'' \in \mathbb{R}^q$ . For any  $n \in \mathbb{N}$ , let us introduce the CRF function  $\Gamma^n : [0, T] \times \mathcal{C}([0, T], \mathbb{R}^q) \rightarrow \mathbb{R}^p$ :

$$\Gamma^n(t, \eta) \triangleq \sum_{i=0}^{n-1} (Q^n \circ \Gamma(\eta(iT/n))) \chi_{[i\frac{T}{n}, (i+1)\frac{T}{n}]}(t), \quad (3.25)$$

where  $\chi_A(t)$  denotes as usual the characteristic function of the set  $A$ , and  $Q^n$  is a countable range function  $Q^n : \mathbb{R}^p \rightarrow \mathbb{R}^p$  such that:

$$\sup_{v \in \mathbb{R}^p} \|Q^n(v) - v\| \leq \frac{1}{2^n}. \quad (3.26)$$

In the following, we will refer to such a function  $Q^n$  as the *quantizer*. Using  $\Gamma^n$  as the feedback law of system (1.1), (1.2), it result to be well defined the processes  $X^n$  and  $Y^n$  as the corresponding solutions of the system:

$$dX^n(t) = f(t, X^n(t), u^n(t))dt + dW^1(t), \quad (3.27)$$

$$dY^n(t) = h(t, X^n(t), u^n(t))dt + dW^2(t), \quad (3.28)$$

$$u^n(t) = \Gamma^n(t, Y^n) \quad (3.29)$$

Let us define the aggregated processes:

$$Z = \begin{bmatrix} X \\ Y \end{bmatrix}, \quad Z^n = \begin{bmatrix} X^n \\ Y^n \end{bmatrix}, \quad W = \begin{bmatrix} W^1 \\ W^2 \end{bmatrix},$$

and the composed system function:

$$H(x, u) = \begin{bmatrix} f(x, u) \\ h(x, u) \end{bmatrix}.$$

From (3.24) the system function  $H$  results to be Lipschitz  $\gamma_H = (\gamma_f^2 + \gamma_h^2)^{1/2}$ . With the above notation, systems (1.1)-(1.4) and (3.27)-(3.29), can be rewritten as:

$$dZ(t) = H(X(t), \Gamma(Y(t)))dt + dW(t), \quad (3.30)$$

$$dZ^n(t) = H(X^n(t), \Gamma^n(t, Y^n))dt + dW(t), \quad (3.31)$$

with  $Z(0) = Z^n(0)$ . We can prove the following Lemma.

**Lemma 3.1.** *There exist positive constants  $K_1, K_2, K_3$  and an integer  $\bar{n}$  such that for  $n \geq \bar{n}$  and  $\tau \in [iT/n, (i+1)T/n]$ :*

$$E(\|Y^n(\tau) - Y^n(iT/n)\|^2) \leq K_1 \frac{T}{n} + K_2 \frac{T}{n} \int_{iT/n}^{\tau} E(\|X^n(\theta)\|^2) d\theta + K_3 \frac{T^2}{n^2} E(\|Y^n(\tau)\|^2). \quad (3.32)$$

*Proof.* Using (3.31)

$$\begin{aligned} E(\|Y^n(\tau) - Y^n(iT/n)\|^2) &= E\left(\left\|\int_{iT/n}^{\tau} dY^n(\theta)\right\|^2\right) \\ &= E\left(\left\|\int_{iT/n}^{\tau} h(X^n(\theta), \Gamma^n(\theta, Y^n))d\theta + W^2(\tau) - W^2(iT/n)\right\|^2\right) \\ &\leq 2E(\|W^2(\tau) - W^2(iT/n)\|^2) + 2E\left(\int_{iT/n}^{\tau} \|h(X^n(\theta), Q^n \circ \Gamma(Y^n(iT/n)))\| d\theta\right)^2 \end{aligned}$$

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where definition (3.25) has been used. Recalling that  $W^2(\tau)$  is a standard Wiener process of dimension  $p$  and using Schwarz inequality, we have

$$E(\|Y^n(\tau) - Y^n(iT/n)\|^2) \leq 2p(\tau - iT/n) + 2(\tau - iT/n)E \int_{iT/n}^{\tau} \|h(X^n(\theta), Q^n \circ \Gamma(Y^n(iT/n)))\|^2 d\theta. \quad (3.33)$$

Using the Lipschitz property of function  $h$  and adding and subtracting the function  $\Gamma(iT/n)$  the previous inequality becomes

$$\begin{aligned} E(\|Y^n(\tau) - Y^n(iT/n)\|^2) &\leq 2p(\tau - iT/n) + \\ &+ 2(\tau - iT/n)E \int_{iT/n}^{\tau} \gamma_h^2 (\|X^n(\theta)\|^2 \\ &+ \|Q^n \circ \Gamma(Y^n(iT/n)) + \Gamma(Y^n(iT/n)) - \Gamma(Y^n(iT/n))\|^2) d\theta. \end{aligned} \quad (3.34)$$

Thanks to the property of the quantizer  $Q^n$  the previous inequality transforms into

$$\begin{aligned} E(\|Y^n(\tau) - Y^n(iT/n)\|^2) &\leq 2p(\tau - iT/n) + 2(\tau - iT/n)\gamma_h^2 E \int_{iT/n}^{\tau} \|X^n(\theta)\|^2 d\theta + \\ &+ \frac{4(\tau - iT/n)^2 \gamma_h^2}{2^{2n}} + 4(\tau - iT/n)\gamma_h^2 E \int_{iT/n}^{\tau} \|\Gamma(Y^n(iT/n))\|^2 d\theta. \end{aligned} \quad (3.35)$$

