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OBJECTIVE-DERIVATIVE-FREE METHODS FOR CONSTRAINED OPTIMIZATION

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Abstract

We propose feasible descent methods for constrained minimization that do not make explicit use of objective derivative information. The methods at each iteration sample the objective function value along a finite set of feasible search arcs and decrease the sampling stepsize if an improved objective function value is not sampled. The search arcs are obtained by projecting search direction rays onto the feasible set and the search directions are chosen such that a subset approximately generates the cone of first-order feasible variations at the current iterate. We show that these methods have desirable convergence properties under certain regularity assumptions on the constraints. In the case of linear constraints, the projections are redundant and the regularity assumptions hold automatically. Numerical experience with the methods in the linear constraint case is reported.
Key words. Constrained optimization, derivative-free method, feasible descent, stationary point, metric regularity, MFCQ.

1. Introduction

We consider a nonlinear program of the form

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0, \quad i = 1, \ldots, m, \end{array} \tag{1}$$

where $f : \Re^n \to \Re$ and $g_i : \Re^n \to \Re$, $i = 1, \ldots, m$, are continuously differentiable functions, and $x \in \Re^n$. We are interested in the situation where the first-order derivatives of f cannot be explicitly calculated or approximated, while the first-order derivatives of the constraint functions $g_1, \ldots, g_m$ are available. Such a situation arises when the constraints are linear, i.e., $g_1, \ldots, g_m$ are affine. It can also arise when, for example, $f(x)$ is the output of a simulation with input parameter x and, for a certain value $\hat{x}$, the simulation crashes or outputs nonsense. Then a (nonconvex) constraint of the form $\|x - \hat{x}\|^2 \geq \delta$, where $\delta > 0$ is a chosen scalar and $\|\cdot\|$ denotes the 2-norm, might be introduced. For simplicity, we consider only inequality constraints, though much of our results can be extended to handle equality constraints with minor modifications.

In the unconstrained case of $m = 0$, there has been proposed many methods for solving (1) that use the objective function value, but not its derivatives. Such methods, called derivative-free or direct-search methods, iteratively sample the objective function value along search directions and decrease the sampling stepsize if an improved objective function value is not sampled (see the book [4], the survey papers [5, 14, 17] and references therein). By using coordinate directions as search directions, some of the methods have been extended to the case of bound constraints [6, 9, 11]. For the linearly constrained case, finite-difference quasi-Newton methods [13] and pattern search methods [10] have been developed.

In this paper, we propose two methods for the general problem (1) that may be viewed as (nontrivial) extensions of the method in [11]. The methods at each iteration sample the objective function value along a finite set of feasible search arcs and decrease the sampling stepsize if an improved objective function value is not sampled. The search arcs are obtained by projecting search direction rays onto the feasible set, and the search directions are chosen such that a subset approximately generates the cone of first-order feasible variations at the current iterate. More precisely, at the beginning of each iteration, we define a cone of first-order ϵ-feasible variations (based on ϵ-active constraints) for some $\epsilon > 0$, and a set of "reference" search directions D is chosen such that a subset of them generates this cone. During the iteration, the iterate is updated and the search directions are determined by projecting the reference search directions onto the cone of first-order feasible variations at the updated iterate. In the case of linear constraints, the projections are redundant and the search directions can be drawn from a finite set. We show that these methods have desirable convergence properties under certain regularity assumptions on the constraints (Propositions 5.2 and 6.2). These regularity assumptions (Assumptions B and C) are satisfied when $g_1, \ldots, g_m$ are affine or when the MFCQ holds at every feasible point. Some preliminary numerical experience with the methods is also reported.

In the linearly constrained case, our methods share some common features with the pattern search methods of [10] in the way that the search directions are constructed. In particular, the second method (Algorithm 2) uses a fixed ϵ and sets of search directions satisfying an assumption reminiscent of the strong hypothesis on exploratory moves of [10]. In contrast to the methods of [10], convergence of our methods are based on sufficient descent rather than simple descent on a lattice. Although our methods use ϵ-active constraints to construct the search directions, they are quite different from traditional ϵ-active set methods which are SQP based and require first-order derivatives of the objective function f. In particular, the methods of [13] are of this type, with the derivatives approximated by finite differences. Our methods' use of arc search is also reminiscent of gradient projection methods (see, e.g., [2, Chapter 2] and references therein). However, the directions used are quite different from the gradient direction $-\nabla f(x)$ used in gradient projection methods and, accordingly, the criteria for stepsize acceptance are different. In addition, gradient projection methods are typically studied in the case of a convex feasible set.

Throughout, we denote by $\mathcal{F}$ the *feasible set*, namely,

$$\mathcal{F} = \{x \in \Re^n : g_i(x) \leq 0,\ i = 1, \ldots, m\}.$$

For every feasible point $x \in \mathcal{F}$, we define the *set of indices of active constraints*:

$$I(x) = \{i \in \{1, \ldots, m\} : g_i(x) = 0\},$$

and the *cone of first-order feasible variations* [2, page 332]:

$$T(x) = T_{I(x)}(x),$$

where, for each $I \subseteq \{1, \ldots, m\}$, we define the convex polyhedral cone:

$$T_I(x) = \{d \in \Re^n : \nabla g_i(x)^T d \leq 0\ \forall i \in I\}.$$

The cone $T(x)$ contains the tangent cone of $\mathcal{F}$ at x and, in the case where $g_1, \ldots, g_m$ are affine or certain regularity conditions hold (see, e.g., [2, pages 336–341]), the two cones are equal. For any scalar $\epsilon > 0$, we define the *set of indices of ϵ-active constraints*:

$$I(x; \epsilon) = \{i \in \{1, \ldots, m\} : g_i(x) \geq -\epsilon\},$$

and the *cone of first-order ϵ-feasible variations*:

$$T(x; \epsilon) = T_{I(x;\epsilon)}(x).$$

By definition, we have $I(x) \subseteq I(x; \epsilon)$ and hence $T(x; \epsilon) \subseteq T(x)$.

We define a *stationary point* of problem (1) to be any point $\bar{x} \in \mathcal{F}$ that satisfies the following first-order optimality condition:

$$\nabla f(\bar{x})^T d \geq 0 \qquad \forall d \in T(\bar{x}). \tag{2}$$

For our convergence analysis, we will consider the following standard assumption:

Assumption A Given a point $x_0 \in \mathcal{F}$, the level set

$$\mathcal{L} = \{x \in \mathcal{F} : f(x) \leq f(x_0)\}$$

is *compact*.

Finally, we denote the Euclidean ball $\mathcal{B}_\delta(\bar{x}) = \{x \in \Re^n :\ \|x - \bar{x}\| \leq \delta\}$ and, for any finite set $D = \{d^1, \ldots, d^r\} \subset \Re^n$, we denote $cone\{D\} = \{d^1\beta^1 + \cdots + d^r\beta^r : \beta_1 \geq 0, \ldots, \beta_r \geq 0\}$. We write $\{x_k\}$ as a shorthand for $\{x_k\}_{k=0,1,\ldots}$.

2. Estimating Active Constraints

The following proposition describes some properties of $I(x;\epsilon)$ as an estimate of $I(\bar{x})$ when x is near $\bar{x}$.

