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GENERAL POLYHEDRAL PROPERTIES OF INTEGER BLOCK STRUCTURED PROBLEMS

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Abstract

In this paper we investigate the relations between the polyhedron described by the inequalities of a block structured problem and the polyhedra described by the inequalities of its blocks. We identify certain classes of block structured problems for which zero lifting on facet inducing inequalities of the single blocks preserves the facet inducing property in the complete problem. Some applications are discussed.

Key words: Integer Programming; Block structured problems; Lifting procedures

1. Introduction

In this paper we consider block structured problems and discuss some conditions to lift the facet inducing inequalities of the polyhedra described by the solutions of the single blocks to inequalities that induce facets also for the complete problem.
We show the existence of block structured problems for which facet inducing inequalities of the polyhedron of a block can be lifted to facet inducing inequalities for the complete problems using zero coefficients for the variables in the other blocks. This is not true in general, as shown by the simple example at the end of this section.
In the next sections we give some sufficient conditions to characterize problems where all non trivial facet inducing inequalities for the polyhedra described by the single blocks maintain that property for the block structured problem with disjoint variable constraints. For brevity, hereafter we call a polyhedron described by the convex hull of solutions of a block structured formulation a *block structured polyhedron.*

Example 1.1. Let $n \geq 4$ be even and $e \in \mathbb{R}^n$ be the vector with all entries equal to one; define a set of points S as

$$x^1 \in S \subset \{0,1\}^n \Leftrightarrow e^T x^1 = n/2 \text{ or } e^T x^1 = n/2+1,$$

and let

$$Q = conv(S)$$

be the polyhedron defined by the convex hull of points in S.

Consider the following matrix:

$$A = \begin{bmatrix} I_{n/2} & J_{n/2} \\ J_{n/2} & I_{n/2} \end{bmatrix}$$

where $I_{n/2}$ is the identity matrix of order $n/2$ and $J_{n/2}$ is the square matrix of order $n/2$ with all entries equal to 1. Let $d = (n/2)^2 - 1$, one can verify that the following matrix is the inverse of the previous one:

$$\frac{1}{d}\begin{bmatrix} dI_{n/2} - (n/2)J_{n/2} & J_{n/2} \\ J_{n/2} & dI_{n/2} - (n/2)J_{n/2} \end{bmatrix}.$$

Therefore, the points described by the rows of A are affinely indepedent roots of the inequality $e^T x^1 \leq n/2+1$. As $e^T x^1 \leq n/2+1$ is trivially valid for Q and has n roots affinely independent, then it is a facet inducing inequality for Q.
Now, consider the polyhedron P defined as the convex hull of pairs of disjoint points in the set S, that is

$$P = conv\{(x^1, x^2) : x^1, x^2 \in S, x^1 + x^2 \leq e\}.$$

Hence, all facet inducing inequalities for the polyhedron Q are valid inequalities for P, and we can give an integer programming formulation for P considering the block structured system of inequalities given by the facet inducing inequalities for Q on both the variable vectors x^1 and x^2, plus the disjoint variable constraints $x^1 + x^2 \leq e$:

$$\begin{array}{ccccc} Ax^1 & & & \leq & b \\ & & Ax^2 & \leq & b \\ x^1 & + & x^2 & \leq & e \\ x^1 & , & x^2 & \in & \{0,1\}^n, \end{array}$$

where $Ax^i \leq b$ describes all the facets of Q.
The question that may arise is: when a facet inducing inequality for Q is also facet inducing for P? Clearly this is false for inequalities $x^1_j \leq 1$ for all j, as they are dominated by the disjunction conditions $x^1_j + x^2_j \leq 1$. Moreover, consider other facet inducing inequalities for Q: inequality $e^T x^1 \leq n/2 + 1$, for instance, is not facet inducing for P, as there are no pairs of disjoint points $x^1, x^2 \in S$ with $n/2 + 1$ entries equal to 1 in vector x^1. Thus, that inequality does not define even a proper face of P.
Conversely, $e^T x^1 \geq n/2$ is facet inducing for Q and also for P.
Thus, the above described block structured polyhedron presents certain facet inducing inequalities of the single blocks that cannot be lifted with zero coefficients to facet inducing inequalities of the whole polyhedron, and others that can. This concludes the example.