Let  $K_1$  a constant such that

$$2p(\tau - iT/n) + \frac{4(\tau - iT/n)^2 \gamma_h^2}{2^{2n}} \leq \frac{K_1 T}{2 n} \quad (3.36)$$

(recall that  $\tau \in [iT/n, (i+1)T/n]$ ) and define

$$K_2 = 4\gamma_h^2.$$

Then

$$\begin{aligned} E(\|Y^n(\tau) - Y^n(iT/n)\|^2) &\leq \frac{K_1 T}{2 n} + \frac{K_2 T}{2 n} E \int_{iT/n}^{\tau} \|X^n(\theta)\|^2 d\theta + \\ &+ 4\gamma_h^2 \gamma_{\Gamma}^2 \frac{T^2}{n^2} E \|Y^n(iT/n)\|^2. \end{aligned} \quad (3.37)$$

From this, defining  $K_3 = 8\gamma_h^2 \gamma_{\Gamma}^2$ , it results

$$\begin{aligned} \left(1 - \frac{K_3 T^2}{2 n^2}\right) E(\|Y^n(\tau) - Y^n(iT/n)\|^2) &\leq \frac{K_1 T}{2 n} + \\ &+ \frac{K_2 T}{2 n} E \int_{iT/n}^{\tau} \|X^n(\theta)\|^2 d\theta + \frac{K_3 T^2}{2 n^2} E \|Y^n(\tau)\|^2. \end{aligned} \quad (3.38)$$

Let  $\bar{n}$  such that  $\forall n > \bar{n}$ ,

$$K_3 \frac{T^2}{n^2} \leq 1, \quad (3.39)$$

then for  $n > \bar{n}$  formula (3.32) follows.

□

**Lemma 3.2.** *There exists a constant  $R$  and an integer  $\bar{n}$  such that  $\forall n \geq \bar{n}$*

$$E\|Z^n(t)\|^2 \leq R, \quad \forall t \in [0, T].$$

*Proof.* From the definition of the process  $Z^n$  given in (3.31) it is

$$\begin{aligned} E\|Z^n(t)\|^2 &= 3E\left\|Z(0) + \int_0^t H(X^n(\tau), \Gamma^n(\tau, Y^n))d\tau + W(t)\right\|^2 \\ &\leq E\|Z(0)\|^2 + 3pT + 3T\gamma_H^2 E\left\{\int_0^t (\|X^n(\tau)\|^2 + \|\Gamma^n(\tau, Y^n)\|^2) d\tau\right\} \leq \\ &\leq C_1 + 3T\gamma_H^2 E \int_0^t \|X^n(\tau)\|^2 d\tau + \\ &\quad + 6T\gamma_H^2 E \int_0^t (\|\Gamma(Y^n(\tau))\|^2 + \|\Gamma^n(\tau, Y^n) - \Gamma(Y^n(\tau))\|^2) d\tau \leq \\ &\leq C_1 + 3T\gamma_H^2(1 + 2\gamma_\Gamma^2) E \int_0^t \|Z^n(\tau)\|^2 d\tau + \\ &\quad + 6T\gamma_H^2 \sum_{i=0}^{n-1} \int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}} \|\Gamma^n(\tau, Y^n) - \Gamma(Y^n(\tau))\|^2 \chi_{[0,t]}(\tau) d\tau. \end{aligned} \tag{3.40}$$

Let us define  $C_2 = 3T\gamma_H^2(1 + 2\gamma_\Gamma^2)$ , and let us add and subtract the term  $\Gamma(Y^n(iT/n))$  in the integrals inside the summation. Recalling also the definition (3.25) of  $\Gamma^n$  one obtains

$$\begin{aligned} E\|Z^n(t)\|^2 &\leq C_1 + C_2 E \int_0^t \|Z^n(\tau)\|^2 d\tau \\ &\quad + 6T\gamma_H^2 \sum_{i=0}^{n-1} \int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}} E\|Q^n \circ \Gamma(Y^n(iT/n)) - \Gamma(Y^n(iT/n)) \\ &\quad \quad + \Gamma(Y^n(iT/n)) - \Gamma(Y^n(\tau))\|^2 \chi_{[0,t]}(\tau) d\tau. \end{aligned} \tag{3.41}$$

Recalling the property (3.26) of the quantizer  $Q^n$  it is also

$$\begin{aligned} E\|Z^n(t)\|^2 &\leq C_1 + C_2 E \int_0^t \|Z^n(\tau)\|^2 d\tau \\ &\quad + 12T\gamma_H^2 \sum_{i=0}^{n-1} \left( \frac{1}{2^{2n}} \frac{T}{n} + \gamma_\Gamma^2 \int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}} E\|Y^n(\tau) - Y^n(iT/n)\|^2 \chi_{[0,t]}(\tau) d\tau \right) \end{aligned} \tag{3.42}$$

and

$$\begin{aligned} E\|Z^n(t)\|^2 &\leq C_1 + C_2 E \int_0^t \|Z^n(\tau)\|^2 d\tau + \\ &\quad + \frac{12T^2\gamma_H^2}{2^{2n}} + 12T\gamma_H^2\gamma_\Gamma^2 \sum_{i=0}^{n-1} \int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}} E\|Y^n(\tau) - Y^n(iT/n)\|^2 \chi_{[0,t]}(\tau) d\tau. \end{aligned} \tag{3.43}$$

