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SQUARE BIPARTITE DESIGNS

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Abstract

A *square bipartite design* is a pair of square 0-1 matrices A and B satisfying the following matrix equation:

$$A^T B = \lambda J + diag(d),$$

where J is a matrix filled with all ones, λ is a positive integer, and d is a vector. We characterize all non-regular square bipartite designs that satisfy some mild hypotheses. Our results generalize some earlier results of de Bruijn and Erdős [3], Lehman [10], and Gasparyan [6].

Key words: 0-1 matrices, design theory, linear algebra

1. Introduction

Let A and B be two 0-1 matrices of size $n \times m$, with $n, m \geq 3$, such that $A^T B = \lambda J + diag(d)$, where J is a matrix of appropriate size filled with all ones, λ is a positive integer and $diag(d)$ is the diagonal matrix of size $m \times m$ with the vector d in the diagonal; the pair (A, B) is called a (d, λ) *bipartite design*. A (d, λ) bipartite design with $m = n$ is called a (d, λ) *square bipartite design* (SBD); a SBD is *regular* if there exist positive integers r and s such that

$$
\begin{aligned}
AJ &= JA = rJ, \\
BJ &= JB = sJ, \\
A^T B &= BA^T = \lambda J + (rs - \lambda n)I.
\end{aligned}
$$

If (A, A) is a SBD then A is called a *square design*; a *symmetric design* is a regular square design. A symmetric design with $\lambda = 1$ is called a *projective plane*.

An important class of square zero-one matrices is the class of the incidence matrices of *degenerate projective planes* (DPP), i.e.,

$$
\begin{bmatrix} 0 & \mathbf{1}^T \\ \mathbf{1} & I \end{bmatrix},
$$

where $\mathbf{1}$ denotes a vector whose components are all equal to one. We say that the pair (A, B) is a *DPP design* when both matrices are DPP (up to rearranging rows and columns).

The problem of characterizing the SBD's is usually divided into two subproblems: characterization of the non-regular SBD's and characterization of the regular ones. The first important result on the direction of the problem of characterizing the non-regular SBD's was obtained by de Bruijn and Erdős in 1948:

Theorem 1.1. [3] *If (A, A) is a non-regular $(d, 1)$ square design with $d \geq \mathbf{1}$, then (A, A) is a DPP design.*

This was generalized by Lehman in 1979:

Theorem 1.2. [10] *If (A, B) is a non-regular $(d, 1)$ square bipartite design with $d \geq \mathbf{1}$, then (A, B) is a DPP design.*

Theorem 1.2 has some important consequences in polyhedral combinatorics: it completely characterizes the square minimally non-ideal matrices and it is one of the main arguments in the proof of the famous result of Lehman [11] on the structure of minimally non-ideal polyhedra (see also [13]).

Theorem 1.2 has been recently generalized in [6]:

Theorem 1.3. [6] *Let (A, B) be a non-regular $(d, 1)$ square bipartite design with $n > 3$. If the vectors d, Ad^{-1}, and Bd^{-1} have full supports, then (A, B) is a DPP design.*

(Here d^{-1} denotes the vector whose i-th component is equal to the inverse of the i-th component of d.) To see that Theorem 1.3 generalizes Theorem 1.2, note that $d \geq \mathbf{1}$ implies that d has full support; moreover, since $A^T B$ is non-singular (see Corollary 2.4 in the following section), it follows that A and B have no zero-rows, and so $Ad^{-1} > 0$ and $Bd^{-1} > 0$.

Some applications of Theorem 1.3 in the theory of perfect graphs and in polyhedral combinatorics, are discussed in [6].

The *extremal bipartite designs*, that is pairs (A, B) of matrices that define (d, λ) bipartite designs and in which $m = n + 1$, are studied in [7]. The extremal bipartite designs are related to the 0-1 *simplices* (full-dimensional simplices with 0-1 vertices, which can be described by using 0-1 constraints with λ as right hand side).

The theory of regular bipartite designs includes two important subjects of combinatorics: the classical theory of symmetric block designs and the theory of partitionable clutters. The first is one of the most developed topics of combinatorics; the second is mainly related to the theory of perfect graphs (see, for instance, [2] and [12]) and to the theory of ideal clutters [11]. For more applications of the theory of SBD's in polyhedral combinatorics, graph theory, and combinatorics we refer to [1], [4], [5], [8], [9], and [15].

The goal of this paper is to investigate square bipartite designs in a more general context than the one considered by de Bruijn and Erdős, Lehman, and Gasparyan (λ is not constrained to be equal to one). Our main result (Theorem 4.3) is the characterization of all non-regular SBD's, under some mild hypotheses. Theorem 4.3 generalizes Theorem 1.1 of de Bruijn and Erdős, Theorem 1.2 of Lehman, and Theorem 1.3 of Gasparyan.

We close this section with some definitions and notations.

As usual, I denotes an identity matrix of appropriate size, J denotes a matrix filled with all ones of appropriate size; $\mathbf{1}$ and $\mathbf{0}$ denote vectors of appropriate dimension whose components are all equal to one and zero, respectively. A *one-row* (*one-column*) of a matrix is a row (column) whose components are all equal to one; a *zero-row* (*zero-column*) of a matrix is a row (column) whose components are all equal to zero.

Let $A = [a_{ij}]$ be a 0-1 matrix. We denote by $a_{i\cdot}$ and $a_{\cdot j}$ the vectors corresponding to the i-th row of A and to the j-th column of A, respectively. We denote by r_i^A the number of the components of $a_{i\cdot}$ that are equal to one and by c_j^A the number of components of $a_{\cdot j}$ that are equal to one. We say that A is r-*regular* (or regular) if $r_i^A = r$ for every i and $c_j^A = r$ for every j.

A *full support* vector is a vector whose components are all different from zero. For every full support vector d, we denote by d^{-1} the vector whose i-th component is equal to the inverse of the i-th component of d. The scalar product of two vectors u and v of equal dimension is denoted by $u \cdot v$.

Let A and B be two matrices; if there exist two permutation matrices P_1 and P_2 such that $B = P_1 A P_2$, then we write $A \cong B$. Let (A, B) and (C, D) be two pairs of matrices; if there exist two permutation matrices P_1 and P_2 such that $(C, D) = (P_1 A P_2, P_1 B P_2)$ or $(D, C) = (P_1 A P_2, P_1 B P_2)$, then we write $(A, B) \cong (C, D)$.

Finally, we say that two integers h and k are *relatively prime*, if they have no common divisor, that is $gcd(h, k) = 1$.

2. Some properties of zero-one matrices

In this section we shall give some properties of 0-1 matrices that define square bipartite designs.

Property 2.1. *Let A be a 0-1 matrix of size $n \times m$ and let r be a positive integer. If each row of A has at least r components equal to one, and if each column of A has at most r components equal to one, then $m \geq n$. Moreover, A is r-regular if and only if $m = n$.*

(To see the validity of this property, let N denote the total number of elements of A that are equal to one. By assumption, $rn \leq N \leq rm$, and so $m \geq n$. Thus A is r-regular if and only if $m = n$.)