Proposition 2.1. *Let $\{x_k\}$ be a sequence of points in $\Re^n$ converging to some $\bar{x} \in \mathcal{F}$. Then*

(i) there exists a scalar $\bar{\epsilon} > 0$ (depending on $\bar{x}$ only) such that for every $\epsilon \in (0, \bar{\epsilon}]$ there exists an index k_ϵ for which

$$I(x_k;\epsilon) = I(\bar{x}) \qquad \forall k \geq k_\epsilon;$$

(ii) when $\epsilon_k \to 0$, we have

$$I(x_k;\epsilon_k) \subseteq I(\bar{x}) \qquad \forall k \textit{ sufficiently large.}$$

Proof. The proof below only requires the constraint functions $g_1, \ldots, g_m$ to be continuous. *(i)* Let

$$\begin{aligned} N(\bar{x}) &= \{i \in \{1,\ldots,m\} : g_i(\bar{x}) < 0\}, \\ N(x;\epsilon) &= \{i \in \{1,\ldots,m\} : g_i(x) < -\epsilon\}, \end{aligned}$$

and let

$$\bar{\epsilon} = \frac{1}{3} \min_{i \in N(\bar{x})} -g_i(\bar{x}).$$

By the definition of $\bar{\epsilon}$, we have for each $\epsilon \in (0, \bar{\epsilon}]$ that

$$-g_i(\bar{x}) \geq 3\epsilon > 2\epsilon \qquad \forall i \in N(\bar{x}). \tag{3}$$

For any $\epsilon \in (0, \bar{\epsilon}]$, the continuity property of g_i, $i = 1, \ldots, m$, implies there exists a scalar $\delta_\epsilon > 0$ such that

$$\max_{i=1,\ldots,m} |g_i(x) - g_i(\bar{x})| \leq \epsilon \qquad \forall x \in \mathcal{B}_{\delta_\epsilon}(\bar{x}). \tag{4}$$

Since $x_k \to \bar{x}$, there exists an index k_ϵ such that for all $k \geq k_\epsilon$ we have $x_k \in \mathcal{B}_{\delta_\epsilon}(\bar{x})$. Fix any $k \geq k_\epsilon$. We then have from (3) and (4) that, for each $i \in N(\bar{x})$,

$$g_i(x_k) = g_i(\bar{x}) + g_i(x_k) - g_i(\bar{x}) < -\epsilon,$$

which implies $i \in N(x_k;\epsilon)$. We also have from (4) that, for each $i \in I(\bar{x})$,

$$g_i(x_k) = g_i(x_k) - g_i(\bar{x}) \geq -\epsilon,$$

which implies $i \in I(x_k;\epsilon)$. This proves assertion (i).

(ii) Let $\delta_{\bar{\epsilon}}$ be the scalar associated with $\bar{\epsilon}$ as defined in the proof of assertion (i). For any $\epsilon \in [0, \bar{\epsilon}]$, we have $I(x_k;\epsilon) \subseteq I(x_k;\bar{\epsilon})$. Thus, since $\epsilon_k \to 0$, there exists an index $\bar{k}$ such that

$$I(x_k;\epsilon_k) \subseteq I(x_k;\bar{\epsilon}) \qquad \forall k \geq \bar{k}. \tag{5}$$

On the other hand, by assertion (i), there exists an index $k_{\bar{\epsilon}}$ such that

$$I(x_k;\bar{\epsilon}) = I(\bar{x}) \qquad \forall k \geq k_{\bar{\epsilon}}. \tag{6}$$

Combining (5) and (6) yields

$$I(x_k;\epsilon_k) \subseteq I(x_k;\bar{\epsilon}) = I(\bar{x})$$

for all k sufficiently large, which proves assertion (ii). ∎

3. Search Directions

In this section, we discuss the choice of the search directions. Since $T(x)$ is polyhedral, we can characterize a stationary point of problem (1) by the following simple proposition.

Proposition 3.1. *A point $\bar{x} \in \mathcal{F}$ is a stationary point of problem* (1) *if and only if*

$$\nabla f(\bar{x})^T d^j \geq 0, \qquad j = 1, \ldots, r, \tag{7}$$

where the directions d^j, $j = 1, \ldots, r$, are such that:

$$cone\{d^1, \ldots, d^r\} = T(\bar{x}).$$

Propositions 2.1 and 3.1 motivate a method whereby, at a feasible point x near $\bar{x}$, we estimate $I(\bar{x})$ by $I(x;\epsilon)$ and estimate $T(\bar{x})$ by $T(x;\epsilon)$ for some suitably chosen $\epsilon > 0$. Then we choose a set of "reference" search directions D at x satisfying

$$cone\{D \cap T(x;\epsilon)\} = T(x;\epsilon),$$

based on which the search arcs, along which f is sampled, are constructed. As with active-set methods, setting $\epsilon > 0$ is crucial, since $\epsilon = 0$ would not work. We specify two choices of D below.

Assumption 1 Given x_k and $\epsilon_k > 0$, the set of search directions

$$D_k = \{d_k^j, \quad j = 1, \ldots, r_k\}, \qquad \text{with} \qquad \|d_k^j\| = 1,$$

satisfies: r_k is uniformly bounded and

$$cone\{D_k\} = T(x_k;\epsilon_k).$$

Assumption 2 Given x_k, the set of search directions

$$D_k = \{d_k^j, \quad j = 1, \ldots, r_k\}, \qquad \text{with} \qquad \|d_k^j\| = 1,$$

satisfies (for some constant $\epsilon^* > 0$): r_k is uniformly bounded and

$$cone\{D_k \cap T(x_k;\epsilon)\} = T(x_k;\epsilon) \qquad \forall \epsilon \in [0, \epsilon^*].$$

There are many choices of D_k that satisfy Assumption 1. For example, we can take D_k to comprise the extreme directions (normalized to have unit length) of $T(x_k;\epsilon_k)$. To satisfy Assumption 2, we can take D_k to comprise the union of the extreme directions (normalized to have unit length) of $T(x_k;\epsilon)$ for $\epsilon \in [0, \epsilon^*]$, with ϵ^* arbitrarily chosen. Since the number of distinct $I(x_k;\epsilon)$, for $\epsilon \in [0, \epsilon^*]$, is at most $m + 1$, the cardinality of D_k is uniformly bounded. Although more effort may be needed to construct D_k satisfying Assumption 2, compared to Assumption 1, the method based on such D_k has a stronger convergence property (see Proposition 6.2).

In the case of linearly independent ϵ-active constraint gradients, the following proposition established in [10, Proposition 8.2] (also see [13]) suggests a simpler way to compute the sets of search directions satisfying Assumption 1 or Assumption 2. [Although [10] considers the linearly constrained case, this proposition also applies to the general case.]

Proposition 3.2. *Suppose that, for a given $\epsilon_k > 0$ and $x_k \in \mathcal{F}$, the matrix $A_k = [\nabla g_i(x_k)]_{i \in I(x_k;\epsilon_k)}$ has full column rank. Let*

$u_k^1, \dots, u_k^{s_k}$ *be nonzero vectors such that* $\text{cone}\{u_k^j\}_{1 \le j \le s_k} = (\text{null space of } A_k^T)$,

$v_k^1, \dots, v_k^{t_k}$ *be the columns of the matrix* $A_k(A_k^T A_k)^{-1}$.

Then

(i) the set $D_k = \left\{u_k^j/\|u_k^j\|\right\}_{1 \le j \le s_k} \bigcup \left\{-v_k^j/\|v_k^j\|\right\}_{1 \le j \le t_k}$ *satisfies Assumption 1;*

(ii) if $\epsilon_k = \epsilon^*$, *the set* $D_k = \left\{u_k^j/\|u_k^j\|\right\}_{1 \le j \le s_k} \bigcup \left\{v_k^j/\|v_k^j\|, -v_k^j/\|v_k^j\|\right\}_{1 \le j \le t_k}$ *satisfies Assumption 2.*

4. Regularity Assumptions for Convergence Analysis

For our convergence analysis, we need the following continuity assumption on $T_I(x)$:

Assumption B If $\{x_k\}$ and $\{y_k\}$ are sequences of points in $\mathcal{F}$ converging to some $\bar{x}$ and $I_k \subseteq I(\bar{x})$ for all k, then

$$\{dist\,(T_{I_k}(x_k), T_{I_k}(y_k))\} \to 0,$$

where we define $dist(T_1, T_2) = \max_{d_1 \in T_1 : \|d_1\|=1} \{\min_{d_2 \in T_2} \|d_1 - d_2\|\}$.

