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A. Germani, G. Mavelli

THE POLYNOMIAL APPROACH TO THE LQ NONGAUSSIAN REGULATOR PROBLEM

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Alfredo Germani – Dipartimento di Ingegneria Elettrica, Università dell'Aquila, 67100 Monteluco (L'Aquila), Italy and Istituto di Analisi dei Sistemi ed Informatica del CNR, viale Manzoni 30 - 00185 Rome, Italy. Email: germani@iasi.rm.cnr.it.

Gabriella Mavelli – Istituto di Analisi dei Sistemi ed Informatica del CNR, viale Manzoni 30 - 00185 Rome, Italy and Dipartimento di Informatica e Sistemistica, Università di Roma "La Sapienza," via Eudossiana 18 - 00184 Rome, Italy. Email: mavelli@iasi.rm.cnr.it.

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Abstract

In this work we propose a new approach to solve the regulator problem when the process is a linear discrete time dynamical systems with non-Gaussian state and output noises. It generalizes previous results concerning quadratic optimal regulator problem [1]. An optimal recursive control law is achieved for systems with partial state information and when a quadratic cost function is considered. Numerical simulations show the high performances of the proposed method with respect to the classical linear regulation techniques.

Key words: discrete-time systems, Kronecker algebra, LQG optimal control, Kalman filter, nonlinear filtering, separation principle, stochastic control, non-Gaussian systems.

1. Introduction

In this paper the optimal regulator problem for linear stochastic non-Gaussian discrete time systems, with partial state informations, is treated. A quadratic performance index is considered. In a considerable number of technical areas, the widely used Gaussian assumption must be removed to obtain a more realistic statistical description. As shown in various papers (e.g. [2] and [3]) increasing attention has been paid to non-Gaussian systems in control engineering, where both parameter and state estimation problems are important in designing feedback control laws. Non-Gaussian problems often arise in digital communications when the noise interference includes noise components that are essentially non-Gaussian (this is a common situation below 100 MHZ) [4]. Neglecting these components is a major source of error in communication system design.

Many papers have appeared which consider the optimal control problem (in particular the optimal regulator problem) of linear systems when the performance criterion is a quadratic form in the state and control. This problem has been extensively studied by control engineers, scientists and mathematicians. In literature there are many papers that deal with the case of deterministic systems (see, for example, [5-9]). For stochastic systems, some authors assume that only partial informations about the noises statistics are known (minimax [9,10,11] or $H\infty$ [12] approaches). Other authors, on the contrary, assume that the noises statistics are Gaussian (LQG control problem [13-17]). For stochastic systems with incomplete state observation, this problem presented itself as a control and filtering simultaneous problem. The separation principle [18] states, in this case, an important decoupling result between the two problems of filtering and stochastic control.

The purpose of this paper is to present a new algoritm to solve the optimal stochastic regulator problem for non-Gaussian systems with incomplete state observation (i.e. only noisy state observation are available) when a quadratic index is considered. The resulting solution is given by the same feedback control law as in the linear optimal regulator [19,20] and a filtering stage given by the optimal polynomial filter [21]. We, therefore, will prove also an extension of the separation principle which is valid for non-Gaussian control systems when a suitable class of sub-optimal state estimator (with respect to the conditional expectation) is considered.

In the following first section the precise statement and formulation of the problem are given. Then, we recall the optimal solution of the problem with Gaussian state and output noises [19,20] which represents a sub-optimal linear solution in the non-Gaussian case.

The main result of the paper is given in section 3 in which the optimal polynomial regulator for a linear discrete-time non-Gaussian system is given. In section 4, some numerical examples of application are presented, showing high performance of the proposed polynomial optimal control with respect to the linear one. An appendix is included where the main definitions and properties about Kronecker algebra are reported together with some new results. The paper ends with a concluding remark in section 5.

2. Problem formulation and preliminaries

Let us consider the control problem relative to the class of stochastic discrete-time linear systems described by the following equations:

$$x(k+1) = A(k)x(k) + B(k)u(k) + F(k)N_k + d(k), \quad x(0) = \bar{x} \tag{2.1}$$

$$y(k) = C(k)x(k) + G(k)N_k \tag{2.2}$$

with the corresponding quadratic criterion:

$$J = \frac{1}{2}E\{x^T(N)S(N)x(N) + \sum_{k=0}^{N-1}(x^T(k)Q(k)x(k) + u^T(k)R(k)u(k))\}, \tag{2.3}$$

where , for any k,

$$x(k) \in \mathrm{IR}^n, \quad y(k) \in \mathrm{IR}^q, \quad u(k) \in \mathrm{IR}^p, \quad d(k) \in \mathrm{IR}^n.$$

The noise N_k forms a sequence of non-Gaussian random vector variables whose all moments up to the 2ν-th order are known:

$$E(N_k) = 0, \tag{2.4}$$

$$E(N_k^{[i]}) = \Psi_{N,i}, \quad i = 2, 3, ..., 2\nu. \tag{2.5}$$

Without less of generality, let us assume the following equality:

$$st^{-1}\Psi_{N,2} = E(N_k \cdot N_k^T) = I. \tag{2.6}$$

The sequence $\{N_k\}$ forms with the initial state random vector $\bar{x}$ a family of independent random variables. The initial state $\bar{x}$ is endowed too with statistical moments, namely

$$E(\bar{x}) = m_0, \tag{2.7}$$

$$E((\bar{x} - m_0)^{[i]}) = \Psi_{x,i}, \quad i = 2, 3, ..., 2\nu. \tag{2.8}$$

We assume independence of the state noise $F(k)N_k$ with respect to the measurement noise $G(k)N_k$, therefore the relation $F(k)G(j)^T = 0$, $\forall k, \forall j$, is verified. Moreover the covariance matrix $G(k)G(k)^T$ is assumed full rank. The forcing term $d(k)$ is assumed to be known at any time k.

The cost index weight matricies are such that:

$$S(N) \in \mathrm{IR}^{n\times n}, \quad Q(k) \in \mathrm{IR}^{n\times n}, \quad R(k) \in \mathrm{IR}^{p\times p},$$

where $R(k)$ is assumed to be positive definite, whereas the matrices $Q(k)$ and $S(N)$ are at least nonnegative definite.

Remark 2.1. Note that in order to synthetize the optimal control it is sufficient to consider $d(k) = 0$ in that any deterministic forcing term contributes to the cost index as an additive constant.

If we assume $\{N_k\}$ to be a sequence of Gaussian random variables, problem (2.1)-(2.3) becomes the well known classical LQG regulator problem. The cost-minimizing controller $\hat{u}(k)$ (i.e. the output feedback optimal control) has been shown to be in this case [19,20]:

$$\hat{u}(k) = -M(k)\hat{x}(k/k-1), \tag{2.9}$$

if only incomplete, noise-corrupted measurements are available and are processed by the appropriate Kalman filter [22]. If perfect knowledge of the state vector is assumed available at each sample time, the state feedback optimal controller has the same gain sequence $\{M(k)\}$ as in the previous case and this is also equal to the associated optimal deterministic controller one (*certainty equivalence* applies [19]). For all three cases, these gains can be generated from the solution of the backward Riccati recursive equations:

$$M(k) = R(k)^{-1}B(k)^T P_c(k+1)[I + B(k)R(k)^{-1}B(k)^T P_c(k+1)]^{-1}A(k), \tag{2.10}$$
$$P_c(k) = A(k)^T P_c(k+1)[I + B(k)R(k)^{-1}B(k)^T P_c(k+1)]^{-1}A(k) + Q(k), \tag{2.11}$$
$$P_c(N) = S(N). \tag{2.12}$$

For the Kalman predictor part, we have:

$$\begin{aligned}\hat{x}(k/k-1) = {} & A(k-1)\hat{x}(k-1/k-2) + A(k-1)K(k-1) \\ & \cdot [y(k-1) - C(k)\hat{x}(k-1/k-2)] + B(k)\hat{u}(k-1) + d(k-1),\end{aligned} \tag{2.13}$$
$$K(k) = P(k)C(k)^T[G(k)G(k)^T]^{-1}, \tag{2.14}$$
$$P(k) = [I + H(k-1)C(k)^T[G(k)G(k)^T]^{-1}C(k)]^{-1}H(k-1), \tag{2.15}$$
$$H(k-1) = A(k)P(k-1)A(k)^T + F(k)F(k)^T. \tag{2.16}$$

It is well known that the control (2.10)-(2.16) represents the optimal solution for our problem (2.1)-(2.3) (i.e. non-Gaussian case) if we seek the solution among all linear output transformations. This is, therefore, a sub-optimal solution for the problem we treat.

In the following, for any given sequence of random vectors $\{w(l)\}$ in $\mathbb{R}^r$, we will use the short notation w_k to indicate the aggregate vector generated by $w(j)$, $j \leq k$, namely:

$$w_k = \{w(j), \quad 0 \leq j \leq k\}. \tag{2.17}$$

Moreover, the set of all random variables defined as affine transformations of w_k in $\mathbb{R}^s$ will be indicated

$$\begin{aligned}\mathcal{L}(w_k, \mathbb{R}^s) = \big\{ & u \in \mathbb{R}^s : u = l + Lw_k, \\ & L : \mathbb{R}^{r(k+1)} \to \mathbb{R}^s, \ l \in \mathbb{R}^s \big\}.\end{aligned} \tag{2.18}$$

By the use of this notation, the feedback control (2.9)-(2.16) (LQG) is exactly the element of the admissible set

$$\mathcal{U}^1 = \mathcal{L}(y_{k-1}, \mathbb{R}^p) \tag{2.19}$$

which minimizes the index (2.3) with constraints (2.1), (2.2). From the classical theory of Kalman filtering, it is well known that $\hat{x}(k/k-1)$, as given by (2.13), is precisely the orthogonal projection of the state $x(k)$ on $\mathcal{U}^1$, that is

$$\hat{x}(k/k-1) = \Pi(x(k)/\mathcal{U}^1) \tag{2.20}$$

where $\Pi(v/S)$ denotes the orthogonal projection operator of the vector v on the affine subspace S, which is well defined when $\|v\| < \infty$. In the non-Gaussian case this solution, in many cases, is not satisfactory in that the relevant index assumes a value too much far from the optimal one. In order to improve such solution is therefore necessary to enlarge the admissible set (2.19). The idea is to consider also polynomial transformation of the measurements.