We shall often use the following result in linear algebra (see, for instance, Lemma 6 in [6]):

Lemma 2.2. *Let D be a non-singular real matrix of size $n \times n$, and let U and W be two $n \times m$ real matrices such that U or W has full column rank. Then the matrix $D + UW^T$ is non singular if and only if the matrix $W^T D^{-1} U + I$ is non-singular.*

Two instant corollaries of Lemma 2.2, are the following:

Corollary 2.3. *Let A be a non-singular 0-1 matrix. Then the matrix $J - A$ is non-singular if and only if the sum of the elements of A^{-1} is not equal to one.*

Corollary 2.4. *Let (A, B) be a (d, λ) square bipartite design, where d is a full support vector. Then $A^T B$ is non singular if and only if $\lambda \sum_{i=1}^{n} d_i^{-1} \neq -1$.*

(Corollary 2.3 easily follows from Lemma 2.2 with $D = -A$ and $U = W = \mathbf{1}$; Corollary 2.4 easily follows from Lemma 2.2 with $D = diag(d)$, $U = \lambda \mathbf{1}$, and $W = \mathbf{1}$.)

The following lemma gives properties of pairs of matrices that define SBD's:

Lemma 2.5. *If (A, B) is a (d, λ) square bipartite design with d full support, then the columns of A (B) are affinely independent. Moreover, if A (B) is singular then $Ad^{-1} = \mathbf{0}$ $(Bd^{-1} = \mathbf{0})$.*

Proof. By assumption, $A^T B = \lambda J + diag(d)$ with d full support. To show that the columns of A are affinely independent, assume the contrary: there exists a nonzero vector μ such that $\mu^T A^T = \mathbf{0}^T$ and $\mu^T \mathbf{1} = 0$. Hence, $\mu^T A^T B = \mathbf{0}^T$, that is $\lambda \mu^T J + \mu^T diag(d) = \mathbf{0}^T$. But then, $\mu^T diag(d) = \mathbf{0}^T$ (because $\mu^T J = \mathbf{0}^T$), contradicting the assumption that d has full support. Similarly, one can show that the columns of B are affinely independent.

Now, assume that A is singular. Then there exists a nonzero vector μ such that $\mu^T A^T = \mathbf{0}^T$; clearly, $\mu^T \mathbf{1} \neq 0$, for otherwise the columns of A would be affinely dependent. Since $\mu^T A^T B = \mathbf{0}^T$, we have $\lambda \mu^T J + \mu^T diag(d) = \mathbf{0}^T$, and so $\mu^T = -\lambda \mu^T J diag(d^{-1}) = -\lambda \mu^T \mathbf{1}(d^{-1})^T$. Since $\mu^T A^T = \mathbf{0}^T$, and since $\mu^T \mathbf{1} \neq 0$, it follows that $Ad^{-1} = \mathbf{0}$, and we are done. In a similar way, one can show that if B is singular then Bd^{-1} is a zero vector. ∎

Lemma 2.5 generalizes Lemma 8 in [6].

An important class of 0-1 matrices is the class of the so-called *de Bruijn-Erdős (DE) matrices*.

Definition 2.6. *A zero-one matrix is a DE matrix if, for each zero element, the number of ones in the corresponding row equals the number of ones in the corresponding column.*

Observation 2.7. *Every square DE matrix can always be decomposed in the following way:*

$$\begin{bmatrix} X_1 & J & \dots & J \\ J & X_2 & \dots & J \\ \vdots & \vdots & \ddots & \vdots \\ J & J & \dots & X_t \end{bmatrix},$$

where each diagonal block X_i is a square regular matrix and each off-diagonal block has all elements equal to one. In particular, the number of one-columns is equal to the number of one-rows.

To see the validity of the above statement, consider a square DE matrix $A = [a_{ij}]$. Build the bipartite graph $G = (U, V, E)$ corresponding to A: U is the set of the indices of all rows of A, V is the set of the indices of all columns of A, and there exists an edge ij in E if and only if $a_{ij} = 0$. Since A is a square DE matrix, it follows that the endpoints of each edge have the same degree in G, and so each connected component of G is regular. Moreover, the number of isolated vertices of G that belong to U is equal to the number of isolated vertices of G that belong to V, and so the number of one-columns of A is equal to the number of one-colums of A.

The following lemma of de Brujin and Erdős [3] gives a sufficient condition for a 0-1 matrix to be a DE matrix.

Lemma 2.8. [3] *Let $A = [a_{ij}]$ be an $n \times m$ zero-one matrix, with $n \geq m$, having no one-columns. If, for every $a_{ij} = 0$, $r_i^A \geq c_j^A$, then A is a square DE matrix.*

We close this section, by proving a property of a DE matrix that will be used to prove one of the main results.

Lemma 2.9. *Let A be a square DE matrix of size $n \times n$, where each column has at most $n - 2$ components equal to one. Then A is non-singular if and only if $J - A$ is non-singular.*

Proof. By Observation 2.7, A has a block decomposition where each diagonal block X_i is an r_i-regular matrix of size $n_i \times n_i$, and each off-diagonal block has all elements equal to one; clearly, $r_i \leq n_i - 2$ (because $c_j^A \leq n - 2$). It follows that the matrix $J - A$ has a block decomposition where each diagonal block F_i is equal to $J - X_i$ and each off-diagonal block has all elements equal to zero. Clearly, F_i is an $(n_i - r_i)$-regular matrix of size $n_i \times n_i$.

First, assume that A is non-singular. If $J - A$ is singular then there exists some F_i, say F_1, that is singular. Then, there exists a nonzero vector μ such that $F_1\mu = \mathbf{0}$, and so $\mathbf{1}^T F_1 \mu = 0$. Since $\mathbf{1}^T F_1 = (n_1 - r_1)\mathbf{1}^T$ (because F_1 is $(n_1 - r_1)$-regular) with $n_1 > r_1$, it follows that $\mathbf{1}^T \mu = 0$. Now, $\mathbf{1}^T \mu = 0$ and $F_1\mu = \mathbf{0}$ imply that $(J - F_1)\mu = \mathbf{0}$, that is $X_1\mu = \mathbf{0}$. But then, there exists a nonzero vector δ with $\delta^T = [\mu^T \ \mathbf{0}_{n-n_1}^T]$ such that $A\delta = \mathbf{0}$, and so the columns of A are linearly dependent, contradicting the assumption that A is non-singular.

Next, assume that $J - A$ is non-singular, and so each F_i is non-singular. Let $\sigma(J - A)$ denote the sum of all elements of $(J - A)^{-1}$, and let $\sigma(F_i)$ denote the sum of all elements of F_i^{-1}. Corollary 2.3 implies that the matrix A is non-singular if and only if $\sigma(J - A) \neq 1$. Now, since $\sigma(J - A) = \sum_i \sigma(F_i)$, we only need show that each $\sigma(F_i) > 1$. For this purpose, set $Y_i = JF_iF_i^{-1}J = n_iJ$; on the other hand, $Y_i = (n_i - r_i)JF_i^{-1}J = (n_i - r_i)\sigma(F_i)J$. Hence, $\sigma(F_i) = n_i/(n_i - r_i)$. Now, since $r_i \geq 1$ (for otherwise, the matrix F_i would be singular), $\sigma(F_i) > 1$, and so we are done. ∎

3. λ-pairs

In this section, we shall introduce the concept of λ-*pairs* and we shall give some properties of them. The λ-pairs play a special role in the study of square bipartite designs; as we shall see, every $(d, 1)$ SBD studied by Lehman is a λ-pair.

Definition 3.1. *Let A and B be two zero-one matrices of size $n \times m$, with $n \geq m \geq 3$, and let λ be a positive integer. The pair (A, B) is a λ-pair if, for each $i \neq p$ ($i, p = 1, \ldots, n$) and for each $j \neq q$ ($j, q = 1, \ldots, m$),*

$$a_{.j} \cdot b_{.q} \leq \lambda \leq a_{i.} \cdot b_{p.}. \tag{1}$$

In the following, we shall give two properties of a λ-pair.