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From Lemma 3.1, there exists an integer  $\bar{n}$  and constants  $K_1, K_2, K_3$  such that for any  $t$  inequality (3.32) holds for any  $n \geq \bar{n}$ . Substitution in (3.43) gives

$$\begin{aligned} E\|Z^n(t)\|^2 &\leq C_1 + C_2 E \int_0^t \|Z^n(\tau)\|^2 d\tau + \frac{12T^2\gamma_H^2}{2^{2n}} + 12T\gamma_H^2\gamma_\Gamma^2 K_1 \frac{T}{n} t \\ &\quad + 12T\gamma_H^2\gamma_\Gamma^2 \sum_{i=0}^{n-1} \int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}} \left( K_2 \frac{T}{n} \int_{iT/n}^\tau E\|X^n(\theta)\|^2 d\theta + K_3 \frac{T^2}{n^2} E\|Y^n(\tau)\|^2 \right) \chi_{[0,t]}(\tau) d\tau \end{aligned} \quad (3.44)$$

Let us define the constant

$$C_3 = \frac{12T^2\gamma_H^2}{2^{2\bar{n}}} + 12\gamma_H^2\gamma_\Gamma^2 K_1 \frac{T^3}{\bar{n}}. \quad (3.45)$$

Breaking the integral in the summation into two terms yields, for any  $n \geq \bar{n}$ :

$$\begin{aligned} E\|Z^n(t)\|^2 &\leq C_1 + C_3 + C_2 \int_0^t E\|Z^n(\tau)\|^2 d\tau + \\ &\quad + 12T\gamma_H^2\gamma_\Gamma^2 K_2 \frac{T}{n} \sum_{i=0}^{n-1} \int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}} d\tau \int_0^t E\|X^n(\theta)\|^2 d\theta + \\ &\quad + 12T\gamma_H^2\gamma_\Gamma^2 K_3 \frac{T^2}{n^2} \int_0^t E\|Y^n(\tau)\|^2 d\tau \end{aligned} \quad (3.46)$$

from which

$$E\|Z^n(t)\|^2 \leq C_1 + C_3 + \left( 12T^3\gamma_H^2\gamma_\Gamma^2 \left( K_2 + \frac{K_3}{\bar{n}} \right) + C_2 \right) \int_0^t E\|Z^n(\tau)\|^2 d\tau. \quad (3.47)$$

Finally, an application of the Gronwall's Lemma to the previous inequality gives:

$$E\|Z^n(t)\|^2 \leq (C_1 + C_3) \exp \left( 12T^3\gamma_H^2\gamma_\Gamma^2 \left( K_2 + \frac{K_3}{\bar{n}} \right) + C_2 \right) = R.$$

□

**Lemma 3.3.** *There exist an integer  $\bar{n}$  and a constant  $K_4$  such that*

$$E\|Y^n(\tau) - Y^n(iT/n)\|^2 \leq \frac{K_4}{n}, \quad \forall n \geq \bar{n}, \quad \forall \tau \in [iT/n, (i+1)T/n]. \quad (3.48)$$

*Proof.* Since  $E\|X^n(t)\|^2 \leq E\|Z^n(t)\|^2$  and  $E\|Y^n(t)\|^2 \leq E\|Z^n(t)\|^2$ , and therefore, from Lemma 3.2,  $E\|X^n(t)\|^2 \leq R$  and  $E\|Y^n(t)\|^2 \leq R$ , from Lemma 3.1 it is

$$E(\|Y^n(\tau) - Y^n(iT/n)\|^2) \leq K_1 \frac{T}{n} + K_2 \frac{T}{n} \int_{iT/n}^\tau R d\theta + K_3 \frac{T^2}{n^2} R, \quad \forall n \geq \bar{n} \quad (3.49)$$

and therefore

$$E(\|Y^n(\tau) - Y^n(iT/n)\|^2) \leq K_1 \frac{T}{n} + (K_2 + K_3) \frac{T^2}{n^2} R, \quad \forall n \geq \bar{n} \quad (3.50)$$

From this we can find a constant  $K_4$  such that inequality (3.48) holds for  $n \geq \bar{n}$ .

□

**Theorem 3.4.** *The sequence of processes  $\{Z^n\}$ , solutions of eq. (3.31), converges towards the solution  $Z$  of eq. (3.30) in  $L_2([0, T] \times \Omega, \mathcal{B}_T \times \mathcal{F}, d\lambda \times dP)$ . Moreover,  $\forall t \in [0, T]$   $Z^n(t) \rightarrow Z(t)$  in mean square.*

*Proof.* Using equations (3.30) and (3.31) we have

$$\begin{aligned}
E \int_0^T \|Z(t) - Z^n(t)\|^2 dt &= \\
&= E \int_0^T \left\| \int_0^t \left( H(X(\tau), \Gamma(Y(\tau))) - H(X^n(\tau), \Gamma^n(\tau, Y^n)) \right) d\tau \right\|^2 dt \leq \\
&\leq E \int_0^T \left( \int_0^t \|H(X(\tau), \Gamma(Y(\tau))) - H(X^n(\tau), \Gamma^n(\tau, Y^n))\| d\tau \right)^2 dt \leq \\
&\leq T \cdot E \int_0^T \int_0^t \|H(X(\tau), \Gamma(Y(\tau))) - H(X^n(\tau), \Gamma^n(\tau, Y^n))\|^2 d\tau dt.
\end{aligned} \tag{3.51}$$