Definition 2.2. For any integer $\nu \geq 1$, the admissible set of order ν is defined as

$$\mathcal{U}^\nu = \mathcal{L}(y_{k-1}^\nu, \mathbb{R}^p) \tag{2.21}$$

where y_{k-1}^ν is generated by the sequence

$$y^\nu(j) = \begin{pmatrix} y(j) \\ y^{[2]}(j) \\ \vdots \\ y^{[\nu]}(j) \end{pmatrix}, \quad j = 0, 1, ..., k-1, \tag{2.22}$$

in which $y^{[i]}(j)$ denotes the i-th Kronecker power (see Appendix).

Remark 2.3. From the previous definition it readily results

$$\mathcal{U}^1 \subseteq \mathcal{U}^2 \subseteq ... \subseteq \mathcal{U}^\nu. \tag{2.23}$$

For the following we need to use the property

Lemma 2.4. *Let $w_d(j)$, $j = 0, 1, ..., k-1$, be any given deterministic sequence in $\mathbb{R}^r$. Then, defining $w_s(j) = w(j) - w_d(j)$, $w(j)$ as in (2.17), and*

$$\tilde{w}^\nu(j) = \begin{pmatrix} w_s(j) \\ w_s^{[2]}(j) \\ \vdots \\ w_s^{[\nu]}(j) \end{pmatrix}, \tag{2.24}$$

we have

$$\mathcal{L}(w_{k-1}^\nu, \mathbb{R}^s) = \mathcal{L}(\tilde{w}_{k-1}^\nu, \mathbb{R}^s). \tag{2.25}$$

Proof. Let $u_k \in \mathcal{L}(w_{k-1}^\nu, \mathbb{R}^s)$. Then there exists a vector d_k and matrices $\beta_{j,l}$ such that

$$u_k = d_k + \sum_{j=0}^{k-1} \sum_{l=1}^{\nu} \beta_{j,l} \cdot w^{[l]}(j).$$

Moreover, because $w(j) = w_d(j) + w_s(j)$, we have, using "Newton formula" (see appendix),

$$\begin{aligned} u_k &= d_k + \sum_{j=0}^{k-1} \sum_{l=1}^{\nu} \beta_{j,l} \cdot [w_d(j) + w_s(j)]^{[l]} \\ &= d_k + \sum_{j=0}^{k-1} \sum_{l=1}^{\nu} \beta_{j,l} \sum_{m=0}^{l} M_m^l \cdot [w_d^{[l-m]}(j) \otimes w_s^{[m]}(j)] \\ &= \tilde{d}_k + \sum_{j=0}^{k-1} \sum_{m=1}^{\nu} \Big[\sum_{l=m}^{\nu} \beta_{j,l} \cdot M_m^l \cdot [w_d^{[l-m]}(j) \otimes I_{q^m}] \Big] \cdot w_s^{[m]}(j) \\ &= \tilde{d}_k + \sum_{j=0}^{k-1} \sum_{m=1}^{\nu} \tilde{\beta}_{j,m} \cdot w_s^{[m]}(j). \end{aligned}$$

Then $u_k \in \mathcal{L}(\tilde{w}^{\nu}_{k-1}, \mathbb{R}^s)$.

Viceversa, let $u_k \in \mathcal{L}(\tilde{w}^{\nu}_{k-1}, \mathbb{R}^s)$, then

$$u_k = \bar{d}_k + \sum_{j=0}^{k-1} \sum_{l=1}^{\nu} \bar{\beta}_{j,l} \cdot w_s^{[l]}(j).$$

Because $w_s(j) = w(j) - w_d(j)$, by following the same process as before, it follows $u_k \in \mathcal{L}(w^{\nu}_{k-1}, \mathbb{R}^s)$. ∎

Lemma 2.5. *Let $\{\mathcal{V}(j)\}$ be any deterministic sequence in $\mathbb{R}^{r_\nu}$, $r_\nu = r\dfrac{r^\nu - 1}{r-1}$. Then, defining w_k as in (2.17), we have:*

$$\mathcal{L}(w_k, \mathbb{R}^s) = \mathcal{L}(w_k - \mathcal{V}_k, \mathbb{R}^s). \tag{2.26}$$

Proof. It is immediate from the definition (2.18). ∎

We conclude this section by recalling [21] the following lemma, whose proof can be find in [21] for the time invariant case, which will result useful in the next section. In the sequel we will denote with $M_i^j(l)$ the binomial matrices (see appendix) highlighting the dependence by the dimension l of the vectors, and the symbol $I_{i,j}$ will denote the identity in $\mathbb{R}^{i^j}$.

Lemma 2.6. *Let, on the same probability space, $\{z(k),\ k \geq 0\}$ and $\{N(k),\ k \geq 0\}$ be random sequences in $\mathbb{R}^\alpha$ and $\mathbb{R}^\beta$ respectively, such that $\forall k$ $N(k)$ is independent by $\{z(k), z(j), N(j),\ j < k\}$. Moreover let us assume*

$$w(k) = \Gamma(k)z(k) + \Psi(k)N(k) \tag{2.27}$$

where $w(k) \in \mathbb{R}^\gamma$ and $\Gamma(k)$, $\Psi(k)$ are subsequently dimensioned deterministic matrices. Consider the Kronecker powers of $w(k)$ and $z(k)$ up to the ν-th order aggregated in the vectors

$$\mathcal{W}(k) = \begin{bmatrix} w(k) \\ w^{[2]}(k) \\ \vdots \\ w^{[\nu]}(k) \end{bmatrix}, \quad \mathcal{Z}(k) = \begin{bmatrix} z(k) \\ z^{[2]}(k) \\ \vdots \\ z^{[\nu]}(k) \end{bmatrix},$$

and

$$\mathcal{O}(k) = \begin{bmatrix} \Gamma(k) & 0 & \dots & 0 \\ O_{2,1}(k) & \Gamma^{[2]}(k) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ O_{\nu,1}(k) & O_{\nu,2}(k) & \dots & \Gamma^{[\nu]}(k) \end{bmatrix}, \quad \mathcal{T}(k) = \begin{bmatrix} \Psi(k)E(N(k)) \\ \Psi^{[2]}(k)E(N^{[2]}(k)) \\ \vdots \\ \Psi^{[\nu]}(k)E(N^{[\nu]}(k)) \end{bmatrix},$$

where

$$O_{i,l}(k) = M^i_{i-l}(\gamma)(\Psi^{[i-l]}(k) \otimes \Gamma^{[l]}(k))\left(E(N^{[i-l]}(k)) \otimes I_{\alpha,l}\right) \tag{2.28}$$

then there exists the representation

$$\mathcal{W}(k) = \mathcal{O}(k)\mathcal{Z}(k) + \mathcal{T}(k) + \mathcal{N}(k) \tag{2.29}$$

where

$$\mathcal{N}(k) = \begin{bmatrix} h_1(k) \\ h_2(k) \\ \vdots \\ h_\nu(k) \end{bmatrix} \tag{2.30}$$

and

$$h_i(k) = \sum_{l=0}^{i-1} M_{i-l}^i(\gamma)(\Psi^{[i-l]}(k) \otimes \Gamma^{[l]}(k))\Big(\Big(N^{[i-l]}(k) - E(N^{[i-l]}(k))\Big) \otimes I_{\alpha,l}\Big) z^{[l]}(k),$$

moreover $\{\mathcal{N}(k)\}$ *is a zero mean white sequence, such that* $\forall k$ $\mathcal{N}(k)$ *is uncorrelated with* $\{\mathcal{Z}(j),\ j \leq k\}$, *with covariance* $\mathcal{S}(k)$ *such that its* (r,s)*-block is given by*

$$\begin{aligned} \mathcal{S}_{r,s} &\triangleq E(h_s(k)h_r(k)^T) \\ &= \sum_{l=0}^{r-1}\sum_{m=0}^{s-1} M_{r-l}^r(\gamma)(\Psi^{[r-l]}(k) \otimes \Gamma^{[l]}(k)) \cdot st^{-1}\Bigg((I_{\beta,s-m} \otimes C_{\beta^{r-l},\alpha^m}^T \otimes I_{\alpha,l}) \\ &\quad \cdot \Bigg(\Big(E\big(N^{[s+r-m-l]}(k)\big) - E\big(N^{[s-m]}(k)\big) \otimes E\big(N^{[r-l]}(k)\big)\Big) \\ &\quad \otimes C_{1,\alpha^m} \otimes I_{\alpha,l}\Bigg) \cdot E\big(z^{[l+m]}(k)\big)\Bigg)(\Psi^{[s-m]}(k) \otimes \Gamma^{[m]}(k))^T (M_{s-m}^s(\gamma))^T, \end{aligned} \tag{2.31}$$

provided that there exist finite all the moments involved.

3. Main result

For the following we need to consider the state $x(k)$ of the stochastic system (2.1) as the sum of $x_u(k)$ and $x_N(k)$, respectively the input and the state-noise dependent part of $x(k)$, namely

$$x_u(k+1) = A(k)x_u(k) + B(k)u(k), \tag{3.1}$$

$$x_N(k+1) = A(k)x_N(k) + F(k)N_k, \tag{3.2}$$

with

$$x_u(0) = \Psi_{x,1} = m_0 \tag{3.3}$$

and $x_N(0)$ be described by its first 2ν moments, that is:

$$E(x_N(0)) = 0, \tag{3.4}$$

$$E(x_N^{[i]}(0)) = \Psi_{x,i}, \quad i = 2, ..., 2\nu. \tag{3.5}$$

Moreover, let us define the stochastic part of the output process by

$$y_N(k) = y(k) - C(k)x_u(k). \tag{3.6}$$

A simple use of lemma 2.4 and lemma 2.5 allows to state the following result.

Proposition 3.1. *The polynomial optimal control of (2.1)-(2.3) $\hat{u}(k)$ belongs to the set*

$$\mathcal{L}(Y_{e,k-1}, \mathrm{I\!R}^p) \tag{3.7}$$

where

$$Y_e(j) = \begin{pmatrix} y_N(j) - G(k)E(N_k) \\ y_N^{[2]}(j) - G(k)^{[2]}E(N_k^{[2]}) \\ y_N^{[3]}(j) - G(k)^{[3]}E(N_k^{[3]}) \\ \vdots \\ y_N^{[\nu]}(j) - G(k)^{[\nu]}E(N_k^{[\nu]}) \end{pmatrix}. \tag{3.8}$$

Proof. By definition, the polynomial optimal control belongs to

$$\mathcal{L}(y_{k-1}^{\nu}, \mathrm{I\!R}^p) \tag{3.9}$$

therefore the proposition is proved taking into account definition (3.6) of $y_N(k)$ and lemmas 2.4 and 2.5 that implies the equivalence of the set (3.9) with the set (3.7). ∎

We have only to stress that $x_u(k)$ depends on the past values of the input up to time $(k-2)$ which we assume to be known thanks to the "optimality principle". Now, the key to solve the optimal polynomial control problem is given by the property assured by the following theorem.