Property 3.2. *Let (A, B) be a λ-pair with $A = [a_{ij}]$ and $B = [b_{ij}]$. Then*

$$
\begin{aligned}
a_{ij} = 0 &\quad\Rightarrow\quad r_i^A \geq c_j^A, \\
b_{ij} = 0 &\quad\Rightarrow\quad r_i^B \geq c_j^B.
\end{aligned}
$$

Proof. Since (A, B) is a λ-pair, (1) implies that $r_i^A \geq \lambda$ and $r_i^B \geq \lambda$ for every $i = 1, \ldots, n$. Let i and j be two arbitrary indices such that $a_{ij} = 0$. If $c_j^A = 0$, then we are done. Otherwise, consider the $c_j^A \times r_i^A$ submatrix B' of B obtained by removing all rows $b_{h.}$ such that $a_{hj} = 0$ and all columns $b_{.k}$ such that $a_{ik} = 0$. Since (A, B) is a λ-pair, it follows that the number of components equal to one of each row of B' is at least λ (take the scalar product of the i-th row of A and any row of B of different index) and that the number of components equal to one of each column of B' is at most λ (take the scalar product of the j-th column of A and any column of B of different index). Hence, the total number of elements of B' that are equal to one is at least λc_j^A and at most λr_i^A, and so $r_i^A \geq c_j^A$. A similar argument shows that, $b_{ij} = 0$ implies $r_i^B \geq c_j^B$. ∎

An instant corollary of Property 3.2, Lemma 2.8, and Observation 2.7, is the following:

Corollary 3.3. *Let (A, B) be a λ-pair. If A and B have no one-columns, then they are square DE matrices with no one-rows.*

Lemma 3.4. *Let (A, B) be a λ-pair such that A and B have no one-columns. If $a_{.i} \neq a_{.j}$ or $b_{.i} \neq b_{.j}$ for some $i \neq j$ then $a_{.i} \cdot b_{.j} = \lambda$; similarly, if $a_{i.} \neq a_{j.}$ or $b_{i.} \neq b_{j.}$ for some $i \neq j$ then $a_{i.} \cdot b_{j.} = \lambda$.*

Proof. By Corollary 3.3, A and B are square DE matrices with no one-rows. Since (A, B) is a λ-pair, (1) implies that $r_i^A \geq \lambda$ and $r_i^B \geq \lambda$, for every i, and so neither A nor B has a zero-row; hence neither A nor B has a zero-column (because they are DE matrices).

To prove the first part of the lemma, let i and j be two arbitrary indices, with $i \neq j$, such that $a_{.i} \neq a_{.j}$ or $b_{.i} \neq b_{.j}$; without loss of generality, we can assume that $a_{.i} \neq a_{.j}$. Note that there exists an index, say h, such that $a_{hj} = 1$ and $a_{hi} = 0$: if $a_{hj} = 1$ implied $a_{hi} = 1$ for all h, then either $a_{.i}$ would be a one-column, or there would exist an index k such that $a_{ki} = 0$ and $a_{kj} = 0$, and so $c_i^A = c_j^A$ (because A is a DE matrix); but then $a_{.i} = a_{.j}$. Since $a_{hi} = 0$, it follows that $r_h^A = c_i^A$ (because A is a DE matrix).

Now, consider the submatrix B' of B obtained by removing all rows p such that $a_{pi} = 0$ and all columns q such that $a_{hq} = 0$. Since (A, B) is a λ-pair, the number of components equal to one of each row of B' is at least λ (take the scalar product of the h-th row of A and any row of B of different index) and the number of components equal to one of each column of B' is at most λ (take the scalar product of the i-th column of A and any column of B of different index). Since B' is a square matrix (because $r_h^A = c_i^A$), Property 2.1 implies that B' is λ-regular. But, since in the construction of the matrix B' column j has not been eliminated while column i has been eliminated, it follows that $a_{.i} \cdot b_{.j} = \lambda$, and we are done.

To prove the second part of the lemma, we can apply the same reasoning used for the first part by just interchanging columns and rows. ∎

An instant corollary of Lemma 3.4 is the following:

Corollary 3.5. *Let (A, B) be a λ-pair such that for every $i \neq j$, $a_{.i} \neq a_{.j}$ or $b_{.i} \neq b_{.j}$. If A and B have no one-columns, then (A, B) is a square bipartite design.*

Obviously, every regular square bipartite design is a λ-pair. Hence, every $(d, 1)$ SBD studied by Lehman is a λ-pair with $\lambda = 1$ (if it is not regular, then, by Theorem 1.2, it is a DPP which is obviously a 1-pair). If we remove the hypothesis $d \geq 1$, then not every $(d, 1)$ SBD is a λ-pair. Consider, for instance, the pair of matrices

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

However, we can show that, under some mild hypotheses, every (d, λ) SBD is a λ-pair. These hypotheses are always satisfied by every $(d, 1)$ SBD studied by Lehman.

Lemma 3.6. *Let (A, B) be a (d, λ) square bipartite design verifying the following two properties:*
(a) the vectors d, Ad^{-1}, and Bd^{-1} have full support,
(b) there exists a nonzero integer t, such that t and λ are relatively prime and such that $d_i \equiv t \pmod{\lambda}$, for every $i = 1 \ldots, n$.
Then (A, B) is a λ-pair.

Proof. By assumption, $A^T B = \lambda J + diag(d)$ where d, Ad^{-1}, and Bd^{-1} are full support vectors. By Lemma 2.5, both A and B are non singular, and so $A^T B$ is non singular. Hence, by Corollary 2.4, $\lambda \sum_{i=1}^{n} d_i^{-1} \neq -1$.

To prove the lemma, we only need show that every off-diagonal element of BA^T is greater than or equal to λ. For this purpose, let $X = [x_{ij}]$ be the matrix defined by:

$$x_{ij} = \begin{cases} d_i^{-1} - \dfrac{\lambda d_i^{-1} d_i^{-1}}{1 + \sigma}, & if \quad i = j \\ -\dfrac{\lambda d_i^{-1} d_j^{-1}}{1 + \sigma}, & otherwise, \end{cases}$$

where $\sigma = \lambda \sum_{i=1}^{n} d_i^{-1}$. It is easy to verify that $XA^T B = I$, and so $X = (A^T B)^{-1}$. Write $(A^T B)^{-1} = F - G$, where

$$F = diag(d^{-1}) \quad \text{and} \quad G = \frac{\lambda}{1 + \sigma} d^{-1} (d^{-1})^T.$$

Since $(A^T B)(A^T B)^{-1} = I$ can be rewritten as $B(A^T B)^{-1} A^T = I$, we have

$$BFA^T = I + BGA^T = I + \frac{\lambda}{1 + \sigma} (Bd^{-1})(Ad^{-1})^T.$$

Now, let i and j be two arbitrary indices with $i \neq j$ ($i, j = 1, \ldots, n$); let u_{ij} denote the (i, j)-th element of the matrix BFA^T, and let v_{ij} denote the (i, j)-th element of the matrix BGA^T. Clearly, $u_{ij} = v_{ij}$ (because $BFA^T = I + BGA^T = I$). By definition,

$$u_{ij} = \sum_{k=1}^{n} b_{ik} a_{jk} d_k^{-1},$$

$$v_{ij} = \frac{\lambda}{1 + \sigma} \left(\sum_{k=1}^{n} b_{ik} d_k^{-1} \right) \left(\sum_{k=1}^{n} a_{jk} d_k^{-1} \right).$$

Since, by assumption, Ad^{-1} and Bd^{-1} have full support, $v_{ij} \neq 0$, and so $u_{ij} \neq 0$. Hence there exists an index h such that $b_{ih}a_{jh} \neq 0$. It follows that the (i,j)-th element of BA^T is nonzero, and so it is positive; but then, by integrality, it must be greater than or equal to one. Since i and j were arbitrary indices with $i \neq j$, it follows that every off-diagonal element of BA^T is greater than or equal to one. Thus, if $\lambda = 1$, we are done.