Now

$$\begin{aligned}
\|H(X(\tau), \Gamma(Y(\tau))) - H(X^n(\tau), \Gamma^n(\tau, Y^n))\|^2 &\leq \\
&\leq \gamma_H^2 \|X(\tau) - X^n(\tau)\|^2 + \gamma_H^2 \|\Gamma(Y(\tau)) - \Gamma^n(\tau, Y^n)\|^2 \leq \\
&\leq \gamma_H^2 \|X(\tau) - X^n(\tau)\|^2 + 2\gamma_H^2 \|\Gamma(Y(\tau)) - \Gamma(Y^n(\tau))\|^2 \\
&+ 2\gamma_H^2 \|\Gamma(Y^n(\tau)) - \Gamma^n(\tau, Y^n)\|^2 \leq \\
&\leq \gamma_H^2 (1 + 2\gamma_\Gamma^2) \|Z(\tau) - Z^n(\tau)\|^2 + 2\gamma_H^2 \|\Gamma(Y^n(\tau)) - \Gamma^n(\tau, Y^n)\|^2.
\end{aligned} \tag{3.52}$$

Substituting (3.52) in (3.51) we have

$$\begin{aligned}
E \int_0^T \|Z(t) - Z^n(t)\|^2 dt &= T\gamma_H^2 (1 + 2\gamma_\Gamma^2) \int_0^T E \int_0^t \|Z(\tau) - Z^n(\tau)\|^2 d\tau dt \\
&+ T2\gamma_H^2 \int_0^T E \int_0^t \|\Gamma(Y^n(\tau)) - \Gamma^n(\tau, Y^n)\|^2 d\tau dt.
\end{aligned} \tag{3.53}$$

Let us consider the second term of the right hand side of (3.53). One has

$$\begin{aligned}
T2\gamma_H^2 \int_0^T E \int_0^t \|\Gamma(Y^n(\tau)) - \Gamma^n(\tau, Y^n)\|^2 d\tau dt &\leq \\
&\leq 2T^2\gamma_H^2 \int_0^T E \|\Gamma(Y^n(\tau)) - \Gamma^n(\tau, Y^n)\|^2 d\tau \\
&= 2T^2\gamma_H^2 \sum_{i=0}^{n-1} \int_{iT/n}^{(i+1)T/n} E \|\Gamma(Y^n(\tau)) - Q^n \circ \Gamma(Y^n(iT/n))\|^2 d\tau \\
&\leq 4T^2\gamma_H^2 \sum_{i=0}^{n-1} \int_{iT/n}^{(i+1)T/n} E \|\Gamma(Y^n(\tau)) - \Gamma(Y^n(iT/n))\|^2 d\tau \\
&+ T^2\gamma_H^2 \sum_{i=0}^{n-1} \int_{iT/n}^{(i+1)T/n} E \|\Gamma(Y^n(iT/n)) - Q^n \circ \Gamma(Y^n(iT/n))\|^2 d\tau \\
&\leq 4T^2\gamma_H^2 \sum_{i=0}^{n-1} \int_{iT/n}^{(i+1)T/n} \gamma_\Gamma^2 E \|Y^n(\tau) - Y^n(iT/n)\|^2 d\tau + 4T^2\gamma_H^2 \sum_{i=0}^{n-1} \frac{1}{2^n} \frac{T}{n}
\end{aligned} \tag{3.54}$$

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From Lemma 3.48 we have, for  $n \geq \bar{n}$

$$T2\gamma_H^2 \int_0^T E \int_0^t \|\Gamma(Y^n(\tau)) - \Gamma^n(\tau, Y^n)\|^2 d\tau dt \leq o(n) \quad (3.55)$$

where

$$o(n) = 4T^3\gamma_H^2 \left( \gamma_\Gamma^2 \frac{K_4}{n} + \frac{T}{2^{2n}} \right).$$

Substituting (3.55) in (3.53)

$$E \int_0^T \|Z(t) - Z^n(t)\|^2 dt \leq T\gamma_H^2(1 + 2\gamma_\Gamma^2) \int_0^T E \int_0^t \|Z(\tau) - Z^n(\tau)\|^2 d\tau dt + o(n).$$

Using the Gronwall's Lemma it results,

$$E \int_0^T \|Z(t) - Z^n(t)\|^2 dt \leq \exp(T^2\gamma_H^2(1 + 2\gamma_\Gamma^2))o(n),$$

and since  $o(n)$  goes to zero, the first part of the Theorem is proven.

Following similar calculations as before, it follows that:

$$E \|Z(t) - Z^n(t)\|^2 dt \leq T\gamma_H^2(1 + 2\gamma_\Gamma^2) E \int_0^t \|Z(\tau) - Z^n(\tau)\|^2 d\tau dt + o(n).$$

and since we have just proven that  $Z_t^n \rightarrow Z_t$  in the mean square, the right hand side tends to zero, and the proof is completed.  $\square$

**Remark 3.5.** As a consequence of Lemma 3.2 and Theorem 3.4, it results

$$E \|Z(t)\|^2 \leq R, \quad (3.56)$$

where  $R$  is the same bound as in Lemma 3.2.