Theorem 3.2. *The sequence $Y_e(k)$ is generated by the following stochastic "extended system":*

$$\begin{aligned} X_e(k+1) &= \mathcal{A}(k)X_e(k) + \mathcal{B}(k)u(k) \\ &\quad + \mathcal{D}_e f_e(k) + d_e(k), \quad X_e(0) = \bar{X}_e \end{aligned} \tag{3.10}$$

$$Y_e(k) = \mathcal{C}(k)X_e(k) + g_e(k), \tag{3.11}$$

with

$$X_e(k) = \begin{pmatrix} x_u(k) \\ x_N(k) \\ x_N^{[2]}(k) \\ x_N^{[3]}(k) \\ \vdots \\ x_N^{[\nu]}(k) \end{pmatrix}, \tag{3.12}$$

$$d_e(k) = \begin{pmatrix} F(k)E(N_k) \\ F(k)^{[2]}E(N_k^{[2]}) \\ \vdots \\ F(k)^{[\nu]}E(N_k^{[\nu]}) \end{pmatrix}, \tag{3.13}$$

$$f_e(k) = \begin{pmatrix} f_1(k) \\ f_2(k) \\ \vdots \\ f_\nu(k) \end{pmatrix}, \quad g_e(k) = \begin{pmatrix} g_1(k) \\ g_2(k) \\ \vdots \\ g_\nu(k) \end{pmatrix}, \tag{3.14}$$

where, for $i = 1, ..., \nu$,

$$f_i(k) = \sum_{l=0}^{i-1} M^i_{i-l}(n)(F^{[i-l]}(k) \otimes A^{[l]}(k))\Big(\Big(N_k^{[i-l]} - E(N_k^{[i-l]})\Big) \otimes I_{n,l}\Big)x_N^{[l]}(k),$$

$$g_i(k) = \sum_{l=0}^{i-1} M^i_{i-l}(q)(G^{[i-l]}(k) \otimes C^{[l]}(k))\Big(\Big(N_k^{[i-l]} - E(N_k^{[i-l]})\Big) \otimes I_{n,l}\Big)x_N^{[l]}(k).$$

The coefficients matrices are given by:

$$\mathcal{A}(k) = \begin{bmatrix} A(k) & 0 & \dots & \dots & \dots & 0 \\ 0 & A(k) & 0 & \dots & \dots & 0 \\ 0 & H_{2,1}(k) & A^{[2]}(k) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & H_{\nu,1}(k) & H_{\nu,2}(k) & \dots & H_{\nu,\nu-1}(k) & A^{[\nu]}(k) \end{bmatrix} = \begin{bmatrix} A(k) & 0 \\ 0 & \mathcal{A}_r(k) \end{bmatrix}, \tag{3.15}$$

$$\mathcal{B}(k) = \begin{bmatrix} B(k) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{3.16}$$

$$\mathcal{C}(k) = \begin{bmatrix} 0 & C(k) & 0 & \dots & \dots & 0 \\ 0 & L_{2,1}(k) & C^{[2]}(k) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & L_{\nu,1}(k) & L_{\nu,2}(k) & \dots & L_{\nu,\nu-1}(k) & C^{[\nu]}(k) \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{C}_r(k) \end{bmatrix}, \tag{3.17}$$

$$\mathcal{D}_e = \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{3.18}$$

with

$$H_{i,l}(k) = M^i_{i-l}(n)(F^{[i-l]}(k) \otimes A^{[l]}(k))\Big(E(N_k^{[i-l]}) \otimes I_{n,l}\Big),$$

$$L_{i,l}(k) = M^i_{i-l}(q)(G^{[i-l]}(k) \otimes C^{[l]}(k))\Big(E(N_k^{[i-l]}) \otimes I_{n,l}\Big).$$

The noises sequences $\{f_e(k)\}$, $\{g_e(k)\}$ *are zero mean white sequences such that:*

$$E(f_e(k)g_e(j)^T) = 0, \quad k \neq j. \tag{3.19}$$

Moreover, defining

$$P^{r,s}_{l,m}(k) = st^{-1}\Bigg((I_{n,s-m} \otimes C^T_{n^{r-l},n^m} \otimes I_{n,l}) \cdot \Bigg(\Big(E(N_k^{[s+r-m-l]}) - E(N_k^{[s-m]}) \otimes E(N_k^{[r-l]})\Big) \otimes C_{1,n^m} \otimes I_{n,l}\Bigg) \cdot E\big(x_N^{[l+m]}(k)\big)\Bigg),$$

its auto-covariance and cross-covariance matrices are give by:

$$E(f_e(k)f_e(k)^T) = \Psi_{f_e}(k), \tag{3.20}$$

$$E(g_e(k)g_e(k)^T) = \Psi_{g_e}(k), \tag{3.21}$$

$$E(f_e(k)g_e(k)^T) = \mathcal{J}(k), \tag{3.22}$$

with

$$\begin{aligned}
[\Psi_{f_e}(k)]_{r,s} &= \sum_{l=0}^{r-1}\sum_{m=0}^{s-1} M^r_{r-l}(n)(F^{[r-l]}(k) \otimes A^{[l]}(k))P^{r,s}_{l,m}(k) \\
&\qquad \cdot (F^{[s-m]}(k) \otimes A^{[m]}(k))^T (M^s_{s-m}(n))^T, \\
[\Psi_{g_e}(k)]_{r,s} &= \sum_{l=0}^{r-1}\sum_{m=0}^{s-1} M^r_{r-l}(q)(G^{[r-l]}(k) \otimes C^{[l]}(k))P^{r,s}_{l,m}(k) \\
&\qquad \cdot (G^{[s-m]}(k) \otimes C^{[m]}(k))^T (M^s_{s-m}(q))^T, \\
[\mathcal{J}(k)]_{r,s} &= \sum_{l=0}^{r-1}\sum_{m=0}^{s-1} M^r_{r-l}(n)(F^{[r-l]}(k) \otimes A^{[l]}(k))P^{r,s}_{l,m}(k) \\
&\qquad \cdot (G^{[s-m]}(k) \otimes C^{[m]}(k))^T (M^s_{s-m}(q))^T,
\end{aligned}$$

in which

$$\begin{aligned}
[\Psi_{f_e}(k)]_{r,s} &\doteq E(f_r(k)f_s(k)^T), \\
[\Psi_{g_e}(k)]_{r,s} &\doteq E(g_r(k)g_s(k)^T), \\
[\mathcal{J}(k)]_{r,s} &\doteq E(f_r(k)g_s(k)^T).
\end{aligned}$$

Proof. The proof is readily obtained by using lemma 2.6. ∎

Finally, we can state the main theorem of this paper which states the algorithm for the solution of the polynomial optimal regulator for the system (2.1),(2.2).

Theorem 3.3. *The optimal polynomial regulator for the system (2.1)-(2.2) is given by the equations*

$$\begin{aligned}
\hat{u}(k) &= -R(k)^{-1}B(k)^T P_c(k+1)[I + B(k)R(k)^{-1}B(k)^T P_c(k+1)]^{-1} \\
&\qquad A(k)\hat{x}(k/k-1), && (3.23)\\
P_c(k) &= A(k)^T P_c(k+1)[I + B(k)R(k)^{-1}B(k)^T P_c(k+1)]^{-1}A(k) + Q(k), && (3.24)\\
P_c(N) &= S(N), && (3.25)\\
\hat{x}(k/k-1) &= x_u(k) + \hat{x}_N(k/k-1), && (3.26)
\end{aligned}$$

$$\nu_o(k) = \begin{pmatrix}
[y(k) - C(k)x_u(k)] - C(k)\hat{x}_N(k/k-1) \\
[y(k) - C(k)x_u(k)]^{[2]} - G(k)^{[2]}\Psi_{N,2} - C(k)^{[2]}\widehat{x_N^{[2]}}(k/k-1) \\
[y(k) - C(k)x_u(k)]^{[3]} - G(k)^{[3]}\Psi_{N,3} - L_{3,1}(k)\hat{x}_N(k/k-1) \\
\qquad -C(k)^{[3]}\widehat{x_N^{[3]}}(k/k-1) \\
\vdots \\
[y(k) - C(k)x_u(k)]^{[\nu]} - G(k)^{[\nu]}\Psi_{N,\nu} - \sum_{i=1}^{\nu-2} L_{\nu,i}(k)\widehat{x_N^{[i]}}(k/k-1) \\
\qquad -C(k)^{[\nu]}\widehat{x_N^{[\nu]}}(k/k-1)
\end{pmatrix}, \tag{3.27}$$

$$x_u(k+1) = A(k)x_u(k) + B(k)\hat{u}(k), \qquad x_u(0) = E(x(0)), \tag{3.28}$$

$$\hat{x}_N(k+1/k) = A(k)\hat{x}_N(k/k-1) + A(k)K_1(k)\nu_o(k), \tag{3.29}$$

$$\widehat{x_N^{[2]}}(k+1/k) = A^{[2]}(k)\widehat{x_N^{[2]}}(k/k-1) + A^{[2]}(k)K_2(k)\nu_o(k) + F^{[2]}(k)\Psi_{N,2}, \tag{3.30}$$

$$\begin{aligned}\widehat{x_N^{[3]}}(k+1/k) &= H_{3,1}(k)\hat{x}_N(k/k-1) + A^{[3]}(k)\widehat{x_N^{[3]}}(k/k-1) + F^{[3]}(k)\Psi_{N,3}\\ &\quad + [H_{3,1}(k)K_1(k) + A^{[3]}(k)K_3(k)]\nu_o(k),\end{aligned} \tag{3.31}$$

$$\vdots$$

$$\begin{aligned}\widehat{x_N^{[\nu]}}(k+1/k) &= \sum_{i=1}^{\nu-2} H_{\nu,i}(k)\widehat{x_N^{[i]}}(k/k-1) + A^{[\nu]}(k)\widehat{x_N^{[\nu]}}(k/k-1) + F^{[\nu]}(k)\Psi_{N,\nu}\\ &\quad + \left[\sum_{i=1}^{\nu-2} H_{\nu,i}(k)K_i(k) + A^{[\nu]}(k)K_\nu(k)\right]\nu_o(k),\end{aligned} \tag{3.32}$$