Otherwise, assume that $\lambda \geq 2$. Let y_{ij} denote the (i,j)-th element of BA^T. To prove that $y_{ij} \geq \lambda$, we shall show that $y_{ij} \equiv 0 \ (mod \ \lambda)$, and so y_{ij} is a positive multiple of λ (because $y_{ij} \geq 1$). For this purpose, set: $\pi = \Pi_{k=1}^n d_k$ and $q_k = \pi d_k^{-1}$ for $k = 1, \ldots, n$. Since $\pi^2(1+\sigma)u_{ij} = \pi^2(1+\sigma)v_{ij}$, we can write

$$\pi(1+\sigma) \sum_{k=1}^n b_{ik}a_{jk}q_k = \lambda \left(\sum_{k=1}^n b_{ik}q_k \right) \left(\sum_{k=1}^n a_{jk}q_k \right).$$

By assumption, $d_k \equiv t \ (mod \ \lambda)$ (for every k), and so there exists an integer s_k such that $q_k = t^{n-1} + s_k\lambda$. Replacing this expression for q_k in the left hand side of the above equation, we have

$$\pi(1+\sigma)t^{n-1} \sum_{k=1}^n b_{ik}a_{jk} = \lambda \left[\left(\sum_{k=1}^n b_{ik}q_k \right) \left(\sum_{k=1}^n a_{jk}q_k \right) - \pi(1+\sigma) \sum_{k=1}^n b_{ik}a_{jk}s_k \right].$$

Since $\pi(1+\sigma)$ is an integer (because $\pi(1+\sigma) = \pi + \lambda \sum_{k=1}^n q_k$), it follows that the right hand side of the above equation is an integer multiple of λ. But then, the left hand side of the above equation must be also an integer multiple of λ, i.e.

$$\pi(1+\sigma)t^{n-1} \sum_{k=1}^n b_{ik}a_{jk} \equiv 0 \ (mod \ \lambda).$$

Now, $\pi(1+\sigma)t^{n-1} \neq 0$; moreover, since $\pi(1+\sigma)t^{n-1} = \pi t^{n-1} + \pi\sigma t^{n-1}$, if $\pi(1+\sigma)t^{n-1}$ has a common divisor with λ, such a divisor must divide also πt^{n-1}, contradicting the assumption (b). Hence

$$y_{ij} = \sum_{k=1}^n b_{ik}a_{jk} \equiv 0 \ (mod \ \lambda).$$

The lemma follows. ∎

The previous lemma generalizes a result in [6], where the case $\lambda = 1$ was studied (indeed, when $\lambda = 1$ the assumption (b) is always satisfied).

We close this section by introducing special λ-pairs: the *peculiar* pairs.

Definition 3.7. *A peculiar pair is a λ-pair (A, B) satisfying one of the following four properties:*
(P1) $A = J$ and B is λ-regular (or viceversa),
(P2) $(A, B) \cong (J - I, J - I)$,
(P3) (A, B) is a DPP design,
(P4) (A, B) satisfies one of the following three cases:

Case 1: $\lambda = 1$ and (A, B) is one of the following four types

$$(A, B) \cong (J - I, I), \tag{2}$$

$$(A, B) \cong \left(\begin{bmatrix} J & J \\ J & J - I \end{bmatrix}, I \right), \tag{3}$$

$$(A, B) \cong \left(\begin{bmatrix} 1 & x^T \\ \mathbf{1} & I \end{bmatrix}, \begin{bmatrix} 0 & \mathbf{1}^T \\ \mathbf{1} & Z \end{bmatrix} \right), \tag{4}$$

$$(A, B) \cong \left(\begin{bmatrix} \mathbf{1} & A' \end{bmatrix}, \begin{bmatrix} \mathbf{1} & B' \end{bmatrix} \right), \tag{5}$$

where x denotes an arbitrary 0-1 vector, Z denotes a matrix whose elements are all equal to zero, and A' and B' denote two arbitrary 0-1 matrices with $c_j^{A'} \leq 1$ and $c_j^{B'} \leq 1$, for every j.

Case 2: $\lambda = 2$ and (A, B) is one of the following five types

$$(A, B) \cong \left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \right), \tag{6}$$

$$(A, B) \cong \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \right), \tag{7}$$

$$(A, B) \cong \left(\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \right), \tag{8}$$

$$(A, B) \cong \left(\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \right), \tag{9}$$

$$(A, B) \cong \left(\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \right). \tag{10}$$

Case 3: $\lambda = 3$ and

$$(A, B) \cong \left(\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \right). \tag{11}$$

Note that every peculiar pair, with the possible exception of the pairs satisfying (4) or (5), is a square bipartite design. A peculiar pair (A, B) that satisfies (4) is a SBD only if the vector x has all components equal to one (because $n > 2$); a peculiar pair (A, B) that satisfies (5) is a SBD only if $B' = A'$ and A' has all elements equal to zero, but the elements of a row (that are all equal to one) or if

$$(A, B) \cong \left(\left[\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right], \left[\begin{array}{ccc} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right] \right). \tag{12}$$

Remark 3.8. *The only peculiar pairs (A, B) that define non-regular (d, λ) square bipartite designs with d full support, either are DPP designs (with $n > 3$), or they satisfy one of (6), (7), (10), (11), and (12).*

4. Our results

In this section we shall characterize all non-regular (d, λ) square bipartite designs that satisfy the hypotheses of Lemma 3.6. To do that, first we shall characterize all λ-pairs (A, B) such that $c_j^A \geq n - 1$ or $c_j^B \geq n - 1$ for some j; then we shall show that every other λ-pair is a regular design or it does not satisfy every hypothesys of Lemma 3.6.

Theorem 4.1. *Let (A, B) be a pair of zero-one matrices of size $n \times m$ with $n \geq m \geq 3$, such that $c_j^A \geq n - 1$ or $c_j^B \geq n - 1$ for some j $(j = 1, \ldots, m)$. Then (A, B) is a λ-pair if and only if it is peculiar.*

Theorem 4.2. *Let (A, B) be a λ-pair of zero-one matrices of size $n \times m$ with $n \geq m \geq 3$, such that $c_j^A \leq n - 2$ and $c_j^B \leq n - 2$ for every j $(j = 1, \ldots, m)$. If the following two conditions hold*

$$a_{.i} \neq a_{.j} \text{ or } b_{.i} \neq b_{.j}, \quad \forall \ i \neq j \tag{13}$$

$$a_{.i} \cdot b_{.i} \neq \lambda \quad \forall \ i, \tag{14}$$

then (A, B) is a regular square bipartite design.