**Lemma 3.6.** *The sequence of processes  $\{\Gamma^n(\cdot, Y^n)\}$  converges toward  $\Gamma(Y)$  in  $L_2([0, T] \times \Omega, \mathcal{B}_T \times \mathcal{F}, d\lambda \times dP)$ .*

*Proof.*

$$\begin{aligned} E \int_0^T \|\Gamma^n(\tau, Y^n) - \Gamma(Y(\tau))\|^2 d\tau &= \\ &= E \sum_{i=0}^{n-1} \int_{iT/n}^{(i+1)T/n} \|Q^n \circ \Gamma(Y^n(iT/n)) - \Gamma(Y(\tau))\|^2 d\tau \\ &\leq 2E \sum_{i=0}^{n-1} \int_{iT/n}^{(i+1)T/n} \|Q^n \circ \Gamma(Y^n(iT/n)) - \Gamma(Y^n(iT/n))\|^2 d\tau \\ &\quad + 2E \sum_{i=0}^{n-1} \int_{iT/n}^{(i+1)T/n} \|\Gamma(Y^n(iT/n)) - \Gamma(Y(\tau))\|^2 d\tau \\ &\leq 2\frac{T}{2^{2n}} + 2\gamma_\Gamma^2 \sum_{i=0}^{n-1} \int_{iT/n}^{(i+1)T/n} E \|Y^n(iT/n) - Y(\tau)\|^2 d\tau \\ &\leq \frac{T}{2^{2n-1}} + 4\gamma_\Gamma^2 \sum_{i=0}^{n-1} \int_{iT/n}^{(i+1)T/n} E \|Y^n(iT/n) - Y^n(\tau)\|^2 d\tau \\ &\quad + 4\gamma_\Gamma^2 \sum_{i=0}^{n-1} \int_{iT/n}^{(i+1)T/n} E \|Y^n(\tau) - Y(\tau)\|^2 d\tau. \end{aligned} \quad (3.57)$$

Using Lemma 3.3 previous inequality becomes

$$\begin{aligned} E \int_0^T \|\Gamma^n(\tau, Y^n) - \Gamma(Y(\tau))\|^2 d\tau \\ \leq \frac{T}{2^{2n-1}} + 4\gamma_{\Gamma}^2 K_4 \frac{T}{n} + 4\gamma_{\Gamma}^2 \int_0^T E \|Y^n(\tau) - Y(\tau)\|^2 d\tau. \end{aligned} \quad (3.58)$$

From Theorem 3.4 we have that

$$\lim_{n \rightarrow \infty} \int_0^T E \|Y(\tau) - Y^n(\tau)\|^2 d\tau = 0, \quad (3.59)$$

hence, from inequality (3.58), the thesis follows.  $\square$

Let  $\tilde{P}$  and  $\tilde{P}^n$  be the probability measures on the measure space  $(\Omega, \mathcal{F})$  that transform the processes  $Z$  and  $Z^n$ , respectively, into standard Wiener processes. As well known, these measures are uniquely determined by the Girsanov Theorem applied to the systems (3.30), (3.31) and satisfy the following relations:

$$d\tilde{P} = e^{\Lambda} dP; \quad d\tilde{P}^n = e^{\Lambda^n} dP \quad (3.60)$$

where  $\Lambda = \Lambda(T)$ ,  $\Lambda^n = \Lambda^n(T)$  and

$$\Lambda(t) = -\frac{1}{2} \int_0^t \|H(X(s), \Gamma(Y(s)))\|^2 ds - \int_0^t H^T(X(s), \Gamma(Y(s))) dW(s), \quad (3.61)$$

$$\Lambda^n(t) = -\frac{1}{2} \int_0^t \|H(X^n(s), \Gamma^n(s, Y^n))\|^2 ds - \int_0^t H^T(X^n(s), \Gamma^n(s, Y^n)) dW(s). \quad (3.62)$$

**Theorem 3.7.** *The above defined sequence  $\{\Lambda^n\}$  converges in the  $P$ -mean of order one towards  $\Lambda$  and the sequence  $\{e^{\Lambda^n}\}$  is uniformly integrable, that is*

$$\lim_{K \rightarrow \infty} \sup_n \int_{\{e^{\Lambda^n} > K\}} e^{\Lambda^n} dP = 0. \quad (3.63)$$

*Proof.* Let us first show the convergence in  $P$ -mean of  $\{\Lambda^n\}$ .

$$\begin{aligned} E|\Lambda^n - \Lambda| \\ \leq \frac{1}{2} \int_0^T E \left| \|H(X^n(t), \Gamma^n(t, Y^n))\|^2 - \|H(X(t), \Gamma(Y(t)))\|^2 \right| dt \\ + E \left| \int_0^T \left( H(X^n(t), \Gamma^n(t, Y^n)) - H(X(t), \Gamma(Y(t))) \right)^T dW(t) \right| \\ \leq \int_0^T E \left( (\|H(X^n(t), \Gamma^n(t, Y^n))\| + \|H(X(t), \Gamma(Y(t)))\|) \right. \\ \left. \cdot \|H(X^n(t), \Gamma^n(t, Y^n)) - H(X(t), \Gamma(Y(t)))\| \right) dt \\ + \left( E \int_0^T \|H(X^n(t), \Gamma^n(t, Y^n)) - H(X(t), \Gamma(Y(t)))\|^2 dt \right)^{1/2}. \end{aligned}$$