Assumption B holds when $g_1, \dots, g_m$ are affine. This is because $\nabla g_i(x) = \nabla g_i(y)$ for all x, y and i, so that $T_I(x) = T_I(y)$ for all x, y and I. Assumption B also holds when the well-known *Mangasarian-Fromovitz constraint qualification* (MFCQ) (see, e.g., [2, page 323]) holds at every $\bar{x} \in \mathcal{F}$, i.e., there exists $d \in \Re^n$ satisfying

$$\nabla g_i(\bar{x})^T d < 0 \qquad \forall i \in I(\bar{x}).$$

This is because, for any $I \subset I(\bar{x})$, there exists $d \in \Re^n$ satisfying

$$\nabla g_i(\bar{x})^T d < 0 \qquad \forall i \in I.$$

Then, by the continuity of ∇g_i, $i \in I$, and a perturbation results for linear systems [12, Theorem 3.1], there exist $\delta > 0$ and $\rho > 0$ such that for all $y \in \mathcal{F} \cap \mathcal{B}_\delta(\bar{x})$, we have

$$T_I(y) \ne \{0\} \qquad \text{and} \qquad dist(d, T_I(y)) \le \rho \sum_{i \in I} |[\nabla g_i(y)^T d]_+| \quad \forall d \in \Re^n,$$

where we denote $[\cdot]_+ = \max\{0, \cdot\}$. Then, for any $x, y \in \mathcal{F} \cap \mathcal{B}_\delta(\bar{x})$, we have $T_I(x) \ne \{0\}$ and

$$\begin{aligned}
dist(T_I(x), T_I(y)) &= \max_{d \in T_I(x) : \|d\|=1} dist(d, T_I(y)) \\
&\le \rho \max_{d \in T_I(x) : \|d\|=1} \sum_{i \in I} |[\nabla g_i(y)^T d]_+| \\
&= \rho \max_{d \in T_I(x) : \|d\|=1} \sum_{i \in I} |[\nabla g_i(y)^T d]_+ - [\nabla g_i(x)^T d]_+| \\
&\le \rho \max_{d \in T_I(x) : \|d\|=1} \sum_{i \in I} |\nabla g_i(y)^T d - \nabla g_i(x)^T d| \\
&\le \rho \sum_{i \in I} \|\nabla g_i(y) - \nabla g_i(x)\|,
\end{aligned}$$

where the second equality uses $d \in T_I(x)$ so that $[\nabla g_i(x)^T d]_+ = 0$ for all $i \in I$; the second inequality uses the nonexpansive property of $[\cdot]_+$ with respect to $|\cdot|$; the last inequality uses the Cauchy-Schwarz inequality. Thus $dist(T_I(x), T_I(y)) \to 0$ as $x, y \to \bar{x}$. In particular, Assumption B holds for the example of $m = 1$ and $g_1(x) = \delta - \|x - \hat{x}\|^2$, where $\hat{x} \in \Re^n$ and $\delta > 0$.

The following proposition gives a convergence result, under Assumption B, for a sequence of $x_k \in \mathcal{F}$ and D_k satisfying either Assumption 1 with $\epsilon_k \to 0$ or Assumption 2.

Proposition 4.1. *Let $\{x_k\}$ be a sequence of feasible points and let $\{D_k\}$ with $D_k = \{d_k^1, ..., d_k^{r_k}\}$ be a sequence of sets of directions that either satisfies Assumption 1 with $\epsilon_k \to 0$ or satisfies Assumption 2.*
Suppose $\{x_k\}_K$ converges to a point $\bar{x}$, for some infinite $K \subseteq \{0, 1, \ldots\}$, and

$$\lim_{k\to\infty, k\in K} \max_{j\in J_k} \Big\{ \min\{0, \nabla f(x_k)^T d_k^j\} \Big\} = 0, \tag{8}$$

where $J_k = \{1, \ldots, r_k\}$ if $\{D_k\}$ satisfies Assumption 1 and otherwise $J_k = \{j \in \{1, \ldots, r_k\} : d_k^j \in T(x_k; \epsilon)\}$ with $\epsilon \in (0, \min\{\bar{\epsilon}, \epsilon^*\}]$ and $\bar{\epsilon}$ defined as in Proposition 2.1(i). Then, under Assumption B, $\bar{x}$ is a stationary point of problem (1).

Proof. We argue by contradiction. Suppose that $\bar{x}$ is not a stationary point. Then, by (2), there exists a direction $\bar{d} \in T(\bar{x})$ such that

$$\nabla f(\bar{x})^T \bar{d} < 0. \tag{9}$$

Since r_k is uniformly bounded by Assumption 1 or 2, there exist an infinite $K_1 \subseteq K$ and $J \subseteq \{1, 2, \ldots\}$ and $\bar{d}^j \in \Re^n$, $j \in J$, such that

$$J_k = J \quad \forall k \in K_1 \qquad \text{and} \qquad \lim_{k\to\infty, k\in K_1} d_k^j = \bar{d}^j \quad \forall j \in J. \tag{10}$$

In the case that Assumption 1 holds and $\epsilon_k \to 0$, assertion (ii) of Proposition 2.1 implies that, for k sufficiently large,

$$T_{I(\bar{x})}(x_k) \subseteq T(x_k; \epsilon_k) = cone\{D_k\} = cone\{d_k^j\}_{j\in J_k}. \tag{11}$$

In the case that Assumption 2 holds, assertion (i) of Proposition 2.1 and $\epsilon \le \bar{\epsilon}$ implies that, for $k \in K$ sufficiently large,

$$T_{I(\bar{x})}(x_k) = T(x_k; \epsilon) = cone\{D_k \cap T(x_k; \epsilon)\} = cone\{d_k^j\}_{j\in J_k}. \tag{12}$$

By Assumption B, $\{dist\Big(T(\bar{x}), T_{I(\bar{x})}(x_k)\Big)\}_K \to 0$, therefore (10), (11), (12) imply that

$$\bar{d} = \sum_{j\in J} \beta^j \bar{d}^j,$$

for some $\beta^j \ge 0$, $j \in J$. This together with (8), (10) and the continuity assumption on ∇f imply that

$$\nabla f(\bar{x})^T \bar{d} = \sum_{j\in J} \beta^j \nabla f(\bar{x})^T \bar{d}^j = \lim_{k\to\infty, k\in K_1} \sum_{j\in J_k} \beta^j \nabla f(x_k)^T d_k^j \ge 0,$$

which contradicts (9). ∎

The preceding proposition shows that a set of directions D_k satisfying Assumption 1 or 2 can play a key role in methods for solving (1). In particular, it says that, by performing finer and finer samplings of f along arcs tangent to d_k^j, for $j \in J_k$, it is possible to either detect that x_k is near a stationary point of (1) or find a feasible point near x_k with a sufficiently lower f-value.

For our convergence analysis, we need the following *metric regularity* of the feasible set $\mathcal{F}$:

Assumption C For every $\bar{x} \in \mathcal{F}$ there exist scalars $\delta > 0$ and $\eta > 0$ such that

$$\min_{z \in \mathcal{F}} \|z - x\| \leq \eta \sum_{i=1}^{m} [g_i(x)]_+ \qquad \forall x \in \mathcal{B}_\delta(\bar{x}).$$

Let $P_{\mathcal{F}}[x] = \arg\min_{z \in \mathcal{F}} \|z - x\|$. [We use the 2-norm for simplicity, but any p-norm with $1 \leq p \leq \infty$ may be used. Notice that, for $p = 1$ and $p = \infty$, the projection is not unique even when $\mathcal{F}$ is convex, but may be easier to compute.]

Assumption C has been much studied and is known to hold when either (i) $g_1, \ldots, g_m$ are affine or (ii) $g_1, \ldots, g_m$ are convex and there exists a weak Slater point, i.e., a $\hat{x} \in \mathcal{F}$ such that $g_i(\hat{x}) < 0$ for all i with g_i non-affine or (iii) $g_1, \ldots, g_m$ are convex and an asymptotic constraint qualification of Auslender and Crouzeix holds (see, e.g., [1, Corollary 8 and Theorem 9] and [3, Proposition 3] for detailed discussions).

Assumption C also holds when the MFCQ holds at every $\bar{x} \in \mathcal{F}$. This follows from a result of Robinson [15, Corollary 1] (take $F(p, x) = [g_i(x)]_{i=1}^m - p$, $D = \{\bar{x}\}$, $K = [0, \infty)^m$); also see [16, Example 9.44]. In particular, Assumption C holds for the earlier nonconvex example of $m = 1$ and $g_1(x) = \delta - \|x - \hat{x}\|^2$.