Theorem 4.3. *Let (A, B) be a non-regular (d, λ) square bipartite design satisfying the following two properties:*

(i) the vectors d, Ad^{-1}, and Bd^{-1} have full support,

(ii) there exists a nonzero integer t, such that t and λ are relatively prime and $d_i \equiv t \pmod{\lambda}$, for every $i = 1 \ldots, n$.

Then (A, B) either is a DPP design (with $n > 3$), or it satisfies one of (10), (11), and (12).

Theorem 4.3 implies Theorem 1.3 (when $\lambda = 1$, assumption (ii) is always satisfied), and so it implies also Theorem 1.1 of de Bruijn and Erdős and Theorem 1.2 of Lehman.

5. Proofs of the results

In this section we shall prove the three theorems of the previous section.

Proof of Theorem 4.1. The *if* part is obvious by definition of a peculiar pair.

To prove the *only if* part, we shall distinguish between the case $c_j^A = n$ or $c_j^B = n$, for some j, and the case $c_j^A \leq n - 1$ and $c_j^B \leq n - 1$ for all j ($j = 1, \ldots, m$).

Case 1. *A or B has a one-column.*

Without loss of generality, we can assume that A contains a one-column, say the first. Since (A, B) is a λ-pair, (1) implies that $r_i^A \geq \lambda$ and $r_i^B \geq \lambda$ for every $i = 1, \ldots, n$, and that $c_j^B \leq \lambda$ for every $j = 2, \ldots, n$ (take the scalar product of the first column of A and the j-th column of B).

Now, if A has some other one-column, then $c_1^B \leq \lambda$ (take the scalar product of such a column of A and the first column of B), and so Property 2.1 implies that B is λ-regular and $m = n$. If $A = J$ then (A, B) is peculiar, and we are done. Otherwise, some element of A is equal to zero, say $a_{ij} = 0$. Since (A, B) is a λ-pair, (1) implies that, for every $h \neq i$, $a_i \cdot b_h \geq \lambda$, and so, $b_{hj} = 0$ (because $r_h^B = \lambda$). It follows that $c_j^B \leq 1$, and so $\lambda = 1$. But then $B \cong I$, and so we can assume that $B = I$. Again, (1) implies that every off-diagonal element of A is one, and so (A, B) satisfies (3).

Hence, we can assume that the number of ones of every column of A, but the first, is at most $n - 1$. We shall distinguish between the case $\lambda = 1$ and the case $\lambda \geq 2$. Recall that $r_i^A \geq \lambda$, $r_i^B \geq \lambda$ (for all i), $c_1^A = n$, and $c_j^B \leq \lambda$ for every $j \neq 1$.

Subcase 1.1. $\lambda = 1$.

If the first column of B is a one-column, then $c_j^A \leq 1$ for every $j \neq 1$ (take the scalar product of the first column of B and any column of A of different index), and so (A, B) satisfies (5), and we are done.

Hence, we can assume that $c_1^B \leq n - 1$. Clearly, $c_1^B \geq 1$: if $c_1^B = 0$ then the $n \times (m - 1)$ submatrix of B obtained by removing the first column would have at least one 1 in each row (because $r_i^B \geq 1$ for all i) and at most one 1 in each column (because $c_j^B \leq 1$ for all $j \geq 2$), and so, by Property 2.1, $m - 1 \geq n$, contradicting the assumption that $n \geq m$.

Now, let B' denote the $(n - c_1^B) \times m$ submatrix of B obtained by removing all rows having a one in the first column. Since the first column of B' is a zero-column, Property 3.2 implies that in every row of B' there are at least c_1^B components equal to one. Let N denote the number of elements of B' that are equal to one; clearly $N \geq (n - c_1^B)c_1^B$ and $N \leq m - 1$ (because $c_1^{B'} = 0$ and $c_j^B \leq 1$ for every $j \neq 1$), and so $(n - c_1^B)c_1^B \leq m - 1 \leq n - 1$. But then either $c_1^B = 1$ or $c_1^B = n - 1$; moreover, $m = n$.

If $c_1^B = 1$ then, without loss of generality, we can assume that $b_{11} = 1$. But then, Property 2.1 implies that the submatrix of B obtained by removing the first row and the first column is 1-regular, and so $b_{1j} = 0$ for every $j \neq 1$ (because $c_j^B \leq 1$ for every $j > 1$). Hence, $B \cong I$ and so again (1) implies that (A, B) satisfies (3), and we are done.

If $c_1^B = n - 1$ then, without loss of generality, we can assume that $b_{11} = 0$. Property 3.2 implies that $r_1^B = n - 1$, and so $b_{1j} = 1$ for every $j > 1$. Since $c_j^B \leq 1$ for every $j > 1$, we have $b_{ij} = 0$ for every $i, j > 1$. Now, let A' denote the submatrix of A obtained by removing the first row and the first column. Clearly, $c_j^{A'} \leq 1$ (take the scalar product of the first column of B and any column of A of different index) and $r_i^{A'} \geq 1$ (take the scalar product of the first row of B

and any row of A of different index), and so, by Property 2.1, A' is 1-regular. Hence, (A, B) satisfies (4), and again we are done.

Subcase 1.2. $\lambda \geq 2$.

Consider the $n \times (m-1)$ submatrix B' of B obtained by removing the first column. Since $c_j^B \leq \lambda$ for every $j \geq 2$ and since $n > m - 1$, it follows that there exists a row of B', say the last, whose number of components that are equal to one is at most $\lambda - 1$ and so precisely $\lambda - 1$ (because $r_n^B \geq \lambda$). Without loss of generality, we can assume that these are the last components, i.e. $b_{nj} = 1$ for every $j \geq m - \lambda + 2$. Let A'' denote the $(n-1) \times (\lambda - 1)$ submatrix of A obtained by removing the last row and the first $m - \lambda + 1$ columns. Clearly, $A'' = J$ (take the scalar product of the last row of B and any row of A different from the last). Since, by assumption, $c_j^A \leq n - 1$ for all $j > 1$, it follows that $a_{nj} = 0$ for every $j \geq m - \lambda + 2$. In particular, $a_{nm} = 0$, and so, by Property 3.2, $r_n^A \geq c_m^A = n - 1$. But, $r_n^A \leq m - \lambda + 1$, and so $m = n$ and $\lambda = 2$.

Recall that the submatrix A'' of A obtained by removing the last row and the first $m - 1$ columns is equal to J, and that $b_{nn} = 1$ and $b_{nj} = 0$ for every $2 \leq j \leq n - 1$. Hence, $a_{nn} = 0$ (because $c_n^A \leq n - 1$), $a_{nj} = 1$ for every $j < n$ (by Property 3.2), and $b_{n1} = 1$ (because $r_n^B \geq \lambda$).

Now, consider the square submatrix B'' of B obtained by removing the last row and the last column. Clearly, $r_i^{B''} \geq 2$ for every i (take the scalar product of the last row of A and any row of B different from the last), $c_j^{B''} \leq 2$ for every j (take the scalar product of the last column of A and any column of B different from the last), and so Property 2.1 implies that B'' is 2-regular. Since $c_n^B \leq 2$, without loss of generality, we can assume that $b_{in} = 0$ for every $i = 2, \ldots, n - 1$, and so $r_i^B = 2$ for every $i = 2, \ldots, n - 1$ (because B'' is 2-regular).