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Using the Schwarz inequality in  $L_2([0, T] \times \Omega)$  previous inequality becomes

$$\begin{aligned} & E|\Lambda^n - \Lambda| \\ & \leq \left( 1 + \left( \int_0^T (2E\|H(X^n(t), \Gamma^n(t, Y^n))\|^2 + 2E\|H(X(t), \Gamma(Y(t)))\|^2) dt \right)^{1/2} \right) \\ & \quad \cdot \left( \int_0^T E\|H(X^n(t), \Gamma^n(t, Y^n)) - H(X(t), \Gamma(Y(t)))\|^2 dt \right)^{1/2}. \end{aligned} \quad (3.64)$$

Moreover using inequality (3.56) we have

$$\int_0^T E\|H(X(t), \Gamma(Y(t)))\|^2 \leq \gamma_H^2 \int_0^T E\|X(t)\|^2 dt + \gamma_H^2 \gamma_\Gamma^2 \int_0^T E\|Y(t)\|^2 \leq K_5, \quad (3.65)$$

with

$$K_5 = \gamma_H^2(1 + \gamma_\Gamma^2)RT, \quad (3.66)$$

and

$$\begin{aligned} & \int_0^T E\|H(X^n(t), \Gamma^n(t, Y^n))\|^2 \leq \gamma_H^2 \int_0^T E\|X^n(t)\|^2 dt \\ & \quad + \gamma_H^2 E \sum_{i=0}^{n-1} \int_{iT/n}^{(i+1)T/n} \|Q^n \circ \Gamma(Y^n(iT/n))\|^2 dt \\ & \leq \gamma_H^2 RT + \gamma_H^2 E \sum_{i=0}^{n-1} \int_{iT/n}^{(i+1)T/n} \|Q^n \circ \Gamma(Y^n(iT/n)) - \Gamma(Y^n(iT/n)) + \Gamma(Y^n(iT/n))\|^2 dt \\ & \leq \gamma_H^2 RT + 2\gamma_H^2 E \sum_{i=0}^{n-1} \int_{iT/n}^{(i+1)T/n} \left( \frac{1}{2^{2n}} + \|\Gamma(Y^n(iT/n))\|^2 \right) dt \\ & \leq \gamma_H^2 RT + \frac{2\gamma_H^2 T}{2^{2n}} \\ & \quad + 2\gamma_H^2 E \sum_{i=0}^{n-1} \int_{iT/n}^{(i+1)T/n} \|\Gamma(Y^n(iT/n)) - \Gamma(Y^n(t)) + \Gamma(Y^n(t))\|^2 dt \\ & \leq \gamma_H^2 T \left( R + \frac{1}{2^{2n-1}} \right) + 4\gamma_H^2 E \sum_{i=0}^{n-1} \int_{iT/n}^{(i+1)T/n} \gamma_\Gamma^2 \|Y^n(iT/n) - Y^n(t)\|^2 dt \\ & \quad + 4\gamma_H^2 E \sum_{i=0}^{n-1} \int_{iT/n}^{(i+1)T/n} \gamma_\Gamma^2 \|Y^n(t)\|^2 dt \\ & \leq \gamma_H^2 T \left( R + \frac{1}{2^{2n-1}} \right) + 4\gamma_H^2 \gamma_\Gamma^2 \int_0^T \|Y^n(t)\|^2 dt \\ & \quad + 4\gamma_H^2 E \sum_{i=0}^{n-1} \int_{iT/n}^{(i+1)T/n} \gamma_\Gamma^2 \|Y^n(iT/n) - Y^n(t)\|^2 dt. \end{aligned} \quad (3.67)$$

From this and from Lemma 3.3, for  $n \geq \bar{n}$  it is

$$\int_0^T E\|H(X^n(t), \Gamma^n(t, Y^n))\|^2 \leq K_6 \quad (3.68)$$

where

$$K_6 = \gamma_H^2 T \left( R + \frac{1}{2^{2n-1}} \right) + 4\gamma_H^2 \gamma_\Gamma^2 RT + 4\gamma_H^2 \gamma_\Gamma^2 K_4 T. \quad (3.69)$$

Substituting (3.65) and (3.68) in (3.64) one obtains

$$\begin{aligned} E|\Lambda^n - \Lambda| &\leq \left( 1 + (2(K_5 + K_6))^{1/2} \right) \\ &\quad \cdot \left( \int_0^T E \|H(X^n(t), \Gamma^n(t, Y^n)) - H(X(t), \Gamma(Y(t)))\|^2 dt \right)^{1/2} \\ &\leq \left( 1 + (2(K_5 + K_6))^{1/2} \right) \\ &\quad \cdot \gamma_H \left( \int_0^T (E \|X(t) - X^n(t)\|^2 + E \|\Gamma^n(t, Y^n) - \Gamma(Y(t))\|^2) dt \right)^{1/2} \end{aligned} \quad (3.70)$$

which goes to zero thanks to Lemma 3.6 and Theorem 3.4. This proves the first part of the theorem.