$$K_r(k) = \begin{pmatrix} K_1(k)\\ K_2(k)\\ \vdots\\ K_\nu(k)\end{pmatrix} = P_r(k)\mathcal{C}_r^T(k)\Psi_{g_e}^{-1}(k), \tag{3.33}$$

$$P_r(k+1) = [I + H_r(k)\mathcal{C}_r^T(k)\Psi_{g_e}^{-1}(k+1)\mathcal{C}_r(k)]^{-1}H_r(k), \tag{3.34}$$

$$P_r(0) = \begin{pmatrix} \Psi_{x(0)} & st^{-1}(\Psi_{x,3}) & \dots & st^{-1}(\Psi_{x,\nu+1})\\ [st^{-1}(\Psi_{x,3})]^T & (P_r(0))_{2,2} & \dots & (P_r(0))_{2,\nu}\\ \vdots & \vdots & \ddots & \vdots\\ [st^{-1}(\Psi_{x,\nu+1})]^T & (P_r(0))_{\nu,2} & \dots & (P_r(0))_{\nu,\nu}\end{pmatrix}, \tag{3.35}$$

$$(P_r(0))_{2,2} = st^{-1}(\Psi_{x,4}) - \Psi_{x,2}\cdot\Psi_{x,2}^T, \tag{3.36}$$

$$(P_r(0))_{2,\nu} = st^{-1}(\Psi_{x,\nu+2}) - \Psi_{x,2}\cdot\Psi_{x,\nu}^T, \tag{3.37}$$

$$(P_r(0))_{\nu,2} = [st^{-1}(\Psi_{x,\nu+2}) - \Psi_{x,2}\cdot\Psi_{x,\nu}^T]^T, \tag{3.38}$$

$$(P_r(0))_{\nu,\nu} = st^{-1}(\Psi_{x,2\nu}) - \Psi_{x,\nu}\cdot\Psi_{x,\nu}^T, \tag{3.39}$$

$$H_r(k) = \mathcal{A}_r(k)P_r(k)\mathcal{A}_r^T(k) + \Psi_{f_e}(k), \tag{3.40}$$

where

$$\mathcal{A}_r(k) = \begin{pmatrix} A(k) & 0 & \dots & \dots & 0\\ H_{2,1}(k) & A^{[2]}(k) & \dots & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ H_{\nu,1}(k) & H_{\nu,2}(k) & \dots & H_{\nu,\nu-1}(k) & A^{[\nu]}(k)\end{pmatrix}, \tag{3.41}$$

$$\mathcal{C}_r(k) = \begin{pmatrix} C(k) & 0 & \dots & \dots & 0\\ L_{2,1}(k) & C^{[2]}(k) & \dots & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ L_{\nu,1}(k) & L_{\nu,2}(k) & \dots & L_{\nu,\nu-1}(k) & C^{[\nu]}(k)\end{pmatrix}. \tag{3.42}$$

Proof. The polynomial optimal control for (2.1)-(2.3) is given by the best linear stochastic control of the extended system (3.10),(3.11) endowed with the index

$$J = \frac{1}{2}E\{X_e^T(N)S_e(N)X_e(N) + \sum_{k=0}^{N-1}(X_e^T(k)Q_e(k)X_e(k) + u^T(k)R(k)u(k))\}, \tag{3.43}$$

where

$$S_e(N) = \begin{pmatrix} S(N) & S(N) & 0 & 0 & \dots & 0 \\ S(N) & S(N) & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$Q_e(k) = \begin{pmatrix} Q(k) & Q(k) & 0 & 0 & \dots & 0 \\ Q(k) & Q(k) & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

This statement will be understanded as soon as it is observed that the state of the extended system contains the state of the original system and that the cost criterion (3.43) is the same of (2.3). Then, the optimal linear control for the extended system which is polynomial with respect to the observation is also the optimal polynomial one for the original system.

We need to show that the equations (3.23)-(3.42) correspond to the optimal linear control for the extended system (3.10)-(3.11) endoved with the cost criterion (3.43). This result can be achieved by applying the algorithm given by equations (2.10)-(2.16) to the extended model (3.10)-(3.11). ∎

4. Simulation examples

Numerical simulations have been performed for two examples in order to test the effectiveness of the proposed algorithm for the solution of the stochastic non-Gaussian regulator problem. These examples show the considerable improvement of the resulting feedback control performances by using as filter stage a polynomial predictor instead of the linear (Kalman) one. The improvement of the process state estimate quality obviously implies a better regulator performance. In a first set of plots, we will show the behaviour of the estimator adopted in the respective proof (state and state prediction plots over 50 steps). It can be seen that the polynomial estimator considerably improves the linear one. In the subsequent plots the process state behaviours obtained by the linear regulator and by the polynomial regulator for each state component are shown. In order that it is possible to compare the results of the two methods, numerical simulations have been performed in the same statistical hypothesis and in correspondence of the same noises realization in both linear and polynomial cases. It can be clearly seen that the state evolution of the process is closer to zero if we apply the polynomial optimal control law instead of the linear one. By growing the adopted polynomial estimate order, the control system capability to lead the state of the process toward values that are near to zero becomes more evident. We would like to stress that this improvement is already clear if we make a comparison between the state behaviour of the cubic optimal regulator and the quadratic one. It is reasonable to think that the solution of the proposed algorithm (that is a sub-optimal solution) tends, for high values of the polynomial regulator order, to the optimal solution of the non-Gaussian regulator problem. Simulations of higher then the third order polynomial regulators should require a more sophisticated numerical implementation, which is not the aim of this paper.

In both of the examples, we consider the control problem for the following class of linear, discrete-time, detectable and stabilizable systems in which the state and the output noises are non-Gaussian:

$$x(k+1) = Ax(k) + Bu(k) + FN_k, \quad x(0) = \bar{x} \tag{4.1}$$
$$y(k) = Cx(k) + GN_k, \tag{4.2}$$

where, for all k, we have:

$$x(k) \in \mathbb{R}^2, \quad y(k) \in \mathbb{R}^1, \quad u(k) \in \mathbb{R}^1.$$

In the first example it is assumed

$$A = \begin{pmatrix} 0 & 1 \\ 0.765 & -0.05 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad F = \begin{pmatrix} F_1 & \vdots & 0 \end{pmatrix},$$
$$C = (1 \quad 0), \qquad G = \begin{pmatrix} 0 & \vdots & G_1 \end{pmatrix}, \tag{4.3}$$

with

$$F_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad G_1 = (5). \tag{4.4}$$

We consider the cost criterion belonging to the class of the quadratic functions in the state and control vectors defined as:

$$J = \frac{1}{2} E\{x^T(N)Sx(N) + \sum_{k=0}^{N-1} (x^T(k)Qx(k) + u^T(k)Ru(k))\}, \tag{4.5}$$

where the following weight matrices have been chosen:

$$S = Q = I_2, \qquad R = 0.1. \tag{4.6}$$

$\{FN_k\}$ and $\{GN_k\}$, the state and measure noises sequences respectively, are independent, zero-mean random sequences. The N_k probability distribution is shown in Table I where the same distribution for both the noise components is assumed and in which results $\bar{\sigma} = (11/2)^{1/2}$.

Table I

$(N_k)_1$	$-10/\bar{\sigma}$	$1/\bar{\sigma}$	0		$(N_k)_2$	$-10/\bar{\sigma}$	$1/\bar{\sigma}$	0
$p[(N_k)_1]$	1/20	1/2	9/20		$p[(N_k)_2]$	1/20	1/2	9/20

In the proposed examples it is assumed $x_s(0)$ to be a zero-mean random variable, independent of the noise sequence and which have the probability distribution given in the following Table II:

Table II

$(x_s(0))_1$	2	-1		$(x_s(0))_2$	2	-1
$p[(x_s(0))_1]$	1/3	2/3		$p[(x_s(0))_2]$	1/3	2/3

The steady-state estimate error covariance matrices have been calculated for the extended system, by solving the Riccati equations 3.34-3.40, in the linear, quadratic and cubic regulator algorithm. They contain, in the first two elements of the main diagonal, the steady-state prediction estimate error covariance of each component of the state of the original system: $(\sigma^2_{e,i})_{1,2}$, where the subscript $i = 1, 2, ..., \nu$ represents the regulator order. The following values have been obtained, for the linear, quadratic and cubic case:

$$\begin{aligned}(\sigma^2_{e,1})_1 &= 2.0388,\\ (\sigma^2_{e,1})_2 &= 2.2198;\\ (\sigma^2_{e,2})_1 &= 1.0160,\\ (\sigma^2_{e,2})_2 &= 1.5994;\\ (\sigma^2_{e,3})_1 &= 0.9744,\\ (\sigma^2_{e,3})_2 &= 1.5736.\end{aligned}$$

The estimate error variance values decrease, as expected, by growing the integer i. We have a notable improvement of the quadratic regulator values with respect to the linear ones. A further remarkable improvement is indeed obtained with the cubic control. The percentage reduction values therefore result:

$$\begin{aligned}\big((\sigma^2_{e,2} - \sigma^2_{e,1})/\sigma^2_{e,1}\big) \cdot 100 &= -27.95\%,\\ \big((\sigma^2_{e,3} - \sigma^2_{e,1})/\sigma^2_{e,1}\big) \cdot 100 &= -29.11\%.\end{aligned}$$

The numerical simulations issues confirm the polynomial optimal regulator high performances, with respect to the linear one, as expected from the *a priori* above shown results. The optimal linear, quadratic and cubic algorithms have been implemented. The results are displayed in Figs. 1-10 for 50 iterations. Figs. 1, 2 show respectively the first and the second conponent of the evolution of the *true* state and its prediction obtained by means of the linear optimal regulator algorithm. Figs. 3, 4 and Figs. 5, 6 show the analogous results obtained by means of the quadratic and cubic optimal regulator algorithms respectively. By comparing the polynomial regulator plots with the linear regulator ones, it is clearly seen that the polynomial filter, which utilizes greater statistical informations on the stochastics processes involved than the linear one, follows the true state evolution better than the linear filter. This is, as expected, more evident in the comparison between the linear and cubic plots.

Figs. 7-10 reproduce the sequences of the *true* state realizations. In particular, Figs. 7, 8 show, for 50 iterations, the behaviours of the system state for linear and quadratic regulation system whereas Figs. 9, 10 show the analogous time evolution for linear and cubic regulator. It can be clearly seen that the system state evolution is closer to zero if we apply the quadratic optimal control law instead of the linear one and that we have a further remarkable improvement by means the cubic controller.