We shall show that
$$b_{1j} = 1 \quad \text{for every } j = 2, \ldots, n - 1,$$
and so $n - 2 \leq 2$ (because the $(n-1) \times (n-1)$ matrix B'' is 2-regular), that is $n \leq 4$. For this purpose, assume the contrary: $b_{1h} = 0$ for some $h = 2, \ldots, n - 1$. Since B'' is 2-regular, $b_{ih} = b_{kh} = 1$ for some $i \neq k$ with $i, k = 2, \ldots, n - 1$. But then, since $b_{in} = b_{kn} = 0$, and since $r_i^B = r_h^B = 2$, it follows that $a_{,h} = \mathbf{1}$ (take the scalar product of the i-th row of B and any row of A of different index; and take the scalar product of k-th row of B and the i-th row of A), contradicting the assumption that $c_h^A \leq n - 1$.

Hence, $n \leq 4$. Now, it is easy to verify that, when $n = 3$, A and B satisfy (6), and that when $n = 4$, A and B satisfy (7), and we are done.

Case 2. *A and B have no one-columns.*

Since (A, B) is a λ-pair where both A and B have no one-columns, Corollary 3.3 implies that A and B are square DE matrices with no one-rows.

We distinguish between the case $\lambda = 1$ and the case $\lambda \geq 2$.

Subcase 2.1. $\lambda = 1$.

By assumption, $c_j^A = n - 1$ or $c_j^B = n - 1$ for some j; without loss of generality, we can assume that $c_1^A = n - 1$ and that $a_{11} = 0$; since A is a DE matrix, $a_{1j} = 1$ for every $j > 1$. Consider the submatrix B' of B obtained by removing the first row and the first column. Clearly, each row of B' has at least one component equal to one (take the scalar product of the first row of A and any row of B of different index), and each column of B' has at most one component equal to one (take the scalar product of the first column of A and any column of B of different index), and so Property 2.1 implies that B' is 1-regular. Without loss of generality, we can assume that $B' = I$.

Now, we shall show that, either $b_{i1} = 1$ for every $i > 1$ and $b_{1j} = 1$ for every $j > 1$, or $b_{i1} = 0$ for every $i > 1$ and $b_{1j} = 0$ for every $j > 1$. This is not difficult to verify when $n = 3$. Hence, assume $n > 3$. Since $b_{2j} = 0$ for every $j \geq 3$, and since $B' = I$, it follows that $b_{21} = b_{1j}$ for every $j \geq 3$ (because B is a DE matrix); applying the same argument, we have: $b_{13} = b_{i1}$ for every $i \neq 1$ and $i \neq 3$, $b_{14} = b_{31} = b_{12}$. But then, it is easy to verify that either (A, B) is a DPP design, or (A, B) satisfies (2), and we are done.

Subcase 2.2. $\lambda \geq 2$.

Claim 1. $r_i^A > \lambda$ and $r_i^B > \lambda$, for every $i = 1, \ldots, n$.

To show that $r_i^A > \lambda$ for every i, assume the contrary: $r_h^A \leq \lambda$ for some h, and so $r_h^A = \lambda$ (because (A, B) is a λ-pair). Without loss of generality, we can assume that $a_{hj} = 1$ for every $j \geq n - \lambda + 1$. Consider the $(n - 1) \times \lambda$ submatrix $\hat{B}$ of B obtained by removing the h-th row and all first $n - \lambda$ columns. Clearly, the number of components equal to one of each row of B' is at least λ (take the scalar product of the i-th row of A and any row of B of different index), and so it is equal to λ. Hence $\hat{B} = J$, and so $b_{hj} = 0$, for every $j \geq n - \lambda + 1$ (because B has no one-columns). Since B is a DE matrix, $r_h^B = c_n^B = n - 1$. But $r_h^B \leq n - \lambda$, and so $\lambda \leq 1$, contradicting the assumption that $\lambda \geq 2$. A similar argument shows that $r_i^B > \lambda$, for every i. Thus, the claim follows.

Now, since $r_i^A > \lambda$ for every i, and since A has no one-rows, it follows that $\lambda \leq n - 2$. By assumption, A or B has a column with precisely $n - 1$ ones; without loss of generality, we can assume that $c_1^A = n - 1$ and that $a_{11} = 0$, and so $r_1^A = n - 1$ (because A is a DE matrix). Now, consider the submatrix A' of A obtained by removing the first row and the first column, and the submatrix B' of B obtained by removing the first row and the first column.

Claim 2. A' and B' are λ-regular, $b_{1.} = a_{1.}$ and $b_{.1} = a_{.1}$.

The λ-regularity of B' it is immediately implied by Property 2.1 by observing that the number of components equal to one of each column of B' is at most λ (take the scalar product of the first column of A and any column of B different from the first) and that the number of components equal to one of each row of B' is at least λ (take the scalar product of the first row of A and any row of B different from the first). Hence $b_{i1} = 1$ for every $i \geq 2$ (because $r_i^B > \lambda$), and so $b_{11} = 0$ (for otherwise, B would have a one-column). But then, $b_{1j} = 1$ for every $j \geq 2$ (because B is a DE matrix). Hence, applying the same argument used to prove the λ-regularity of B', we can show that also A' is λ-regular. The Claim follows.

Now, if $\lambda = n - 2$ then $(A, B) \cong (J - I, J - I)$, and we are done. Hence, we can assume that $2 \leq \lambda \leq n - 3$.

Claim 3. Either $\lambda = 2$ and $n = 5$, or $\lambda = 2$ and $n = 6$, or $\lambda = 3$ and $n = 6$.

To prove the validity of the claim, let $a'_{.j}$ and $b'_{.j}$ denote the j-th column of the matrix A' and B', respectively; similarly, let $a'_{i.}$ and $b'_{i.}$ denote the i-th row of the matrix A' and B', respectively. Since (A, B) is a λ-pair, and since $a_{1.} = b_{1.} = \mathbf{1}$ and $a_{.1} = b_{.1} = \mathbf{1}$, it follows that (A', B') is a $(\lambda - 1)$-pair, that is, for every $j \neq q$, and $i \neq p$,

$$a'_{.j} \cdot b'_{.q} \leq \lambda - 1 \leq a'_{i.} \cdot b'_{p.} \tag{15}$$

Now, set $n' = n - 1$. Since A' is λ-regular, without loss of generality, we can assume that $a'_{1j} = 1$ for every $j \geq n' - \lambda + 1$ and that $a'_{i1} = 1$ for every $i \geq n' - \lambda + 1$. Let $\hat{B}'$ denote the submatrix of B' obtained by removing the last λ rows and the first $n' - \lambda$ columns; and let

$\tilde{B}'$ denote the submatrix of B' obtained by removing the first $n' - \lambda$ rows and the first $n' - \lambda$ columns. Clearly, (15) implies that the number of components equal to one of each row of $\tilde{B}'$ and of each row of $\hat{B}'$, but the first, is at least $\lambda - 1$ (take the scalar product of the first row of A' and any row of B' different from the first) and that the number of components equal to one of each column of $\tilde{B}'$ is at most $\lambda - 1$ (take the scalar product of the first column of A' and any column of B' different from the first). Hence, by Property 2.1, the $\lambda \times \lambda$ matrix $\tilde{B}'$ is $(\lambda - 1)$-regular, and so, without loss of generality, we can assume that $\tilde{B}' = J - I$. Since B' is λ-regular, in each column of $\hat{B}'$ there is exactly one component equal to one, and so $\hat{B}'$ has precisely λ elements equal to one. But then, since in each row of $\hat{B}'$, but the first, there are at least $\lambda - 1$ components equal to one, it follows that $\hat{B}'$ has at most three rows (i.e., $n' - \lambda \leq 3$) and that in the first row of $\hat{B}'$ there is at most one component equal to one. Then, $\lambda = r_1^{B'} \leq n' - \lambda + 1$; in particular, $\lambda \leq 4$.