2. Lifting theorems

Consider the polyhedron P defined by the convex hull of all the $\{0,1\}$ solutions of the following block structured system:

$$\begin{array}{ccccccccc} A_1x^1 & & & & & & & \leq & b^1 \\ & & A_2x^2 & & & & & \leq & b^2 \\ & & & \ddots & & & & & \\ & & & & & & A_kx^k & \leq & b^k \\ x^1 & + & x^2 & \cdots & + & & x^k & \leq & e \\ x^i & \in & \{0,1\}^n & \forall\, i & = & & 1,\ldots,k. & & \end{array}$$

Let $P_i = conv\{x^i : A_i x^i \leq b^i, x^i \in \{0,1\}^n\}$ be the convex hull of the solutions of the i^{th} block of inequalities. In the following, we define a class of block structured polyhedra that show an appealing behaviour in lifting the facet defining inequalities of the polyhedra P_i.

Definition 2.1. *Let $\mathcal{C}_A$ be the class of block structured polyhedra for which the following conditions are verified by each polyhedron P_i:*

$$x^i = e_j \in P_i \quad \forall\, j = 1, \ldots, n, \tag{1}$$

$$x^i = \mathbf{0} \in P_i, \tag{2}$$

where e_j are the vectors of the canonical base of $\mathbb{R}^n$, and $\mathbf{0}$ is the zero vector in $\mathbb{R}^n$.

Note that if $P \in \mathcal{C}_A$, then all polyhedra P_i are full dimensional.

Theorem 2.1. *Let P be a block structured polyhedron belonging to class $\mathcal{C}_A$. If $\alpha x^h \leq \alpha_0$ is a facet inducing inequality for P_h and it is not equivalent to the inequalities $x^h_j \leq 1$, then $\alpha x^h \leq \alpha_0$ is also facet inducing for P.*

Proof. Without loss of generality we suppose $h = 1$; to simplify the notation, we drop the index h when appropriate.
Let $\bar{x}_1,\ \bar{x}_2,\ \ldots,\ \bar{x}_n$ be n affinely independent solutions in P_1 such that $\alpha\bar{x}_t = \alpha_0$ for all $t = 1,\ \ldots,\ n$.
First, we note that for each index $j = 1,\ \ldots,\ n$, there exists at least a solution $\bar{x}_t$ with the j^{th} component $(\bar{x}_t)_j = 0$; otherwise, all the solutions $\bar{x}_1,\ \bar{x}_2,\ \ldots,\ \bar{x}_n$ would satisfy the inequality $x^1_j \leq 1$ at equality, and the facet inducing inequality $\alpha x^1 \leq \alpha_0$ would be equivalent to $x^1_j \leq 1$, contradicting the hypothesis of the theorem.
Second, we prove that $\alpha x^1 \leq \alpha_0$ is facet inducing for P by sequentially lifting the variables of the other blocks. Let x^i_j be the variable to lift. Inequality $\alpha x^1 \leq \alpha_0$ is facet inducing for the face of P where all the variables in the blocks $2,\ \ldots,\ k$ are constrained to be zero. So we have to lift x^i_j on the polyhedron P with the condition $x^i_j = 1$.
The lifting coefficient is found by applying the procedure for sequential lifting (see e.g. [10], [11], [9]). Let η^i_j be the optimal value of the following problem:

$$\begin{aligned} \eta^i_j \quad = \quad & \max \alpha x^1 \\ & (x^1, x^2, \ldots, x^k) \in P' \\ & x^i_j = 1, \end{aligned} \tag{3}$$

where P' is the projection of P obtained by setting to zero all non lifted variables except x^i_j.
The lifting coefficient for x^i_j is equal to $\alpha^i_j = \alpha_0 - \eta^i_j$.
As $P \in \mathcal{C}_A$, $x^i = e_j$ is a feasible solution for P_i, and it satisfies the constraint $x^i_j = 1$. Hence there exists a solution $x^1 = \bar{x}_t$ with $(\bar{x}_t)_j = 0$, and so completing with zeroes for the already lifted variables, x^1 is feasible for the lifting problem and $\eta^i_j = \alpha\bar{x}_t = \alpha_0$, i.e. $\alpha^i_j = 0$.
By induction on the variables to lift we get that the lifting coefficients $\alpha^i_j = 0$ for all $j = 1,\ \ldots,\ n$, and for all $i = 2,\ \ldots,\ k$, and therefore $\alpha x^1 \leq \alpha_0$ is facet inducing for P. ∎