The following proposition gives a preliminary convergence result, under Assumptions B and C, for a sequence of feasible points x_k and y_k^j, $j \in J_k$, and a sequence of sets of directions D_k satisfying either Assumption 1 with $\epsilon_k \to 0$ or Assumption 2. This proposition will be used in the next two sections to obtain convergence results for our methods.

Proposition 4.2. *Let $\{x_k\}$ be a sequence of feasible points, $\{D_k\}$ with $D_k = \{d_k^1, \ldots, d_k^{r_k}\}$ be a sequence of sets of directions, and suppose that the following conditions hold:*

(a) the sets D_k either satisfy Assumption 1 with $\epsilon_k \to 0$ or satisfy Assumption 2;

(b) $\{x_k\}_K$ converges to a point $\bar{x}$ for some infinite $K \subseteq \{0, 1, \ldots\}$ and, for each $k \in K$ and $j \in J_k$, where J_k is defined as in Proposition 4.1, there exist $y_k^j \in \mathcal{F}$ and scalar $\xi_k^j > 0$ satisfying

$$g_i(y_k^j) + \xi_k^j \nabla g_i(y_k^j)^T \hat{d}_k^j \leq 0 \qquad \forall i,\ \forall j \in J_k,\ \forall k \in K, \tag{13}$$

$$f(w_k^j) \geq f(y_k^j) - o(\xi_k^j) \qquad \forall j \in J_k,\ \forall k \in K, \tag{14}$$

$$\lim_{k \to \infty, k \in K} \max_{j \in J_k} \{\xi_k^j\} = 0, \tag{15}$$

$$\lim_{k \to \infty} \max_{j \in J_k} \|x_k - y_k^j\| = 0, \tag{16}$$

where $\lim_{\xi \to 0} o(\xi)/\xi = 0$, $w_k^j \in P_{\mathcal{F}}[y_k^j + \xi_k^j \hat{d}_k^j]$, $\hat{d}_k^j = P_{T_{I_k}(y_k^j)}[d_k^j]$, and $I_k = I(x_k; \epsilon_k)$ if $\{D_k\}$ satisfies Assumption 1 and otherwise $I_k = I(x_k; \epsilon)$ with $\epsilon \in (0, \min\{\bar{\epsilon}, \epsilon^*\}]$ and $\bar{\epsilon}$ defined as in Proposition 2.1(i).

Then, under Assumptions B and C, $\bar{x}$ is a stationary point of problem (1).

Proof. Since r_k is uniformly bounded, there exist an infinite subset $K_1 \subseteq K$ and $J \subseteq \{1, 2, \ldots\}$ and $\bar{d}^j \in \Re^n$, $j \in J$, such that

$$J_k = J \quad \forall k \in K_1 \qquad \text{and} \qquad \lim_{k\to\infty, k\in K_1} d_k^j = \bar{d}^j \quad \forall j \in J. \tag{17}$$

For each $i \in \{1, \ldots, m\}$ and $j \in J$ and $k \in K_1$, we have from (13) and the continuous differentiability of g_i that

$$g_i(y_k^j + \xi_k^j \hat{d}_k^j) = g_i(y_k^j) + \xi_k^j \nabla g_i(y_k^j)^T \hat{d}_k^j + o_i(y_k^j; \xi_k^j \hat{d}_k^j) \le |o_i(y_k^j; \xi_k^j \hat{d}_k^j)|,$$

where $\sup_{\|d\|\le 2} o_i(y; \xi d)/\xi \to 0$ as $y \to \bar{x}$, $\xi \to 0$. Then, $\{x_k\}_K \to \bar{x}$, (15), (16) and (17) imply that $\{y_k^j + \xi_k^j \hat{d}_k^j\}_{K_1} \to \bar{x}$ for all $j \in J$, so Assumption C yields

$$\|w_k^j - (y_k^j + \xi_k^j \hat{d}_k^j)\| \le \eta \sum_{i=1}^m |o_i(y_k^j; \xi_k^j \hat{d}_k^j)| \tag{18}$$

for all $k \in K_1$ sufficiently large, where $\eta > 0$. For each such k, we have from (14) and (17) that

$$f(w_k^j) - f(y_k^j) \ge -o(\xi_k^j) \qquad \forall j \in J. \tag{19}$$

Also, by the mean-value theorem, we can write for each $j \in J$,

$$\begin{aligned} f(w_k^j) - f(y_k^j) &= \nabla f(u_k^j)^T (w_k^j - y_k^j) \\ &= \xi_k^j \nabla f(u_k^j)^T \hat{d}_k^j + \nabla f(u_k^j)^T (w_k^j - (y_k^j + \xi_k^j \hat{d}_k^j)) \\ &\le \xi_k^j \nabla f(u_k^j)^T \hat{d}_k^j + \|\nabla f(u_k^j)\| \eta \sum_{i=1}^m |o_i(y_k^j; \xi_k^j \hat{d}_k^j)|, \end{aligned} \tag{20}$$

where $u_k^j = (1 - \lambda_k^j) y_k^j + \lambda_k^j w_k^j$ for some $\lambda_k^j \in (0, 1)$, and the inequality follows from (18). Combining (19) and (20), we obtain for all $k \in K_1$ sufficiently large that

$$\nabla f(u_k^j)^T \hat{d}_k^j \ge -\left(o(\xi_k^j) + \|\nabla f(u_k^j)\| \eta \sum_{i=1}^m |o_i(y_k^j; \xi_k^j \hat{d}_k^j)| \right) / \xi_k^j \qquad \forall j \in J. \tag{21}$$

Since $0 \in T_{I_k}(y_k^j)$ so that $\|\hat{d}_k^j - d_k^j\| \le \|0 - d_k^j\| = 1$, we see that $\|\hat{d}_k^j\| \le 2$ for all $k \in K_1$ and $j \in J$. Then, it follows from (15), (16) and (18) that $u_k^j \to \bar{x}$, as $k \to \infty$ and $k \in K_1$, for all $j \in J$. Also, Proposition 2.1 implies that $I_k \subseteq I(\bar{x})$ for all $k \in K_1$ sufficiently large, so Assumption B and (using definition of I_k and Assumption 1 or 2) $d_k^j \in T_{I_k}(x_k)$ yield

$$\max_{j \in J_k} \|\hat{d}_k^j - d_k^j\| \to 0 \qquad \text{as } k \to \infty,\ k \in K_1.$$

Hence, using the continuity of ∇f and recalling $\lim_{k\to\infty, k\in K_1} u_k^j = \bar{x}$ for all $j \in J$, we obtain from (15), (17) and (21) that

$$0 \le \lim_{k\to\infty, k\in K_1} \nabla f(u_k^j)^T \hat{d}_k^j = \lim_{k\to\infty, k\in K_1} \nabla f(x_k)^T d_k^j = \nabla f(\bar{x})^T \bar{d}^j \qquad \forall j \in J.$$

Thus, (8) holds with K_1 in place of K and, by Proposition 4.1, $\bar{x}$ is a stationary point of (1). ∎

5. Feasible Descent Method: I

In this section, we describe our first method for solving problem (1). It uses D_k satisfying Assumption 1 with $\epsilon_k \to 0$ and, under Assumptions A, B, C, the generated iterates have at least one cluster point that is a stationary point of (1).

Algorithm 1

Parameters. $x_0 \in \mathcal{F}$, $\tilde{\alpha}_0 > 0$, $\gamma > 0$, $\theta_1 \in (0,1)$, $\theta_2 \in (0,1)$, $\epsilon_0 > 0$.

Step 0. Set $k = 0$.

Step 1. *(Computation of reference search directions)*
Choose a set of directions $D_k = \{d_k^1, \dots, d_k^{r_k}\}$ satisfying Assumption 1.

Step 2. *(Minimization on the cone$\{D_k\}$)*

Step 2.1. *(Initialization)*
Set $j = 1$, $y_k^j = x_k$, $\tilde{\alpha}_k^j = \tilde{\alpha}_k$.