Since $\lambda \leq n - 3$, it follows that $\hat{B}'$ has more than one row. If $\hat{B}'$ has two rows ($n' - \lambda = 2$), then $\lambda \leq 3$; hence, the first row of $\hat{B}'$ has precisely one component equal to one (because $\hat{B}'$ has precisely λ elements equal to one), and so either $\lambda = 2$ and $n = 5$, or $\lambda = 3$ and $n = 6$. If $\hat{B}'$ has three rows ($n' - \lambda = 3$), then since the second and the third row have at least $\lambda - 1$ components equal to one, it follows that the total number (λ) of elements equal to one of $\hat{B}'$ is at least $2(\lambda - 1)$; but then $\lambda = 2$, and $n = 6$. The claim follows.

By Claim 3, we only need verify the validity of the theorem for the three cases $n = 5$ and $\lambda = 2$, $n = 6$ and $\lambda = 2$, and $n = 6$ and $\lambda = 3$.

First, assume that $n = 5$ and $\lambda = 2$. We claim that A' or B' has two equal rows. Indeed, if A' has no two equal rows, then, without loss of generality, we can assume that

$$A' = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(because A' is 2-regular). But then, it is easy to verify that no row of B' has two consecutive ones or two consecutive zeroes (by applying (15)), and so B' has two equal rows. Hence, without loss of generality, we can assume that A' has two equal rows and that

$$A' = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Now, (15) implies that in each row and in each column of B' there is exactly one component equal to one in the first two positions and there is exactly one component equal to one in the last two positions. But then it is easy to see that (A, B) satisfies (8) or (9), and we are done.

Next, assume that $n = 6$ and $\lambda = 2$. We claim that neither A' nor B' has two equal rows: if A' had two equal rows, then, without loss of generality, we could assume that such two rows were equal to $[00011]$; but then (15) would imply that in every row of B' there is at least one component equal to one in the last two positions, and so in the last two columns of B' there would exist at least five ones, contradicting the 2-regularity of B'. Hence, without loss of

generality, we can assume that

$$A' = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now, again (15) implies that no row of B' has two consecutive ones or three consecutive zeroes. But then, since B' is 2-regular, it is easy to verify that (A, B) satisfies (10), and again we are done.

Finally, assume that $n = 6$ and $\lambda = 3$. We claim that neither A' nor B' has two equal rows: if A' had two equal rows, then, without loss of generality, we could assume that such two rows were equal to $[00111]$; but then (15) would imply that in every row of B' there are at least two components equal to one in the last three positions, and so in the last three columns of B' there would exist at least ten ones, contradicting the 3-regularity of B'. Hence, without loss of generality, we can assume that

$$A' = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Again (15) implies that no row of B' has three consecutive ones or two consecutive zeroes. But then, since B' is 3-regular, it is easy to verify that (A, B) satisfies (11), and again we are done. Thus, the theorem follows. $\blacksquare$

Proof of Theorem 4.2. Since (A, B) is a λ-pair where both A and B have no one-columns, Corollary 3.3 implies that A and B are square DE matrices with no one-rows; since $c_j^A \le n - 2$ for every j, it follows that

$$r_i^A \le n - 2 \quad \textit{for every } i. \tag{16}$$

Now, assumption (13) implies that (A, B) satisfies the hypotheses of Corollary 3.5, and so (A, B) is a SBD. Hence, there exists a vector d such that $A^T B = \lambda J + diag(d)$; assumption (14) implies that d has full support, and so A and B have not two equal columns. By Observation 2.7, we can write

$$A = \begin{bmatrix} X_1 & J & \dots & J \\ J & X_2 & \dots & J \\ \vdots & \vdots & \ddots & \vdots \\ J & J & \dots & X_t \end{bmatrix}, \tag{17}$$

where each diagonal block X_i is an r_i-regular matrix of size $n_i \times n_i$ and each off-diagonal block has all elements equal to one. Clearly, for every i, $n_i \ge 2$ and $r_i \le n_i - 2$ (because, by assumption, $c_j^A \le n - 2$ for every j); moreover, $r_i \ge 1$ (because A has not two equal columns), and so $n_i \ge 3$ for every i.

First, we shall show that A and B are non-singular. To show that A is non-singular, assume the contrary: A is singular, and so its columns are linearly dependent. Hence, there exists a nonzero vector δ such that $\delta^T A^T = \mathbf{0}^T$; clearly $\delta^T \mathbf{1} \ne 0$ (for otherwise, the columns of A would

be affinely dependent, contradicting Lemma 2.5). On the other hand, it is easy to verify that the vector

$$
x = \frac{1}{\sigma}
\begin{bmatrix}
\dfrac{1}{n_1 - r_1}\mathbf{1}_{n_1} \\
\dfrac{1}{n_2 - r_2}\mathbf{1}_{n_2} \\
\vdots \\
\dfrac{1}{n_t - r_t}\mathbf{1}_{n_t}
\end{bmatrix},
$$

where $\sigma = \sum_{i=1}^{t} n_i/(n_i - r_i) - 1$, is a solution to the equation $A^T x = \mathbf{1}$, and so $\delta^T A^T x = \delta^T \mathbf{1}$. But then, $0 = \mathbf{0}^T x = \delta^T A^T x = \delta^T \mathbf{1} \neq 0$, which is impossible. Hence, A is non-singular. In a similar way, one can show that also B is non-singular (by using a similar block decomposition for B).

Now, Lemma 2.9 implies that both matrices $J - A$ and $J - B$ are non-singular. Moreover, each matrix X_i is non-singular: indeed, the non-singularity of $J - A$ implies the non-singularity of each matrix $J - X_i$, and so, Lemma 2.9 (applied to the DE matrix X_i) implies that X_i is non-singular.

To prove the theorem, we only need show that there exist integers r and s such that A is r-regular and B is s-regular. Indeed, as soon this is established, we can write:

$$
rsJ = JA^T B = \lambda nJ + J\,diag(d),
$$

and so $d = (rs - \lambda n)\mathbf{1}$. But then,

$$
BA^T = BA^T BB^{-1} = B(\lambda J + (rs - \lambda n)I)B^{-1} = \lambda J + (rs - \lambda n)I = A^T B.
$$

Hence (A, B) is a regular square bipartite design.

To show that the matrices A and B are regular, assume the contrary: A or B is not regular. Without loss of generality, we can assume that A is non-regular, and so $t \geq 2$ in (17). Write:

$$
B =
\begin{bmatrix}
Y_{11} & Y_{12} & \ldots & Y_{1t} \\
Y_{21} & Y_{22} & \ldots & Y_{2t} \\
\vdots & \vdots & \ddots & \vdots \\
Y_{t1} & Y_{t2} & \ldots & Y_{tt}
\end{bmatrix},
$$

We shall show that $Y_{ij} = J$, for every $i \neq j$. To prove this, let i and j be two arbitrary indices, with $i \neq j$ $(i, j = 1, \ldots, t)$. Without loss of generality, we can assume that $i = 1$. Let $b_{.h}$ be an arbitrary column of B with $h > n_1$. Write

$$
b_{.h} =
\begin{bmatrix}
y \\
z
\end{bmatrix},
$$

where y is a vector with n_1 components. To show that $Y_{1j} = J$ for every $j \neq 1$, we only need show that $y = \mathbf{1}_{n_1}$.