The best performances obtained by means of the new approach to the LQ non-Gaussian regulator problem may be remarked also in a quantitative way. The cost criterion (4.5)-(4.6) has, as a matter of fact, been calculated in the various simulation proofs. In the proposed example the index values result:

$$\begin{aligned}J_1 &= 38.2705;\\ J_2 &= 25.9492;\\ J_3 &= 25.3888.\end{aligned}$$

For the quadratic and cubic algorithms respectively, the percentage reduction of the cost criterion is therefore given by:

$$\begin{aligned}\big((J_1 - J_2)/J_1\big) \cdot 100 &= 32.19\%\\ \big((J_1 - J_3)/J_1\big) \cdot 100 &= 33.66\%.\end{aligned}$$

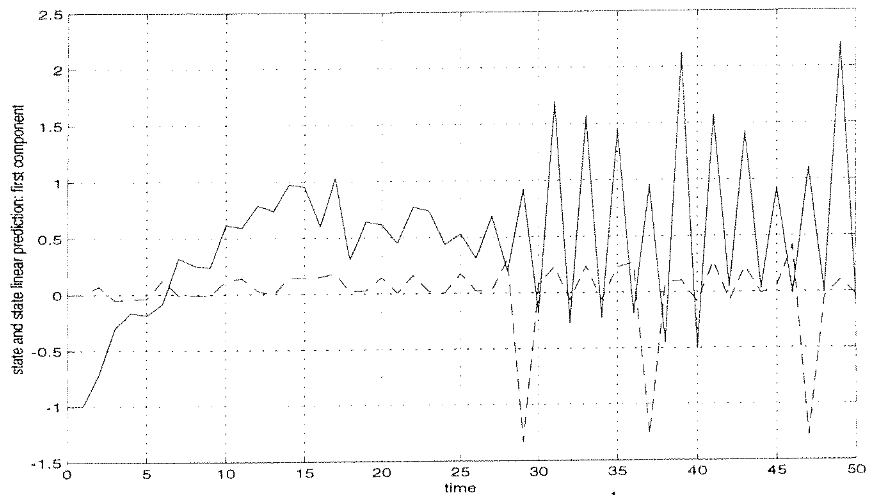

FIG. 1. *State (solid line) and state linear prediction (dashed line) on $N = 50$ steps: first component.*

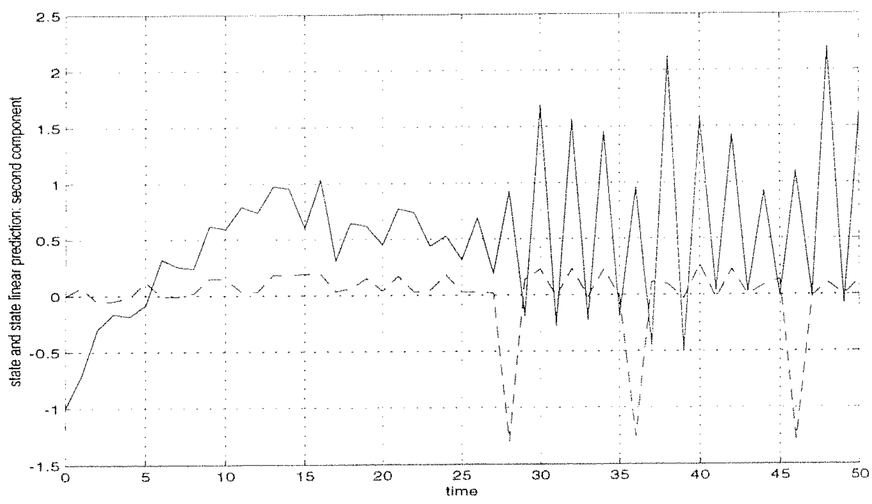

FIG. 2. *State (solid line) and state linear prediction (dashed line) on $N = 50$ steps: second component.*

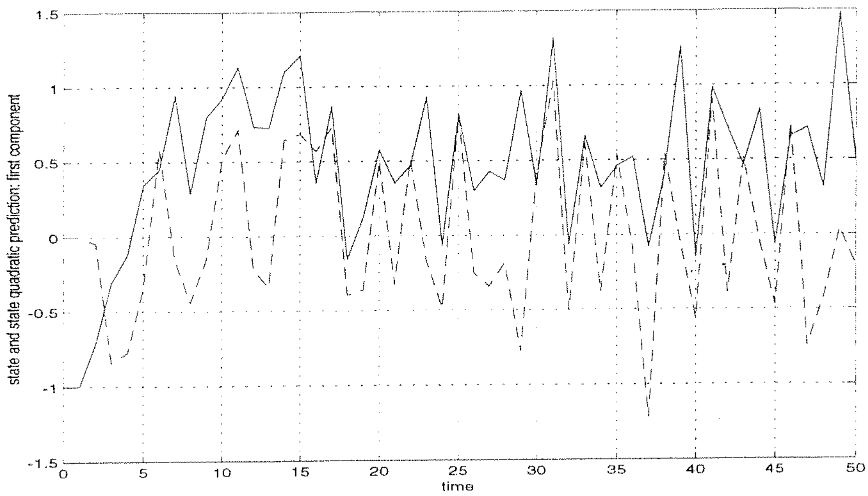

FIG. 3. *State (solid line) and state quadratic prediction (dashed line) on* $N = 50$ *steps: first component.*

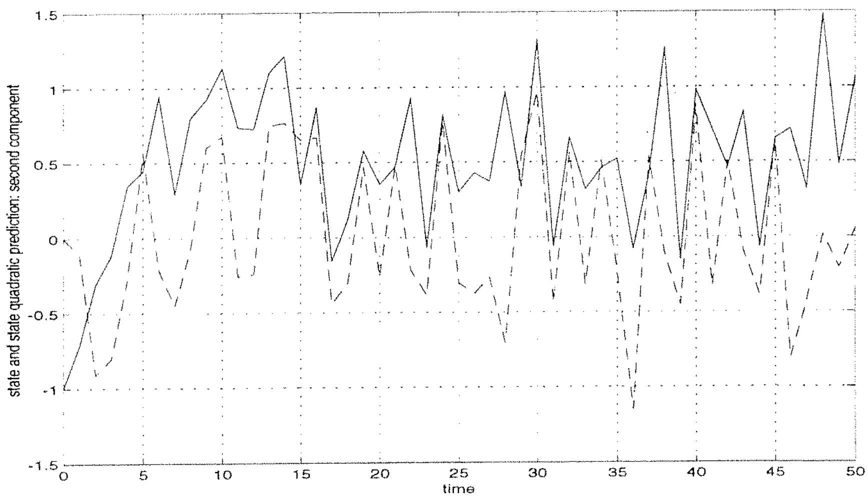

FIG. 4. *State (solid line) and state quadratic prediction (dashed line) on* $N = 50$ *steps: second component.*

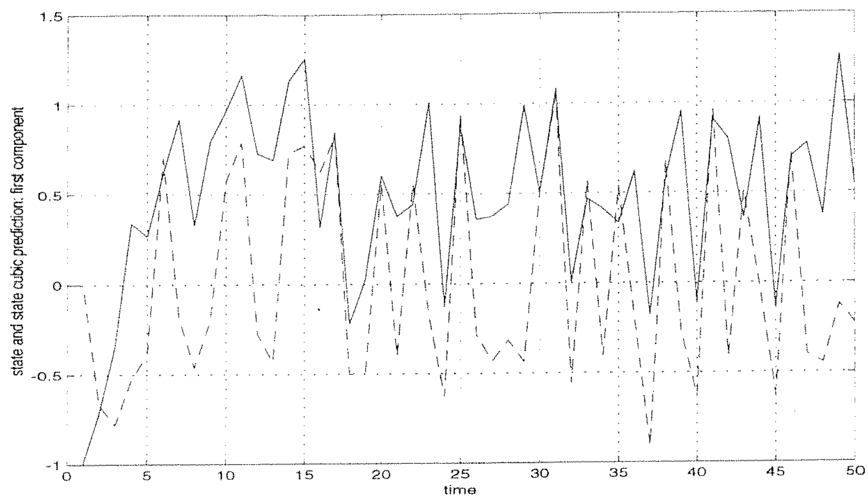

FIG. 5. *State (solid line) and state cubic prediction (dashed line) on $N = 50$ steps: first component.*

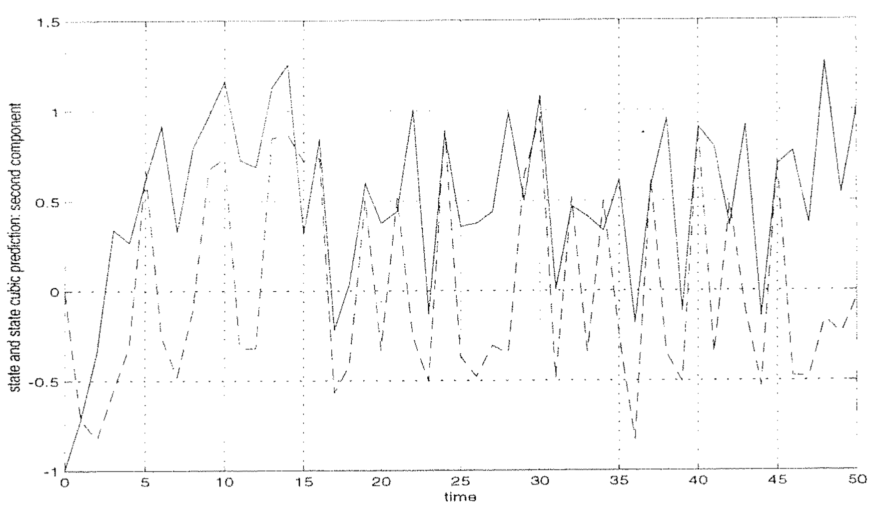

FIG. 6. *State (solid line) and state cubic prediction (dashed line) on $N = 50$ steps: second component.*