Step 2.2. *(Computation of the initial stepsize)*
Set I_k and $\hat{d}_k^j$ as in Proposition 4.2.
Compute the maximum stepsize $\bar{\alpha}_k^j$ such that
$g_i(y_k^j) + \bar{\alpha}_k^j \nabla g_i(y_k^j)^T \hat{d}_k^j \leq 0$ for all i.
Set $\hat{\alpha}_k^j = \min\{\bar{\alpha}_k^j, \tilde{\alpha}_k^j\}$ and $\hat{y}_k^j \in P_{\mathcal{F}}[y_k^j + \hat{\alpha}_k^j \hat{d}_k^j]$.

Step 2.3. *(Test on the search direction)*
If $\hat{\alpha}_k^j > 0$ and $f(\hat{y}_k^j) \leq f(y_k^j) - \gamma(\hat{\alpha}_k^j)^2$, compute α_k^j and y_k^{j+1} by the *Expansion Step*$(\bar{\alpha}_k^j, \hat{\alpha}_k^j, \hat{y}_k^j, y_k^j, \hat{d}_k^j; \alpha_k^j, y_k^{j+1})$ and set $\tilde{\alpha}_k^{j+1} = \alpha_k^j$;
otherwise set $\alpha_k^j = 0$, $y_k^{j+1} = y_k^j$, $\tilde{\alpha}_k^{j+1} = \theta_1 \tilde{\alpha}_k^j$.

Step 2.4. *(Test on the active constraints)*
If $\alpha_k^j = \bar{\alpha}_k^j$, set $\epsilon_{k+1} = \epsilon_k$, and go to Step 3.

Step 2.5 *(Test on the minimization on the cone*$\{D_k\}$)
If $j < r_k$, set $j = j + 1$ and go to Step 2.2;
otherwise set $\epsilon_{k+1} = \theta_2 \epsilon_k$ and go to Step 3.

Step 3. *(Main iteration)*
Find $x_{k+1} \in \mathcal{F}$ such that $f(x_{k+1}) \leq f(y_k^{j+1})$.
Set $\tilde{\alpha}_{k+1} = \tilde{\alpha}_k^{j+1}$, $\tilde{r}_k = j$, $k = k + 1$, and go to Step 1.

Note 1. If $g_1, ..., g_m$ are affine, then $T_{I_k}(y_k^j) = T(x_k; \epsilon_k)$ and hence $\hat{d}_k^j = d_k^j$ and $y_k^j + \bar{\alpha}_k^j \hat{d}_k^j \in \mathcal{F}$ for all j and k. Thus, in this case, the projection operations in Algorithm 1 are redundant. If $\mathcal{F}$ is convex, then the projection mapping $P_{\mathcal{F}}[\cdot]$ (using 2-norm) is single-valued. If $\mathcal{F}$ is nonconvex, then projecting onto $\mathcal{F}$ requires global optimization, which may or may not be practical, depending on the cost of evaluating f.

Note 2. In Step 2.3 and in the Expansion Step, the quadratic function $\gamma(\cdot)^2$ can be replaced by any "forcing function" $\phi(\cdot)$ that is continuous, positive-valued, and satisfies $\lim_{\alpha \to 0} \phi(\alpha)/\alpha = 0$. In Step 3, we can set $x_{k+1} = y_k^{j+1}$ or, if a point $\hat{x}_k \in \mathcal{F}$ with lower f-value than y_k^{j+1} is found, set $x_{k+1} = \hat{x}_k$. Such a $\hat{x}_k$ can possibly be constructed by an interpolation technique based on previously sampled f-values.

Expansion Step $(\bar\alpha_k^j, \hat\alpha_k^j, \hat y_k^j, y_k^j, \hat d_k^j; \alpha_k^j, y_k^{j+1})$.

Parameters. $\gamma > 0$, $\omega \in (0,1)$.

Step 1. Set $\alpha = \hat\alpha_k^j$, $y = \hat y_k^j$.

Step 2. Set $\tilde\alpha = \min\{\bar\alpha_k^j, \alpha/\omega\}$ and $\tilde y \in P_{\mathcal{F}}[y_k^j + \tilde\alpha \hat d_k^j]$.
If $\alpha = \bar\alpha_k^j$ or $f(\tilde y) > f(y_k^j) - \gamma\tilde\alpha^2$, set $\alpha_k^j = \alpha$, $y_k^{j+1} = y$ and exit.

Step 3. Set $\alpha = \tilde\alpha$, $y = \tilde y$ and go to Step 2.

The next two propositions establish the convergence properties of Algorithm 1 under Assumptions A, B, C.

Proposition 5.1. *Let $\{(x_k, \epsilon_k)\}$ and $\{(y_k^j, \tilde\alpha_k^j, \bar\alpha_k^j, \hat\alpha_k^j, \alpha_k^j, d_k^j, \hat d_k^j)_{1\le j\le \tilde r_k}\}$ be sequences produced by Algorithm 1. Then, under Assumptions A, B, C, we have:*

(i) $\{x_k\}$ is well defined;

(ii) $\{x_k\}$ is bounded and every cluster point belongs to $\mathcal{F}$;

(iii) we have

$$\lim_{k\to\infty} \max_{1\le j\le \tilde r_k} \left\{\alpha_k^j\right\} = 0, \tag{22}$$

$$\lim_{k\to\infty} \max_{1\le j\le \tilde r_k} \left\{\tilde\alpha_k^j\right\} = 0, \tag{23}$$

$$\lim_{k\to\infty} \max_{1\le j\le \tilde r_k} \left\|x_k - y_k^j\right\| = 0; \tag{24}$$

(iv) $\bar\alpha_k^j \ge \left(\epsilon_k - \max_{i=1,\dots,m} |g_i(y_k^j) - g_i(x_k)|\right)/\gamma_k$ for all $1 \le j \le \tilde r_k$ and all k, where $\gamma_k = \max_{1\le j\le \tilde r_k} \max_{i=1,\dots,m} |\nabla g_i(y_k^j)^T \hat d_k^j|$.

Proof. To prove assertion (i), it suffices to show that the Expansion Step, when performed along a direction $\hat d_k^j$ from y_k^j, for $j \in \{1,\dots,\tilde r_k\}$, terminates in a finite number of steps. If this were not true, we would have for some k and $j \in \{1,\dots,\tilde r_k\}$ that $\hat\alpha_k^j > 0$ and

$$f(w_k^j[p]) \le f(y_k^j) - \gamma\left(\hat\alpha_k^j/\omega^p\right)^2 \qquad \forall p = 0,1,\dots,$$

where $w_k^j[p] \in P_{\mathcal{F}}[y_k^j + (\hat\alpha_k^j/\omega^p)\hat d_k^j]$, which violates the assumption that f is bounded below on $\mathcal{F}$ (via Assumption A).

To prove assertion (ii), we note that the instructions of Algorithm 1 imply that $x_k \in \mathcal{F}$ and $f(x_{k+1}) \le f(x_k)$ for all k. Hence $x_k \in \mathcal{L}$ for all k and, by Assumption A, $\{x_k\}$ is bounded. Since $x_k \in \mathcal{F}$ for all k and $\mathcal{F}$ is a closed set, the assertion follows.