Now, since (A, B) is a (d, λ) SBD, the scalar product of $b_{.h}$ with any of the first n_1 columns of A is λ, that is

$$
\begin{bmatrix}
X_1 \\
J
\end{bmatrix}^T
\begin{bmatrix}
y \\
z
\end{bmatrix} = \lambda \mathbf{1}_{n_1}.
$$

18.

But

$$\left[\begin{array}{c} X_1 \\ J \end{array}\right]^T \left[\begin{array}{c} y \\ z \end{array}\right] = X_1^T y + Jz,$$

and so $X_1^T y + Jz = \lambda \mathbf{1}_{n_1}$. If we denote by p the number of components of z that are equal to one, then $Jz = p\mathbf{1}_{n_1}$, and so

$$X_1^T y = (\lambda - p)\mathbf{1}_{n_1}.$$

First, assume that $\lambda = p$. Then $X_1^T y = \mathbf{0}_{n_1}$, and so $y = \mathbf{0}_{n_1}$ (because X_1 is non-singular). Now, let $\hat{A}$ denote the $n \times (n - n_1 - 1)$ submatrix of A obtained by removing the first n_1 columns and column h. Write

$$\hat{A} = \left[\begin{array}{c} J \\ \hat{A}' \end{array}\right],$$

where J has n_1 rows. Since (A, B) is a (d, λ) SBD, we have

$$\hat{A}^T \left[\begin{array}{c} y \\ z \end{array}\right] = \lambda \mathbf{1}_{n - n_1 - 1},$$

and so $\hat{A}'^T z = \lambda \mathbf{1}_{n - n_1 - 1}$ (because $y = \mathbf{0}_{n_1}$). Without loss of generality, we can assume that

$$z = \left[\begin{array}{c} \mathbf{1}_p \\ \mathbf{0}_{n - n_1 - p} \end{array}\right] = \left[\begin{array}{c} \mathbf{1}_\lambda \\ \mathbf{0}_{n - n_1 - \lambda} \end{array}\right],$$

and so all first λ rows of $\hat{A}'$ must be one-rows. But then, the matrix A has λ rows with at least $n - 1$ components equal to one, contradicting (16).

Next, assume that $\lambda \neq p$, and so

$$\frac{1}{\lambda - p} X_1^T y = \mathbf{1}_{n_1}.$$

Since X_1 is non-singular, it follows that

$$\frac{y}{\lambda - p}$$

is the unique solution to the equation $X_1^T x = \mathbf{1}_{n_1}$. But since X_1 is r_1-regular, it follows that $X_1^T \mathbf{1}_{n_1} = r_1 \mathbf{1}_{n_1}$, and so

$$y = \frac{\lambda - p}{r_1} \mathbf{1}_{n_1}.$$

Now, since y is a 0-1 vector and $\lambda \neq p$, it follows that $\lambda - p = r_1$, and so $y = \mathbf{1}_{n_1}$, and we are done.

Hence, we have shown that $Y_{ij} = J$ for every $i \neq j$, that is

$$B = \left[\begin{array}{cccc} Y_{11} & J & \dots & J \\ J & Y_{22} & \dots & J \\ \vdots & \vdots & \ddots & \vdots \\ J & J & \dots & Y_{tt} \end{array}\right].$$

Finally we shall show that A has a row whose number of components equal to one is greater than $n - 2$, contradicting (16). Thus, as soon as this is established, the theorem will follow.

For this purpose, let $a_{.i}$ be an arbitrary column of A with $i \leq n_1$; let $a_{.j}$ be an arbitrary column of A with $j > n_1$; let $b_{.p}$ be an arbitrary column of B with $p \leq n_1$ and $p \neq i$; and let $b_{.q}$ be an arbitrary column of B with $q > n_1$ and $q \neq j$ (such columns exist, because $n_1 \geq 2$ and $n - n_1 \geq 2$). We have:

$$a_{.i} = \begin{bmatrix} \alpha \\ \mathbf{1}_{n-n_1} \end{bmatrix}, \quad a_{.j} = \begin{bmatrix} \mathbf{1}_{n_1} \\ \beta \end{bmatrix}, \quad b_{.p} = \begin{bmatrix} \gamma \\ \mathbf{1}_{n-n_1} \end{bmatrix}, \quad b_{.q} = \begin{bmatrix} \mathbf{1}_{n_1} \\ \delta \end{bmatrix}.$$

Set

$$\begin{aligned} S_\alpha &= \{t : \alpha_t = 1\}, \\ S_\beta &= \{t : \beta_t = 1\}, \\ S_\gamma &= \{t : \gamma_t = 1\}, \\ S_\delta &= \{t : \delta_t = 1\}. \end{aligned}$$

Since (A, B) is a (d, λ) SBD, we have

$$\lambda = a_{.i} \cdot b_{.p} = a_{.j} \cdot b_{.q} = a_{.i} \cdot b_{.q} = a_{.j} \cdot b_{.p},$$

that is

$$\begin{aligned} \lambda = a_{.i} \cdot b_{.p} &= |S_\alpha \cap S_\gamma| + (n - n_1), \\ \lambda = a_{.j} \cdot b_{.q} &= n_1 + |S_\beta \cap S_\delta|, \\ \lambda = a_{.i} \cdot b_{.q} &= |S_\alpha| + |S_\delta|, \\ \lambda = a_{.j} \cdot b_{.p} &= |S_\beta| + |S_\gamma|. \end{aligned}$$

Now, adding the first two equations, we get

$$2\lambda = |S_\alpha \cap S_\gamma| + |S_\beta \cap S_\delta| + n;$$

adding the last two equations, we get

$$\begin{aligned} 2\lambda &= (|S_\alpha| + |S_\gamma|) + (|S_\beta| + |S_\delta|) \\ &= (|S_\alpha \cup S_\gamma| + |S_\alpha \cap S_\gamma|) + (|S_\beta \cup S_\delta| + |S_\beta \cap S_\delta|). \end{aligned}$$

Hence, $|S_\alpha \cup S_\gamma| + |S_\beta \cup S_\delta| = n$. But, since $|S_\alpha \cup S_\gamma| \leq n_1$ and $|S_\beta \cup S_\delta| \leq n - n_1$, it follows that $|S_\alpha \cup S_\gamma| = n_1$; in particular, there exists no index k $(k = 1, \ldots, n_1)$ such that $\alpha_k = 0$ and $\gamma_k = 0$.

Now, by assumption, $c_p^B \leq n - 2$, and so $b_{kp} = 0$, for some k $(k = 1, \ldots, n_1)$. Hence $a_{ki} = 1$. But then, since $a_{.i}$ was an arbitrary column of A with $i \leq n_1$ and $i \neq p$, it follows that $r_k^A \geq n - 1$. The theorem follows. ∎

Proof of Theorem 4.3. By assumption, (A, B) satisfies the hypotheses of Lemma 3.6, and so it is a λ-pair. Since d has full support, both (13) and (14) are satisfied, and so Theorem 4.2 implies that $c_j^A \geq n - 1$ or $c_j^B \geq n - 1$, for some j. But then, by Theorem 4.1, (A, B) is peculiar. Now, Remark 3.8 implies that (A, B) either is a DPP design (with $n > 3$), or it satisfies one of (6), (7), (10), (11), and (2.9). Since in both cases (6) and (7) Bd^{-1} has not full support, the theorem follows. ∎

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