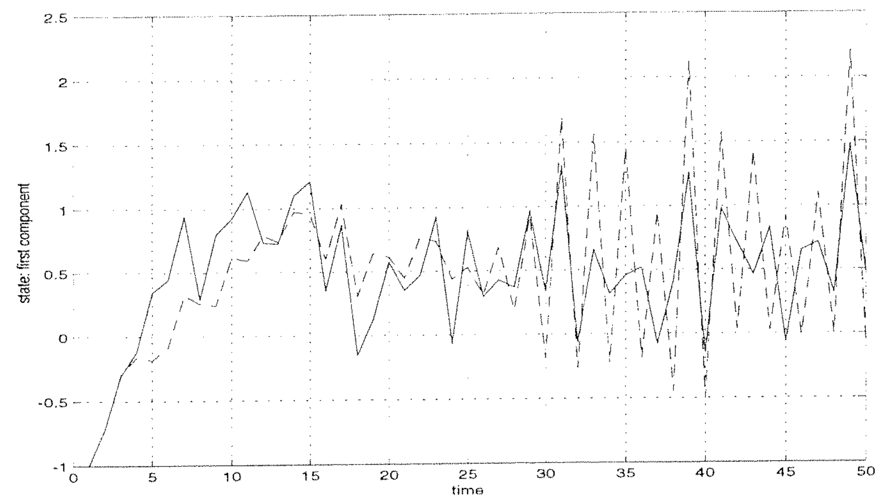

FIG. 7. *State first component on $N = 50$ steps: optimal linear regulator (dashed line) and optimal quadratic regulator (solid line).*

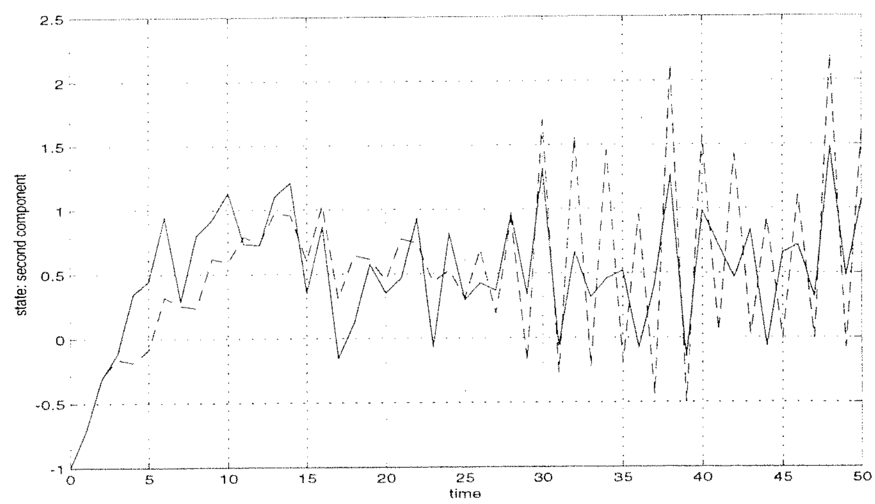

FIG. 8. *State second component on $N = 50$ steps: optimal linear regulator (dashed line) and optimal quadratic regulator (solid line).*

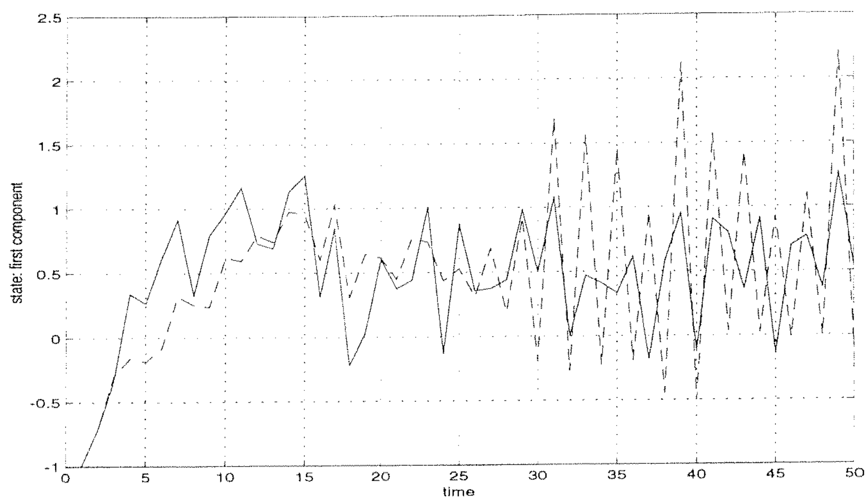

FIG. 9. *State first component on $N = 50$ steps: optimal linear regulator (dashed line) and optimal cubic regulator (solid line).*

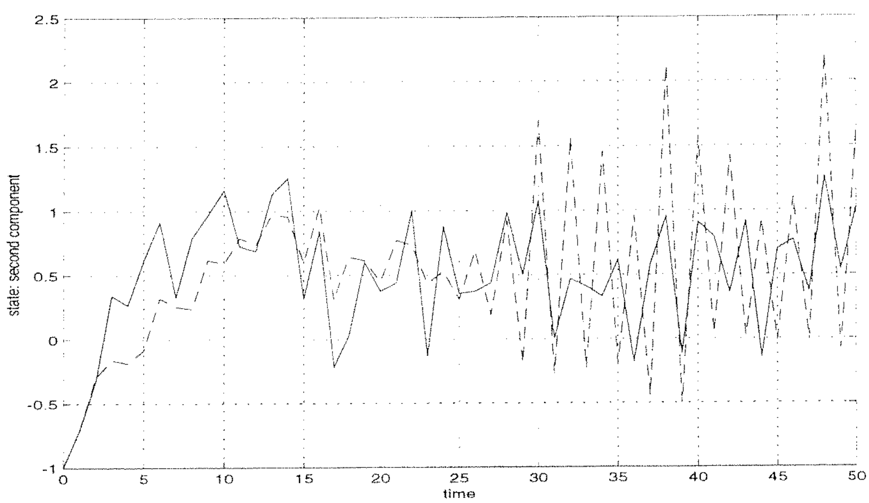

FIG. 10. *State second component on $N = 50$ steps: optimal linear regulator (dashed line) and optimal cubic regulator (solid line).*

In order to test further on the proposed method, numerical simulations have been performed for another example in which an unstable dynamics of the process and a signal-noise ratio decidedly lower than the first example one are considered. So we will see the algorithm issues in a more unfavourable case. The second example data are the following:

$$A = \begin{pmatrix} 0 & 1 \\ -0.909 & 1.910 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad F = \begin{pmatrix} F_1 & \vdots & 0 \end{pmatrix},$$
$$C = (\, 1 \quad 0 \,), \qquad G = \begin{pmatrix} 0 & \vdots & G_1 \end{pmatrix}, \tag{4.7}$$

with

$$F_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad G_1 = (\, 80 \,). \tag{4.8}$$

We will again consider the cost index (4.5)-(4.6) and the same probability distributions for the stochastic variables involved as in Table I and Table II.

The steady-state values of the state prediction error variance for respectively linear, quadratic and cubic controller are shown below:

$$\begin{aligned}
(\sigma^2_{e,1})_1 &= 599.06, \\
(\sigma^2_{e,1})_2 &= 660.93; \\
(\sigma^2_{e,2})_1 &= 438.91, \\
(\sigma^2_{e,2})_2 &= 488.16; \\
(\sigma^2_{e,3})_1 &= 365.17, \\
(\sigma^2_{e,3})_2 &= 410.18.
\end{aligned}$$

The reduction of the error variance for the quadratic case with respect to the linear one has been computed equal to 26.14%. The reduction for the cubic regulator with respect to the linear one is equal to 37.94%.

By comparing Figs. 11-12 with Figs. 13-14 and with Figs. 15-16, it can clearly seen (more clearly in this case than in the previous example) the lack of statistical informations that are utilized by the linear regulator with respect to the polynomial one. The notable effectiveness of the method even in this example results particularly clear from Figs. 17-20 which allow us to make a comparison between the performances of the linear regulator and the polynomial regulator ones respectively of the second and third order.

In the second example the cost criterion values for the three simulations respectively are the following:

$$\begin{aligned}
J_1 &= 4.9249 \cdot 10^4; \\
J_2 &= 1.1901 \cdot 10^4; \\
J_3 &= 7.1131 \cdot 10^3.
\end{aligned}$$

The notable cost reduction we obtain in this case is given by:

$$\begin{aligned}
\big((J_1 - J_2)/J_1\big) \cdot 100 &= 75.84\% \\
\big((J_1 - J_3)/J_1\big) \cdot 100 &= 85.56\%.
\end{aligned}$$

FIG. 11. *State (solid line) and state linear prediction (dashed line) on $N = 50$ steps: first component.*

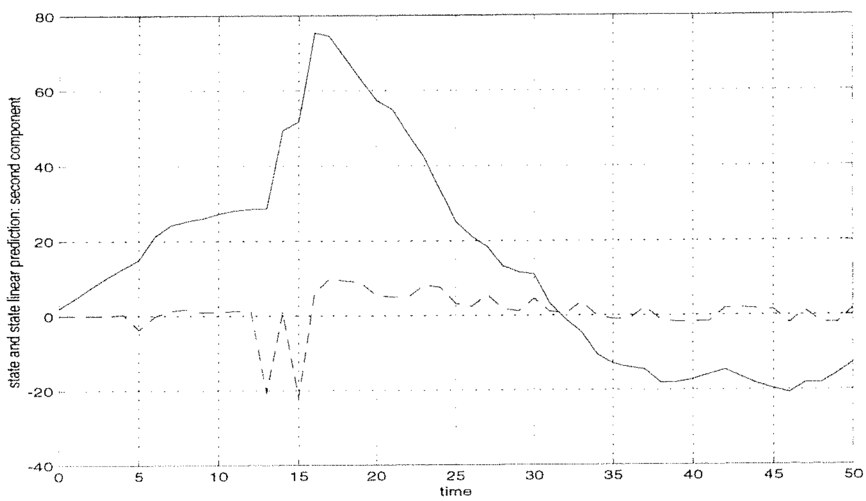

FIG. 12. *State (solid line) and state linear prediction (dashed line) on $N = 50$ steps: second component.*