To prove assertion (iii), we note from the construction of α_k^j and y_k^{j+1} in Step 2.3 that

$$f(y_k^{j+1}) \le f(y_k^j) - \gamma(\alpha_k^j)^2,$$

and from the construction of $\tilde{\alpha}_k^{j+1}$ that

$$\text{either} \quad \tilde{\alpha}_k^{j+1} = \alpha_k^j \quad \text{or} \quad \tilde{\alpha}_k^{j+1} = \theta_1 \tilde{\alpha}_k^j,$$

for each k and each $j \in \{1, \ldots, \tilde{r}_k\}$. Summing the first relation for $j = 1, ..., \tilde{r}_k$ and using the construction of x_{k+1} in Step 3 yields

$$f(x_{k+1}) \leq f(x_k) - \gamma \sum_{j=1}^{\tilde{r}_k} (\alpha_k^j)^2.$$

Since $x_k \in \mathcal{F}$ and $\mathcal{L}$ is compact by Assumption A, this implies $\{f(x_k)\}$ converges and $\{\sum_{j=1}^{\tilde{r}_k} (\alpha_k^j)^2\} \to 0$, thus proving (22). The second relation implies

$$\tilde{\alpha}_k^j = (\theta_1)^{p_k^j} \, \alpha_{m_k^j}^{l_k^j}, \tag{25}$$

for some $0 \leq m_k^j \leq k$ and some $1 \leq l_k^j \leq \tilde{r}_{m_k^j}$, with $p_k^j = j - l_k^j \geq 0$ if $m_k^j = k$ and else $p_k^j = j + \tilde{r}_{k-1} + \cdots + \tilde{r}_{m_k^j} - l_k^j$. Then, letting $K^j = \{k \in \{0, 1, ...\} : j \leq \tilde{r}_k\}$ for an arbitrary j, we see that $\max\{p_k^j, m_k^j\} \to \infty$ as $k \to \infty$, $k \in K^j$, which, together with (22) and (25), yields $\tilde{\alpha}_k^j \to 0$. This then proves (23).

To show (24), we argue by contradiction. Suppose there exist scalar $\delta > 0$ and an infinite $K \subseteq \{0, 1, \ldots\}$ such that

$$\max_{1 \leq j \leq \tilde{r}_k} \|x_k - y_k^j\| \geq \delta \qquad \forall k \in K. \tag{26}$$

Since $\{x_k\}$ and $\{\tilde{r}_k\}$ are bounded (see assertion (ii) and Assumption 1), by passing to a subsequence if necessary, we can assume that

$$\tilde{r}_k = \tilde{r} \quad \forall k \in K \qquad \text{and} \qquad \{x_k\}_K \to \bar{x},$$

for some $\tilde{r}$ and $\bar{x} \in \mathcal{F}$. We claim that $\{y_k^j\}_K \to \bar{x}$ for $1 \leq j \leq \tilde{r}$, which would contradict (26). This claim clearly holds for $j = 1$. Suppose this claim holds for some $1 \leq j < \tilde{r}$. Arguing as in the proof of Proposition 4.2, we have $\{\|\hat{d}_k^j - d_k^j\|\}_K \to 0$. This together with (22) and $\|d_k^j\| = 1$ for all $k \in K$ implies that $\{y_k^j + \alpha_k^j \hat{d}_k^j\}_K \to \bar{x}$. Then, using the fact $y_k^{j+1} \in P_{\mathcal{F}}[y_k^j + \alpha_k^j \hat{d}_k^j]$ and Assumption C, we have that

$$\|y_k^{j+1} - y_k^j\| = \left\| y_k^{j+1} - (y_k^j + \alpha_k^j \hat{d}_k^j) + \alpha_k^j \hat{d}_k^j \right\| \leq \eta \sum_{i=1}^m [g_i(y_k^j + \alpha_k^j \hat{d}_k^j)]_+ + \alpha_k^j \|\hat{d}_k^j\|,$$

for all $k \in K$ sufficiently large, where $\eta > 0$. Since $\{g_i(y_k^j + \alpha_k^j \hat{d}_k^j)\}_K \to g_i(\bar{x}) \leq 0$ for all i, the right-hand side tends to zero as $k \to \infty$, $k \in K$. Thus, $\{\|y_k^{j+1} - y_k^j\|\}_K \to 0$ and so $\{y_k^{j+1}\}_K \to \bar{x}$. The claim then follows by induction.

To prove assertion (iv), notice that $\hat{d}_k^j \in T_{I_k}(y_k^j)$ and the definition of $\bar{\alpha}_k^j$ in Step 2.2 imply either $\bar{\alpha}_k^j = \infty$ or the existence of an index $\bar{i} \notin I_k = I(x_k; \epsilon_k)$ such that

$$g_{\bar{i}}(y_k^j) + \bar{\alpha}_k^j \nabla g_{\bar{i}}(y_k^j)^T \hat{d}_k^j = 0.$$

Then, solving for $\bar{\alpha}_k^j$ and using $0 < \nabla g_{\bar{i}}(y_k^j)^T \hat{d}_k^j \leq \gamma_k$, yields

$$\begin{aligned}
\bar{\alpha}_k^j &= -g_{\bar{i}}(y_k^j)/(\nabla g_{\bar{i}}(y_k^j)^T \hat{d}_k^j) \\
&\geq -g_{\bar{i}}(y_k^j)/\gamma_k \\
&= (-g_{\bar{i}}(x_k) + g_{\bar{i}}(x_k) - g_{\bar{i}}(y_k^j))/\gamma_k \\
&\geq (\epsilon_k + g_{\bar{i}}(x_k) - g_{\bar{i}}(y_k^j))/\gamma_k,
\end{aligned}$$

where the last inequality follows from $\bar{i} \notin I(x_k;\epsilon_k)$ and the definition of $I(x_k;\epsilon_k)$. ∎

Proposition 5.2. *Under Assumptions A, B, C, if $\{x_k\}$ is the sequence produced by Algorithm* 1, *then $\{x_k\}$ is bounded and there exists at least one cluster point which is a stationary point of problem* (1).

Proof. Let $\{\epsilon_k\}$ and $\{(y_k^j, \tilde{\alpha}_k^j, \bar{\alpha}_k^j, \hat{\alpha}_k^j, \alpha_k^j, d_k^j, \hat{d}_k^j)_{1\leq j\leq \tilde{r}_k}\}$ be sequences produced by Algorithm 1 corresponding to $\{x_k\}$. By Proposition 5.1, $\{x_k\}$ is well defined and bounded. We claim that

$$\lim_{k\to\infty} \epsilon_k = 0. \tag{27}$$

We argue this by contradiction. Suppose that (27) is false. Since the sequence $\{\epsilon_k\}$ is not increasing, then there exists an index $\bar{k}$ and a scalar $\epsilon^* > 0$ such that

$$\epsilon_k = \epsilon^* \qquad \forall k \geq \bar{k}. \tag{28}$$

This implies, by the instructions of Steps 2.5 and 2.6, that for each $k \geq \bar{k}$ there is some j_k such that

$$\alpha_k^{j_k} = \bar{\alpha}_k^{j_k}. \tag{29}$$

Moreover, assertion (iv) of Proposition 5.1 implies

$$\bar{\alpha}_k^{j_k} \geq \left(\epsilon_k - \max_{i=1,\dots,m} |g_i(y_k^{j_k}) - g_i(x_k)|\right)/\gamma_k.$$

Since $\{r_k\}$ is uniformly bounded so that, by assertion (iii) of Proposition 5.1, $\lim_{k\to\infty} \alpha_k^{j_k} = \lim_{k\to\infty} \|x_k - y_k^{j_k}\| = 0$, then (28) and (29) and boundedness of $\{x_k\}$ would yield in the limit that $0 \geq \epsilon^*/\sup_{k\geq\bar{k}} \gamma_k > 0$, an obvious contradiction.

By (27), there exists an infinite $K \subseteq \{0, 1, \dots\}$ such that $\epsilon_{k+1} < \epsilon_k$ for all $k \in K$. Then, for $k \in K$, the instructions of the algorithm imply that $\tilde{r}_k = r_k$, i.e., all directions of the set D_k are investigated at iteration k, and $\bar{\alpha}_k^j > 0$, $\hat{\alpha}_k^j > 0$ for $j = 1, \dots, r_k$. Using this together with Steps 2.2 and 2.3 of the algorithm and assertion (iii) of Proposition 5.1, we see that condition (b) of Proposition 4.2 holds for the subsequences $\{x_k\}_K$ and $\{(y_k^j)_{1\leq j\leq r_k}\}_K$, and the subsequence of scalars $\{(\xi_k^j)_{1\leq j\leq r_k}\}_K$ defined by setting $\xi_k^j = \min\{\bar{\alpha}_k^j, \alpha_k^j/\omega\}$ if an Expansion Step is performed in generating α_k^j (ω is the parameter from the Expansion Step) and otherwise setting $\xi_k^j = \hat{\alpha}_k^j$. Also, since the sets D_k satisfy Assumption 1, (27) implies that condition (a) of Proposition 4.2 holds. Then, Proposition 4.2 yields that any cluster point of $\{x_k\}_K$ is a stationary point of (1). ∎

6. Feasible Descent Method: II

In this section, we describe our second method for solving (1). It uses D_k satisfying Assumption 2 and, under Assumptions A, B, C, all cluster points of the generated iterates are stationary points of (1). This method is somewhat more complicated than Algorithm 1, however.