Convergence in the  $P$ -mean of order one of  $\Lambda^n$  to  $\Lambda$  implies convergence in distribution of  $e^{\Lambda^n}$  to  $e^\Lambda$ . Moreover, as well known, it is  $E\{e^{\Lambda^n}\} = E\{e^\Lambda\} = 1$  for any  $n$ . These properties prove uniform integrability of the family  $E\{e^{\Lambda^n}\}$  thanks to theorem 5.4, pag. 32 in [7]. This concludes the proof.  $\square$

Let us consider on the probability spaces  $(\Omega, \mathcal{F}, \tilde{P}^n)$  and  $(\Omega, \mathcal{F}, \tilde{P})$  the processes  $\mathcal{Z}^n$  and  $\mathcal{Z}$  respectively, defined as follows

$$\mathcal{Z}^n(t) = \begin{bmatrix} X^n(t) \\ Y_t^n \\ e^{\Lambda^n} \end{bmatrix}, \quad \mathcal{Z}(t) = \begin{bmatrix} X(t) \\ Y_t \\ e^\Lambda \end{bmatrix}. \quad (3.71)$$

Let  $\mu_t^n$  and  $\mu_t$  be the measures induced on  $\mathbb{R}^d \times L_2([0, t]; \mathbb{R}^m) \times \mathbb{R}$  by  $\mathcal{Z}^n(t)$  and  $\mathcal{Z}(t)$  respectively, that is

$$\mu_t^n(B) = \tilde{P}^n(\omega : \mathcal{Z}^n(t) \in B); \quad \mu_t(B) = \tilde{P}(\omega : \mathcal{Z}(t) \in B);$$

where  $B$  is any set belonging to the Borel  $\sigma$ -algebra of  $\mathbb{R}^d \times L_2([0, T]; \mathbb{R}^m) \times \mathbb{R}$ .

**Lemma 3.8.**  $\mu_t^n$  is weakly convergent to  $\mu_t \forall t \in [0, T]$ .

*Proof.* Let  $\psi^n, \psi$  defined as

$$\psi^n(t) = \begin{bmatrix} X^n(t) \\ Y_t^n \\ \Lambda^n \end{bmatrix}, \quad \psi(t) = \begin{bmatrix} X(t) \\ Y_t \\ \Lambda \end{bmatrix}. \quad (3.73)$$

By using Theorems 3.4 and 3.7 we have that  $\psi^n(t) \rightarrow \psi(t)$  in  $L_1(\Omega, \mathcal{F}, P)$ . Let us prove the convergence of  $\psi^n(t)$  to  $\psi(t)$  in the  $\tilde{P}$  probability, that is:

$$\lim_{n \rightarrow \infty} \tilde{P}(\|\psi^n(t) - \psi(t)\| > \epsilon) = 0, \quad \forall \epsilon > 0. \quad (3.74)$$

Let  $K$  a real positive number,

$$\begin{aligned} \tilde{P}(\|\psi^n(t) - \psi(t)\| > \epsilon) &= \int_{\{\|\psi^n - \psi\| > \epsilon\}} d\tilde{P} = \int_{\{\|\psi^n - \psi\| > \epsilon\}} e^\Lambda dP \\ &= \int_{\{e^\Lambda > K\} \cup \{\|\psi^n - \psi\| > \epsilon\}} e^\Lambda dP + \int_{\{e^\Lambda \leq K\} \cup \{\|\psi^n - \psi\| > \epsilon\}} e^\Lambda dP \\ &\leq KP(\|\psi^n(t) - \psi(t)\| > \epsilon) + \int_{\{e^\Lambda \leq K\}} e^\Lambda dP \end{aligned}$$

from which, by choosing  $K$  such that  $\int_{e^\Lambda > K} e^\Lambda dP < \epsilon/2$  and  $n$  such that  $P(\|\psi^n(t) - \psi(t)\| > \epsilon) < \epsilon/2K$ , we obtain (3.74). Since  $\mathcal{Z}^n(t)$  and  $\mathcal{Z}(t)$  are continuous functions of  $\psi^n(t)$  and  $\psi(t)$  respectively, it results  $\mathcal{Z}^n(t) \rightarrow \mathcal{Z}(t)$  in  $\tilde{P}$  distribution and hence  $\bar{\mu}_t^n \rightarrow \mu_t$  weakly, where  $\bar{\mu}_t^n$  is the measure defined as:  $\bar{\mu}_t^n = \tilde{P}(\mathcal{Z}^n(t) \in B)$ , with  $B$  Borel set of  $\mathbb{R}^d \times L_2([0, t]; \mathbb{R}^m) \times \mathbb{R}$ .

Now, let us prove the weak convergence of  $\mu_t^n$  to  $\mu_t$ . For a suitably large  $n$  it results

$$\begin{aligned} |\mu_t^n(B) - \mu_t(B)| &= |\mu_t^n(B) - \bar{\mu}_t^n(B) + \bar{\mu}_t^n(B) - \mu_t(B)| \\ &\leq |\mu_t^n(B) - \bar{\mu}_t^n(B)| + |\bar{\mu}_t^n(B) - \mu_t(B)| \\ &\leq \frac{\epsilon}{2} + \|\mu_t^n(B) - \bar{\mu}_t^n(B)\| \leq \frac{\epsilon}{2} + |\tilde{P}^n(\mathcal{Z}^n(t) \in B) - \tilde{P}(\mathcal{Z}^n(t) \in B)| \\ &\leq \frac{\epsilon}{2} + \left| \int_{\{\mathcal{Z}^n(t) \in B\}} (e^{\Lambda^n} - e^\Lambda) dP \right| \\ &= \frac{\epsilon}{2} + |E(e^{\Lambda^n} - e^\Lambda)|. \end{aligned} \tag{3.75}$$

Now, since  $e^{\Lambda^n} - e^\Lambda$  is uniformly integrable, by Theorem 3.7, let  $K_\epsilon > 0$  such that

$$\int_{\{|e^{\Lambda^n} - e^\Lambda| > K_\epsilon\}} |e^{\Lambda^n} - e^\Lambda| dP < \frac{\epsilon}{6}, \quad \forall n. \tag{3.76}$$