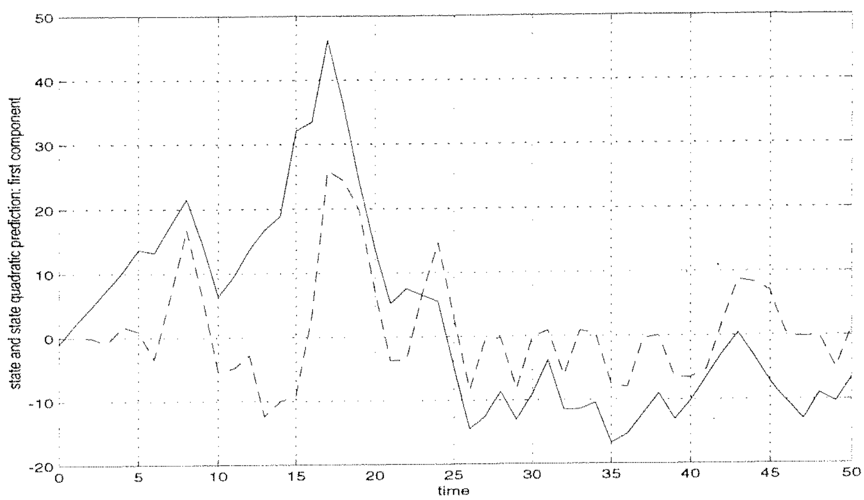

FIG. 13. *State (solid line) and state quadratic prediction (dashed line) on $N = 50$ steps: first component.*

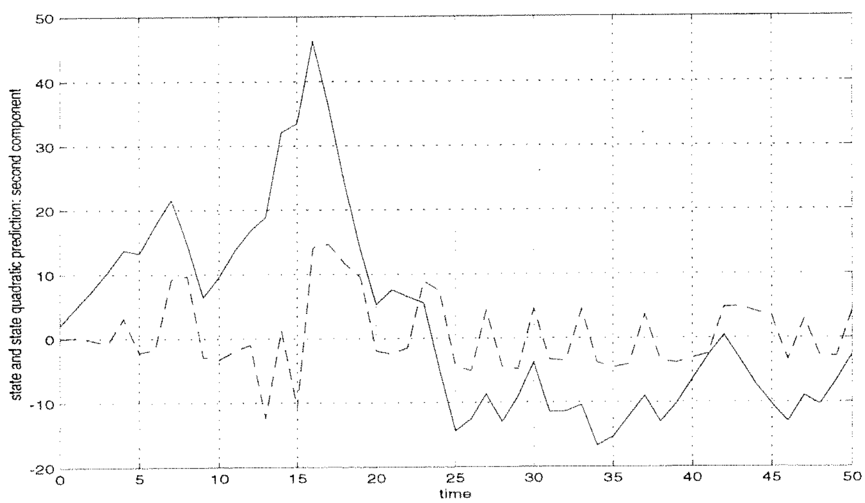

FIG. 14. *State (solid line) and state quadratic prediction (dashed line) on $N = 50$ steps: second component.*

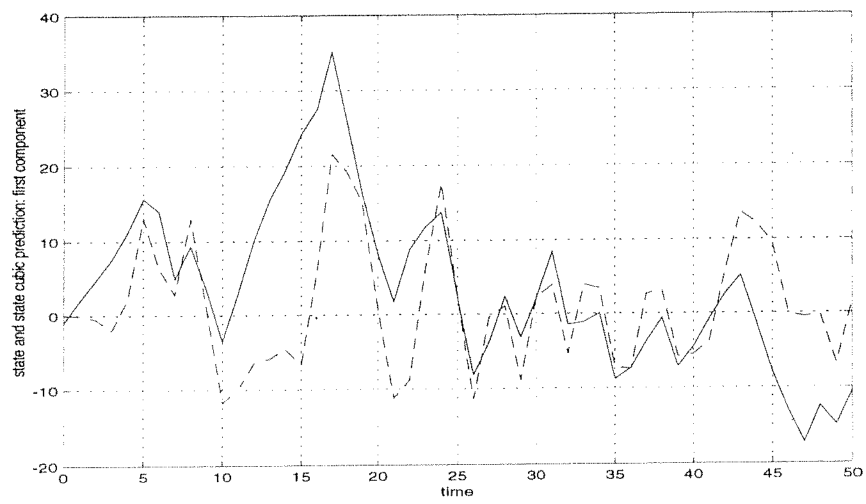

FIG. 15. *State (solid line) and state cubic prediction (dashed line) on $N = 50$ steps: first component.*

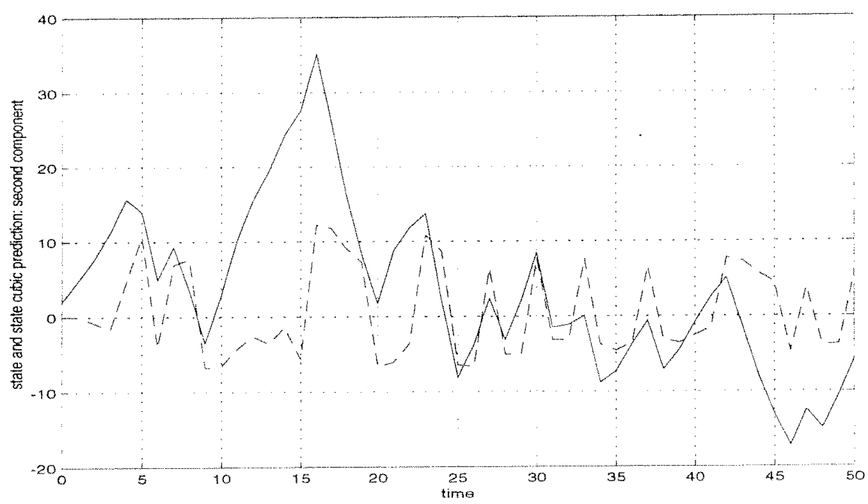

FIG. 16. *State (solid line) and state cubic prediction (dashed line) on $N = 50$ steps: second component.*

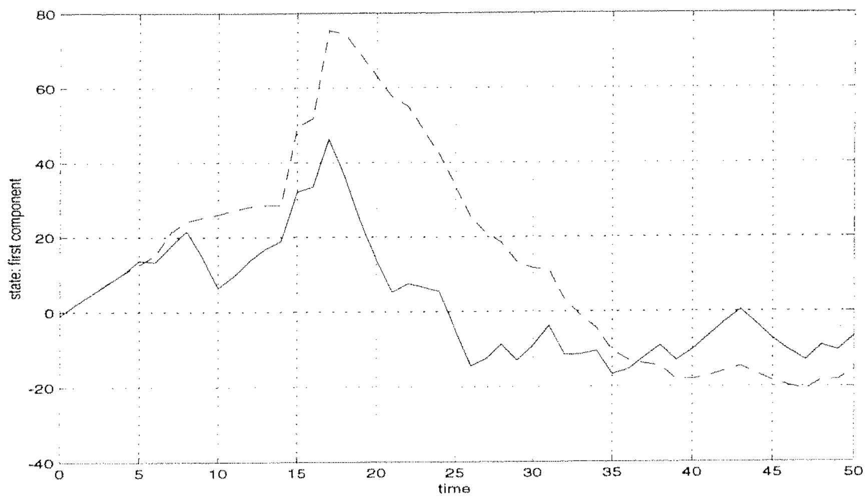

FIG. 17. *State first component on $N = 50$ steps: optimal linear regulator (dashed line) and optimal quadratic regulator (solid line).*

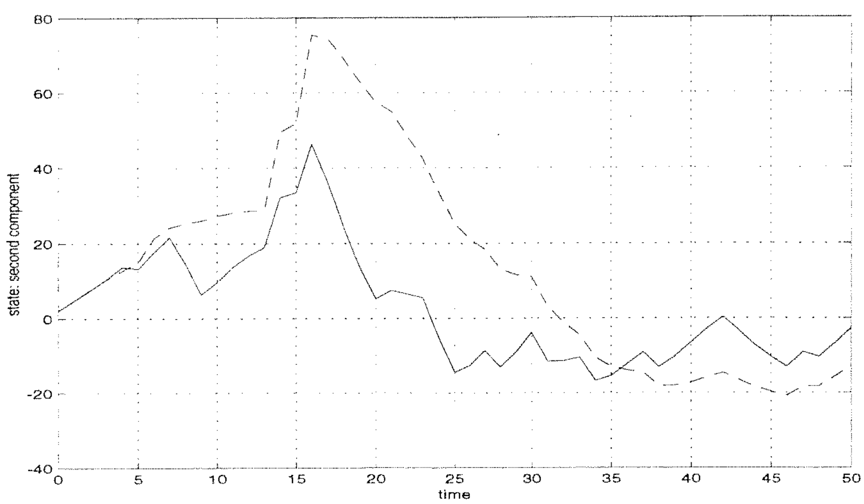

FIG. 18. *State second component on $N = 50$ steps: optimal linear regulator (dashed line) and optimal quadratic regulator (solid line).*

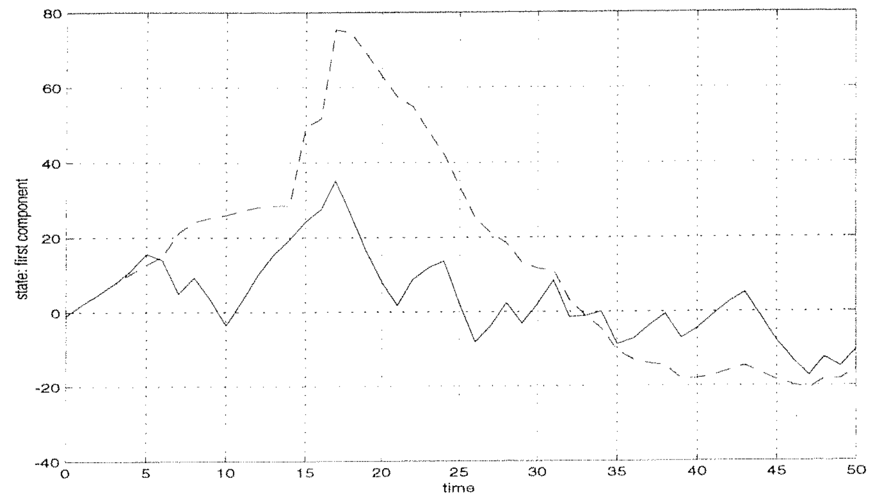

FIG. 19. *State first component on* $N = 50$ *steps: optimal linear regulator (dashed line) and optimal cubic regulator (solid line).*

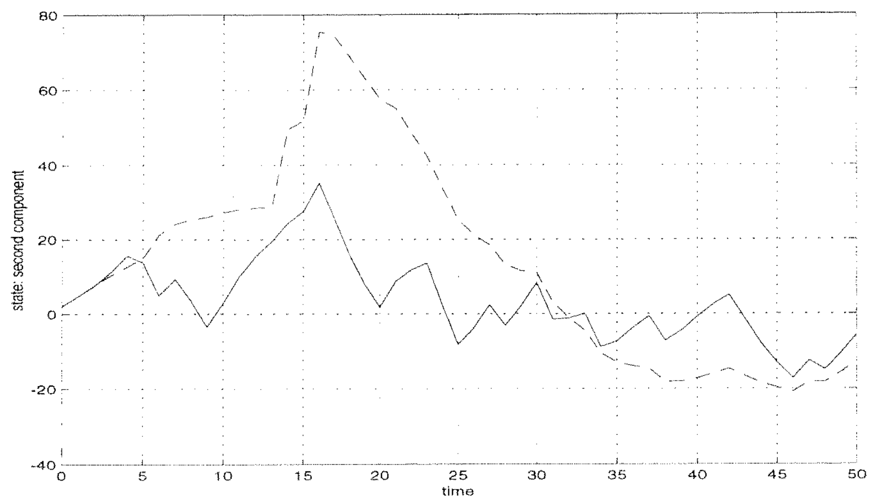

FIG. 20. *State second component on* $N = 50$ *steps: optimal linear regulator (dashed line) and optimal cubic regulator (solid line).*

Concluding remarks

In this paper we have defined the polynomial optimal algorithm of order ν for the solution of the linear quadratic non-Gaussian stochastic regulator problem. The main assumption is the knowledge of the first 2ν moments for the observation and state noises. It has been proved also in this case the validity of the separation principle. Some numerical examples show the effectiveness of the method.