Algorithm 2

Parameters. $x_0 \in \mathcal{F}$, $\tilde{\alpha}_0 > 0$, $\gamma > 0$, $\theta_1 \in (0,1)$, $\epsilon^* > 0$.

Step 0. Set $k = 0$.

Step 1. *(Computation of reference search directions)*
Choose a set of directions $D_k = \{d_k^1, \ldots, d_k^{r_k}\}$ satisfying Assumption 2.
Let $I_k^1 \supset \cdots \supset I_k^{p_k}$ ($p_k \geq 1$) be the distinct elements of $I(x_k;\epsilon)$, $0 \leq \epsilon \leq \epsilon^*$.

Step 2. *(Minimization on the cone$\{D_k\}$)*

Step 2.1. *(Initialization)*
Set $j = 1$, $p = 1$, $q = 1$, $y_k^j = x_k$, $\tilde{\alpha}_k^j = \tilde{\alpha}_k$.

Step 2.2. *(Computation of the initial stepsize)*
Set $I_k^{p,q} = \{i \in I_k^p : \nabla g_i(x_k)^T d_k^q \leq 0\}$ and $\hat{d}_k^j = P_{T_{I_k^{p,q}}(y_k^j)}[d_k^q]$.
Compute the maximum stepsize $\bar{\alpha}_k^j$ such that
$g_i(y_k^j) + \bar{\alpha}_k^j \nabla g_i(y_k^j)^T \hat{d}_k^j \leq 0$ for all i.
Set $\hat{\alpha}_k^j = \min\{\bar{\alpha}_k^j, \tilde{\alpha}_k^j\}$ and $\hat{y}_k^j \in P_{\mathcal{F}}[y_k^j + \hat{\alpha}_k^j \hat{d}_k^j]$.

Step 2.3. *(Test on the search direction)*
If $\hat{\alpha}_k^j > 0$ and $f(\hat{y}_k^j) \leq f(y_k^j) - \gamma(\hat{\alpha}_k^j)^2$, compute α_k^j and y_k^{j+1} by the *Expansion Step*$(\bar{\alpha}_k^j, \hat{\alpha}_k^j, \hat{y}_k^j, y_k^j, \hat{d}_k^j; \alpha_k^j, y_k^{j+1})$ and set $\tilde{\alpha}_k^{j+1} = \alpha_k^j$;
otherwise set $\alpha_k^j = 0$, $y_k^{j+1} = y_k^j$, $\tilde{\alpha}_k^{j+1} = \theta_1 \tilde{\alpha}_k^j$.

Step 2.4 *(Test on the minimization on the cone$\{D_k\}$)*
If $\hat{d}_k^j \neq d_k^q$ and $p < p_k$, set $j = j+1$, $p = p+1$ and go to Step 2.2;
otherwise if $q < r_k$, set $j = j+1$, $p = 1$, $q = q+1$ and go to Step 2.2;
otherwise go to Step 3.

Step 3. *(Main iteration)*
Find $x_{k+1} \in \mathcal{F}$ such that $f(x_{k+1}) \leq f(y_k^{j+1})$.
Set $\tilde{\alpha}_{k+1} = \tilde{\alpha}_k^{j+1}$, $\tilde{r}_k = j$, $k = k+1$, and go to Step 1.

Note 3. Notice that $d_k^q \in T_{I_k^{p,q}}(x_k)$ for all p and q. Thus, if $g_1, ..., g_m$ are affine, then $\hat{d}_k^j = d_k^j$ and $y_k^j + \bar{\alpha}_k^j \hat{d}_k^j \in \mathcal{F}$ for all j and k, so $\tilde{r}_k = r_k$ and the projection operations in Algorithm 2 would be redundant. If $\mathcal{F}$ is convex, then the projection mapping $P_{\mathcal{F}}[\cdot]$ (using 2-norm) is single-valued.

Also, like Algorithm 1, in Step 2.3 and in the Expansion Step, the quadratic function $\gamma(\cdot)^2$ can be replaced by any "forcing function" $\phi(\cdot)$ as discussed in Note 2.

Note 4. The sets $I_k^1 \supset \cdots \supset I_k^{p_k}$ of Step 1 can be determined by ordering the distinct values of $g_i(x_k)$, $i = 1, \ldots, m$, that exceed $-\epsilon^*$ in ascending order and setting $I_k^p = I(x_k; \epsilon_k^p)$, where ϵ_k^p is the pth value in this ordering. If $g_i(x_k) < -\epsilon^*$ for all i, then $p_k = 1$ and $I_k^1 = \emptyset$.

Note 5. In Step 2.2, we can alternatively replace $I_k^{p,q}$ in the projection operation by the simpler index set I_k^p, and the main convergence result (Proposition 6.2) would still hold. However, since d_k^q need not belong to $T_{I_k^p}(x_k)$, it is no longer true that the projection operations would be redundant if $g_1, ..., g_m$ are affine.

The next two propositions establish the convergence properties of Algorithm 2 under Assumptions A, B, C. These properties are stronger than those for Algorithm 1.

Proposition 6.1. *Let $\{x_k\}$, $\{(d_k^q)_{1\le q\le r_k}\}$ and $\{(y_k^j, \tilde{\alpha}_k^j, \bar{\alpha}_k^j, \hat{\alpha}_k^j, \alpha_k^j, \hat{d}_k^j)_{1\le j\le \tilde{r}_k}\}$ be sequences produced by Algorithm 2. Then, under Assumptions A, B, C, we have:*

(i) $\{x_k\}$ is well defined;

(ii) $\{x_k\}$ is bounded and every cluster point belongs to $\mathcal{F}$;

(iii) we have

$$\lim_{k\to\infty} \max_{1\le j\le \tilde{r}_k} \left\{\alpha_k^j\right\} = 0,$$
$$\lim_{k\to\infty} \max_{1\le j\le \tilde{r}_k} \left\{\tilde{\alpha}_k^j\right\} = 0,$$
$$\lim_{k\to\infty} \max_{1\le j\le \tilde{r}_k} \left\|x_k - y_k^j\right\| = 0;$$

(iv) $\bar{\alpha}_k^j \ge (\epsilon - \max_{i=1,\ldots,m} |g_i(y_k^j) - g_i(x_k)|)/\gamma_k$ whenever $\hat{d}_k^j \in T_{I_k}(y_k^j)$ for some $\epsilon > 0$, where $\gamma_k = \max_{1\le j\le \tilde{r}_k} \max_{i=1,\ldots,m} |\nabla g_i(y_k^j)^T \hat{d}_k^j|$ and $I_k = I(x_k; \epsilon)$.

Proof. The proof is nearly identical to the proof of Proposition 5.1. ∎

Proposition 6.2. *Under Assumptions A, B, C, if $\{x_k\}$ is a sequence produced by Algorithm 2, then $\{x_k\}$ is bounded and every cluster point is a stationary point of problem* (1).