Moreover, let us choose  $L_\epsilon > 0$  enough large such that

$$P(e^\Lambda > L_\epsilon) \leq \frac{\epsilon}{6 \cdot K_\epsilon}. \tag{3.77}$$

It results:

$$\begin{aligned} \int_{\Omega} |e^{\Lambda^n} - e^\Lambda| dP &= \int_{\{|e^{\Lambda^n} - e^\Lambda| > K_\epsilon\}} |e^{\Lambda^n} - e^\Lambda| dP + \int_{\{|e^{\Lambda^n} - e^\Lambda| \leq K_\epsilon\} \cap \{e^\Lambda > L_\epsilon\}} |e^{\Lambda^n} - e^\Lambda| dP \\ &\quad + \int_{\{|e^{\Lambda^n} - e^\Lambda| \leq K_\epsilon\} \cap \{e^\Lambda \leq L_\epsilon\}} |e^{\Lambda^n} - e^\Lambda| dP \\ &\leq \frac{\epsilon}{6} + K_\epsilon \cdot P(e^\Lambda > L_\epsilon) + \int_{\{|e^{\Lambda^n} - e^\Lambda| \leq K_\epsilon\} \cap \{e^\Lambda \leq L_\epsilon\}} \left| \int_0^1 e^{\tau\Lambda + (1-\tau)\Lambda^n} d\tau \cdot (\Lambda - \Lambda^n) \right| dP \\ &\leq \frac{\epsilon}{6} + \frac{\epsilon}{6} + \int_{\{|e^{\Lambda^n} - e^\Lambda| \leq K_\epsilon\} \cap \{e^\Lambda \leq L_\epsilon\}} \left| \int_0^1 L_\epsilon^\tau (K_\epsilon + L_\epsilon)^{1-\tau} d\tau \cdot (\Lambda - \Lambda^n) \right| dP \\ &\leq \frac{\epsilon}{6} + \frac{\epsilon}{6} + (K_\epsilon + L_\epsilon) \cdot E(|\Lambda - \Lambda^n|) \leq \frac{\epsilon}{2}, \end{aligned}$$

for a suitably large  $n$ , thanks to Theorem 3.7.

Substituting this in (3.75) we get the thesis. □

Finally we can prove the main theorem (Extension Theorem).

**Theorem 3.9.** *Given the system (1.1)-(1.3), endowed with the feedback law (1.4), it results*

$$E(S(t)/\mathcal{F}_t^Y) = \mathcal{G}(\Gamma_t(Y_t), Y_t),$$

where  $\mathcal{G}$  is the optimal open-loop estimator defined in (1.7).

*Proof.* Let us consider the sequences (3.27), (3.28), obtained with the CRF law defined by (3.25). By Theorem 2.3 it results

$$E(S^n(t)/\mathcal{F}_t^{Y^n}) = \mathcal{G}(\Gamma_t^n(Y_t), Y_t^n), \quad (3.78)$$

where  $S^n = F(X^n)$ , and  $F$  is the bounded continuous function defining the signal in (1.3). By the hypothesis  $\mu_t^n$  is weakly convergent to  $\mu_t$  (remind that  $\mu_t^n$  and  $\mu_t$  are the measures induced by  $Z^n(t)$  and  $Z(t)$  respectively throughout the probability measures  $\tilde{P}^n, \tilde{P}$ ). By construction,  $Z^n$  and  $Z$  are standard Wiener processes with respect to  $\tilde{P}^n$ , and  $\tilde{P}$  respectively, therefore their components are independent processes. This implies that  $X^n(t), Y_t^n$  are mutually independent with respect to  $\tilde{P}^n$  and  $X(t), Y_t$  are mutually independent with respect to  $\tilde{P}$ . Then we can use Theorem 2.1 in [8] and obtain that  $E(S^n(t)/\mathcal{F}_t^{Y^n}) \rightarrow E(S(t)/\mathcal{F}_t^Y)$  in  $P$ -distribution. Therefore from (3.78) it results

$$\mathcal{G}(\Gamma_t^n(Y_t^n), Y_t^n) \rightarrow E(S(t)/\mathcal{F}_t^Y). \quad \text{in } P\text{-distribution.} \quad (3.79)$$

Now, by Theorem 3.4 and Lemma 3.6 it results that  $(\Gamma^n(Y^n), Y^n) \rightarrow (\Gamma(Y), Y)$  in the mean square, and hence in  $P$ -distribution. Since the filter defined by eq.ns (1.5), (1.6) is assumed to be smooth (that is the function  $\mathcal{G}$  is continuous), by Theorem 5.1 of [7], it results that  $\mathcal{G}(\Gamma_t^n(Y_t^n), Y_t^n) \rightarrow \mathcal{G}(\Gamma_t(Y_t), Y_t)$  in  $P$ -distribution and finally, taking into account (3.78), the thesis follows.  $\square$

## 4. Conclusions

The relation between open-loop and closed-loop system state-estimation has been studied. The central result is given by theorem 3.9 asserting that: the same function (of the input and of the current observations), giving the optimal state-estimate for the open-loop system, will give the optimal state-estimate also for the closed-loop system, provided that the input is replaced with the controller-output. This implies, as shown in §3, that the optimal filter of the open-loop system remains optimal when the feedback is closed (whatever the feedback function may be). This is a very common operation in the engineering practice, however any theoretical justification of this operation was up to now given.

The result has been proved under mild hypotheses, verified in all practical cases. We stress that, it may be more simple to derive filter equations for the open loop system.

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