APPENDIX A

The Kronecker algebra.

Definition A.1. Let M and N be matrices of dimensions $r \times s$ and $p \times q$ respectively, then the Kronecker product $M \otimes N$ is defined as the $(r \cdot p) \times (s \cdot q)$ matrix

$$M \otimes N = \begin{bmatrix} m_{11}N & \dots & m_{1s}N \\ \dots & \dots & \dots \\ m_{r1}N & \dots & m_{rs}N \end{bmatrix}$$

where the m_{ij} are the entries of M. Of course this kind of product is not commutative.

Definition A.2. Let M be the $r \times s$ matrix

$$M = [\, m_1 \quad m_2 \quad \dots \quad m_s \,] \tag{A.1}$$

where m_i denotes i-th column of M, then the stack of M is the $r \cdot s$ vector

$$st(M) = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_s \end{bmatrix}. \tag{A.2}$$

Observe that a vector as in (A.2) can be reduced to a matrix M as in (A.1) by considering the inverse operation of the stack denoted by st^{-1}. We refer to [23] for the main properties of the Kronecker product and stack operation. It is easy to verify that for $u \in \mathbb{R}^r$, $v \in \mathbb{R}^s$, the i-th entry of $u \otimes v$ is given by

$$(u \otimes v)_i = u_l \cdot v_m; \quad l = \left[\frac{i-1}{s}\right] + 1, \quad m = |i-1|_s + 1 \tag{A.3}$$

where $[\cdot]$ and $|\cdot|_s$ denote integer part and s-modulo respectively. Moreover, the Kronecker power of M is defined as

$$M^{[0]} = 1 \in \mathbb{R},$$
$$M^{[l]} = M \otimes M^{[l-1]} \quad l \geq 1.$$

Even if the Kronecker product is not commutative in general, the following result holds [24]

Theorem A.3. *For any given pair of matrices $A \in \mathbb{R}^{r\times s}$, $B \in \mathbb{R}^{n\times m}$, we have*

$$B \otimes A = C_{r,n}^T (A \otimes B) C_{s,m} \tag{A.4}$$

where $C_{r,n}$, $C_{s,m}$ are suitable $0-1$ matrices.

It is possible to show that $C_{u,v}$ is the $(u \cdot v) \times (u \cdot v)$ matrix such that its (h, l) entry is given by:

$$\{C_{u,v}\}_{h,l} = \begin{cases} 1, & \text{if } l = (|h-1|_v)u + \left(\left[\frac{h-1}{v}\right] + 1\right); \\ 0, & \text{otherwise.} \end{cases} \tag{A.5}$$

Observe that $C_{1,1} = 1$, hence in the vector case when $a \in \mathbb{R}^r$ and $b \in \mathbb{R}^n$, (A.4) becomes

$$b \otimes a = C_{r,n}^T (a \otimes b). \tag{A.6}$$

Moreover, in the vector case the commutation matrices satisfy also the following recursive formula.

Lemma A.4. *For any $a, b \in \mathbb{R}^n$ and for any $l = 1, 2...$, let $G_l = C_{n,n^l}^T$ so that*

$$b^{[l]} \otimes a = G_l (a \otimes b^{[l]}), \tag{A.7}$$

then the sequence $\{G_l\}$ satisfies the following equations

$$\begin{aligned} G_1 =& C_{n,n}^T & \text{(A.8)} \\ G_l =& (I_1 \otimes G_{l-1}) \cdot (G_1 \otimes I_{l-1}), \quad l > 1 & \text{(A.9)} \end{aligned}$$

where I_r is the identity matrix in $\mathbb{R}^{n^r}$.

Proof. Equation (A.6) assures the existence of the G_l's and implies (A.8). Moreover, using the associative property of the Kronecker product and recalling the identity

$$(A \cdot C) \otimes (B \cdot D) = (A \otimes B) \cdot (C \otimes D)$$

with $A = I_1$, we have

$$\begin{aligned} b^{[l]} \otimes a =& b \otimes b^{[l-1]} \otimes a \\ =& b \otimes \left(G_{l-1}(a \otimes b^{[l-1]}) \right) \\ =& (I_1 \otimes G_{l-1}) \cdot (b \otimes a \otimes b^{[l-1]}) \\ =& (I_1 \otimes G_{l-1}) \cdot \left((G_1(a \otimes b)) \otimes b^{[l-1]} \right) \\ =& (I_1 \otimes G_{l-1}) \cdot (G_1 \otimes I_{l-1}) \cdot (a \otimes b^{[l]}) \end{aligned}$$

then, equation (A.9) follows immediately by using (A.7). ∎

We can also find a binomial formula for the Kronecker power which generalizes the classical Newton one.

Theorem A.5. *For any integer $h \geq 0$ the matrix coefficients of the following binomial power formula:*

$$(a+b)^{[h]} = \sum_{k=0}^{h} M_k^h (a^{[k]} \otimes b^{[h-k]}) \tag{A.10}$$

constitute a set of matrices $\{M_0^h, ..., M_h^h\}$ such that

$$M_h^h = M_0^h = I_h \tag{A.11}$$
$$M_j^h = (M_j^{h-1} \otimes I_1) + (M_{j-1}^{h-1} \otimes I_1) \cdot (I_{j-1} \otimes G_{h-j}), \quad 1 \leq j \leq h-1 \tag{A.12}$$

where G_l and I_l are as in lemma A.4.

Proof. Equation (A.11) is obviously true for any h.

We will prove the (A.12) by induction for $h \geq 2$. For $h = 2$ it results

$$\begin{aligned}(a+b)^{[2]} &= a^{[2]} + a \otimes b + b \otimes a + b^{[2]} \\ &= a^{[2]} + (I_2 + G_1)(a \otimes b) + b^{[2]}\end{aligned} \tag{A.13}$$

where (A.7) has been used. Moreover using (A.12) we obtain

$$M_1^2 = (M_1^1 \otimes I_1) + (M_0^1 \otimes I_1)(I_0 \otimes G_1) = I_2 + I_2 G_1 = I_2 + G_1$$

so that the matrix coefficient of $a \otimes b$ in (A.10) (which is equal to $I_2 + G_1$ by (A.13)) agrees with the matrix M_1^2 computed by using (A.12). Now suppose that (A.12) is true for $h \geq 2$ then we will prove that it is true for $h+1$. We have

$$\begin{aligned}
(a+b)^{[h+1]} &= (a+b)^{[h]} \otimes (a+b) \\
&= \left(\sum_{k=0}^{h} M_k^h (a^{[k]} \otimes b^{[h-k]}) \right) \otimes (a+b) \\
&= \sum_{k=0}^{h} \left((M_k^h \otimes I_1) \cdot (a^{[k]} \otimes b^{[h-k]} \otimes a) + (M_k^h \otimes I_1) \cdot (a^{[k]} \otimes b^{[h+1-k]}) \right) \\
&= \sum_{k=0}^{h} \left((M_k^h \otimes I_1) \cdot \left(a^{[k]} \otimes \left(G_{h-k} (a \otimes b^{[h-k]}) \right) \right) \right. \\
&\qquad \left. + (M_k^h \otimes I_1) \cdot (a^{[k]} \otimes b^{[h+1-k]}) \right) \\
&= \sum_{k=0}^{h} (M_k^h \otimes I_1) \cdot (I_k \otimes G_{h-k}) \cdot (a^{[k+1]} \otimes b^{[h-k]}) \\
&\qquad + \sum_{k=0}^{h} (M_k^h \otimes I_1) \cdot (a^{[k]} \otimes b^{[h+1-k]})
\end{aligned}$$

hence, taking into account (A.10), we have

$$M_j^{h+1} = (M_j^h \otimes I_1) + (M_{j-1}^h \otimes I_1) \cdot (I_{j-1} \otimes G_{h+1-j}) \quad 1 \leq j \leq h.$$

∎

As far as the eigenvalues of a matrix obtained by Kronecker product of two square matrices is concerned, the following theorem has been proved [23].

Theorem A.6. *The eigenvalues of the matrix $A \otimes B$, where $A \in \mathbb{R}^n$ e $B \in \mathbb{R}^m$ are $\lambda_i \mu_j$, where λ_i are the A eigenvalues and μ_j the B ones.*

The relevant eigenvectors are given by:

$$z_{i,j} = \begin{bmatrix} x_1^{(i)} y^{(j)} \\ x_2^{(i)} y^{(j)} \\ \vdots \\ x_n^{(i)} y^{(j)} \end{bmatrix}$$

where $x_k^{(i)} i$, $k = 1, 2, ..., n$, are the components of the eigenvector $x^{(i)}$ of A, whereas $y^{(j)}$ is an eigenvector of B.

From the theorem A.6 it follows that the eigenvalues of a matrix $A^{[h]}$, $h \in \mathbb{N}$, are given by $\lambda_{i_1}^{h_1} \cdot \lambda_{i_2}^{h_2} \cdot ... \cdot \lambda_{i_n}^{h_n}$, where $i_1, ..., i_n \in \{1, ..., n\}$, the non-negative integers $h_1, ..., h_n$ are such that $h_1 + ... + h_n = h$ and $\lambda_{i_1}, ..., \lambda_{i_n}$ are the eigenvalues of the matrix $A \in \mathbb{R}^{n \times n}$.

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