Proof. Let $\{(d_k^q)_{1\le q\le r_k}\}$ and $\{(y_k^j, \tilde{\alpha}_k^j, \bar{\alpha}_k^j, \hat{\alpha}_k^j, \alpha_k^j, \hat{d}_k^j)_{1\le j\le \tilde{r}_k}\}$ be sequences produced by Algorithm 2 corresponding to $\{x_k\}$. By Proposition 6.1, $\{x_k\}$ is well defined and bounded. Let $\bar{x}$ be any cluster point of $\{x_k\}$. Then, there is an infinite $K \subseteq \{0, 1, \ldots\}$ such that

$$\lim_{k\to\infty, k\in K} x_k = \bar{x}.$$

Let $\epsilon \in (0, \min\{\bar{\epsilon}, \epsilon^*\}]$ and J_k, $k \in K$, be defined as in Proposition 4.1, with $\bar{\epsilon}$ defined as in Proposition 2.1(i). Since $I(x_k) \subseteq I(x_k; \epsilon) \subseteq I(x_k; \epsilon^*)$ for each k, we see from Step 1 that there exists $1 \le \bar{p}_k \le p_k$ such that $I_k^{\bar{p}_k} = I(x_k; \epsilon)$. For each $k \in K$ and $q \in J_k$, we have that $\nabla g_i(x_k)^T d_k^q \le 0$ for all $i \in I(x_k; \epsilon) = I_k^{\bar{p}_k}$, so

$$I_k^{\bar{p}_k, q} = I_k^{\bar{p}_k} = I(x_k; \epsilon).$$

We also have from the rule for incrementing p in Step 2.4 that either $\hat{d}_k^{j_q} = d_k^q$ for some j_q and some $p < \bar{p}_k$ or else $\hat{d}_k^{j_q} = P_{T_{I_k^{p,q}}(y_k^{j_q})}[d_k^q]$ for some j_q and $p = \bar{p}_k$. In the former case, since $I_k^{p,q} \supset I_k^{\bar{p}_k,q} = I(x_k;\epsilon)$ so that $T_{I_k^{p,q}}(y_k^{j_q}) \subseteq T_{I(x_k;\epsilon)}(y_k^{j_q})$, we obtain from the definition of $\hat{d}_k^{j_q}$ that

$$\hat{d}_k^{j_q} = P_{T_{I_k}(y_k^{j_q})}[d_k^q], \tag{30}$$

where we set $I_k = I(x_k;\epsilon)$. In the latter case, we have $I_k^{p,q} = I_k^{\bar{p}_k,q} = I_k$ and (30) again holds.

Since (30) implies $\hat{d}_k^{j_q} \in T_{I_k}(y_k^{j_q})$ for all $q \in J_k$ and $k \in K$, assertion (iv) of Proposition 6.1 implies

$$\bar{\alpha}_k^{j_q} \geq \left(\epsilon - \max_{i=1,\ldots,m} |g_i(y_k^{j_q}) - g_i(x_k)|\right) / \gamma_k.$$

Then, assertion (iii) of Proposition 6.1 and boundedness of $\{x_k\}$ imply there exists an index $\bar{k}$ such that for all $k \geq \bar{k}$, $k \in K$, we have

$$\alpha_k^j/\omega < \bar{\alpha}_k^j \quad \text{and} \quad \tilde{\alpha}_k^j < \bar{\alpha}_k^j$$

for each $q \in J_k$ and $j = j_q$ (ω is the parameter from the Expansion Step). Then, the construction of α_k^j in Step 2.3 implies, for each $q \in J_k$ and $j = j_q$, either

$$\alpha_k^j > 0 \quad \text{and} \quad f(w_k^j) > f(y_k^j) - \gamma(\alpha_k^j/\omega)^2$$

if an Expansion Step is performed in generating α_k^j, where $w_k^j \in P_{\mathcal{F}}[y_k^j + (\alpha_k^j/\omega)\hat{d}_k^j]$, or otherwise

$$\hat{\alpha}_k^j > 0 \quad \text{and} \quad f(\hat{y}_k^j) > f(y_k^j) - \gamma(\hat{\alpha}_k^j)^2.$$

Setting $\xi_k^j = \alpha_k^j/\omega$ in the first case and setting $\xi_k^j = \hat{\alpha}_k^j$ in the second case, this together with (30) and assertion (iii) of Proposition 6.1 implies that condition (b) of Proposition 4.2 holds for the subsequences $\{x_k\}_K$ and $\{(y_k^{j_q})_{q\in J_k}\}_K$, and the subsequence of scalars $\{(\xi_k^{j_q})_{q\in J_k}\}_K$ defined as above. Also, since the sets D_k satisfy Assumption 2, condition (a) of Proposition 4.2 holds. Then, Proposition 4.2 yields that $\bar{x}$ is a stationary point of (1). ∎

7. Preliminary Computational Results

In order to gain some understanding of the practical performance of the proposed methods, we have implemented in Fortran Algorithm 2 for the linearly constrained case and have conducted tests on some problems from the Hock-Schittkowski collection [7, 8]. We describe the implementation details and the computational results below.

Choice of parameters

After some tuning, we settled on the following choice of parameters in Algorithm 2:

$$\tilde{\alpha}_0 = 1, \qquad \gamma = 10^{-6}, \qquad \theta_1 = 0.5, \qquad \epsilon^* = 10^{-3}, \qquad \omega = 0.5.$$

In Step 3, we set $x_{k+1} = y_k^{j+1}$, i.e., no scheme for accelerating convergence is exploited.

Choice of the directions

We generate the set D_k as described in Proposition 3.2. In particular, we perform a QR-factorization of A_k, namely

$$A_k = [Y_k \quad Z_k] \begin{bmatrix} R_k \\ 0 \end{bmatrix},$$

where Y_k and Z_k have orthonormal columns, the columns of Y_k form an orthonormal basis for the range of A_k and the columns of Z_k form an orthonormal basis for the null space of A_k^T. Under the assumptions of Proposition 3.2, R_k is nonsingular. Then, we use as the set of search directions D_k the columns (normalized to have unit length) of the following $n \times 2n$ matrix

$$\left[Z_k \quad - Z_k \quad Y_k(R_k)^{-1} \quad - Y_k(R_k)^{-1} \right].$$

Stopping criterion

We use a stopping criterion of

$$\max_{1 \le j \le 2n} \{\tilde{\alpha}_k^j\} \le 10^{-4}.$$

This criterion is reasonable since $\{x_k\} \to \bar{x}$ and $\{\max_{1 \le j \le 2n}\{\tilde{\alpha}_k^j\}\} \to 0$ imply (via the instructions of Algorithm 2 and Note 3) that (14) holds for some infinite $K \subseteq \{0, 1, \ldots\}$, where $o(\cdot) = \gamma(\cdot)^2$ and ξ_k^j equals either $\tilde{\alpha}_k^{j+1}/\omega$ or $\tilde{\alpha}_k^j$. This in turn implies (8), so that upon stopping, $\max_{j \in J_k} \left\{ \min\{0, \nabla f(x_k)^T d_k^j\} \right\}$ would be near zero and x_k would be an approximate stationary point.

In the following table we report the computational results obtained with the above implementation. We denote by n the number of variables, by *lin.* the number of linear constraints, by *box* the number of box constraints, by NF the number of function evaluations required to satisfy the stopping criterion, and by Δ_f the difference between the best f-value found and the optimal f-value.

	n	*lin.*	*box*	NF	Δf
HS21	2	1	4	26	0
HS24	2	3	2	14	0
HS36	3	1	5	12	0
HS37	3	2	6	136	0
HS76	4	3	4	57	10^{-3}
HS224	2	4	4	67	10^{-10}
HS232	2	3	2	13	0
HS250	3	2	6	11	0
HS251	3	1	6	122	10^{-9}
HS331	2	1	3	57	0
HS340	3	1	1	116	10^{-9}
HS354	4	1	4	190	0

We have used as starting points those proposed in [7] and [8], except for the problems HS21 and HS340, where the given starting points are not feasible. In these last cases we have used $x_0 = (2, -1)$ and $x_0 = (0.1, 0.1, 0.1)$ respectively.

From the results reported in the table, we see that the implemented method has been able to find a very good approximation of the optimal solution within an acceptable number of function evaluations. Although our computational experiment is rather limited and the test problems are small–hence no definite conclusion can be drawn, the results seem to indicate that the proposed approach could form the basis for practical objective-derivative-free methods. In particular, the efficiency of the method could be improved by incorporating interpolation techniques in producing the new point at Step 3, thus accelerating the convergence of the method.

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