From the above proof, it is easy to see that conditions (1) and (2) are very strong and are used only partially. Indeed, the essential requirement is that a solution with $x^i_j = 1$ is disjoint with a solution $\bar{x}_t$ and that it can be completed for the variables already lifted. We can thus enlarge class $\mathcal{C}_A$ by the following:

Definition 2.2. *Let $\mathcal{C}_B$ be the class of block structured polyhedra for which the following condition is verified: for all variable indeces $j = 1,\ \ldots,\ n$, for all pairs of blocks $i, h = 1,\ \ldots,\ k$, and for all solutions $x^h \in P_h$ such that $x^h_j = 0$, there exists a solution $(x^1, x^2, \ldots, x^k) \in P$ such that $x^i_j = 1$.*

Note that if $P \in \mathcal{C}_B$, then the polyhedra P_i may not have full dimension.

With some trivial changes we extend the results of theorem 2.1 to the larger class of block structured polyhedra $\mathcal{C}_B$. Indeed, in the proof of theorem 2.1, we substitute the solution $x^i = e_j$ coming from (1), with the solution x^i coming from definition of class $\mathcal{C}_B$; we then consider x^h equal to the same solution $\bar{x}_t$ chosen in the proof and finally complete the solution for the already lifted variables with the values of the solution $(x^1, x^2, \ldots, x^k) \in P$ in the definition of class $\mathcal{C}_B$.

Note that, in theorem 2.1 we have excluded the case of the inequality $x^h_j \leq 1$, as this inequality is dominated by the condition of disjunction on the variables. We can state a similar result also for these inequalities.

Theorem 2.2. *Let P be a block structured polyhedron belonging to class $\mathcal{C}_B$. If, for a block h, $x_j^h \leq 1$ is facet inducing for P_h, then $\sum_{i=1}^k x_j^i \leq 1$ is facet inducing for P.*

Proof. Without loss of generality we can suppose $h = 1$. The proof is again by sequential lifting and induction on the variables to lift.
Consider the following two cases:

a) lifting x_j^i with $i \neq 1$;

b) lifting $x_{j'}^i$ with $i \neq 1$ and $j' \neq j$.

In both cases the inequality obtained after s steps of lifting is facet inducing on the projection of the polyhedron on $x_{j'}^i = 0$ for all not yet lifted variables, and so the lifting procedure is equal to the one described in the proof of theorem 2.1, with objective function equal to $\sum_{l \in S} x_j^l$ where S is the set of indeces l such that x_j^l has already been lifted. Moreover, $\alpha_0 = 1$.
In case (a), we can choose a solution x with $x_j^i = 1$ (it must exist as $P \in \mathcal{C}_B$); the optimum of the lifting problem is then zero and the lifting coefficient is equal to $\alpha_j^i = \alpha_0 - \eta_j^i = 1 - 0 = 1$.
In case (b), we can choose a solution x with $x_{j'}^i = 1$ and $x_j^1 = 1$, as they are not restricted by disjunction constraints, so we get $\eta_{j'}^i = 1$ and $\alpha_j^i = 1 - 1 = 0$. ∎

2.1. Extentions

The results derived in the previous section can be generalized to cases where the constraints joining the blocks are somewhat more general than the disjunction conditions on all the variables. Consider, for instance, the case where the constraints linking the blocks are generalized upper bounds:

$$\sum_{i=1}^{k} x_j^i \leq u_j \quad \forall\, j = 1, \ldots, n.$$

To extend theorem 2.1, we have to lift the variables of the other blocks for all the values they may assume, that is, the values from 1 to the corresponding u_j.
Now theorem 2.1 can be reformulated for general integer programming problems with generalized upper bound constraints as follows:

Theorem 2.3. *Let P be a block structured polyhedron belonging to class $\mathcal{C}_A$. If $\alpha x^h \leq \alpha_0$ is a facet inducing inequality for P_h, and the following conditions hold:*

a) $\alpha x^h \leq \alpha_0$ is not equivalent to an inequality $x_j^h \leq r_j$ with $r_j \geq u_j$,
b) $\alpha x^h \leq \alpha_0$ has no roots with $x_j^h > u_j$,

then $\alpha x^h \leq \alpha_0$ is also facet inducing for P.

The proof is very similar to the one for binary variables, but the lifting procedure needs some changes (see [11]). The lifting coefficient is now defined as

$$\alpha_j^i = \min_{1 \leq r \leq u_j} \frac{\alpha_0 - \eta_j^i(r)}{r},$$

where $\eta_j^i(r)$ is defined as

$$\begin{aligned} \eta_j^i(r) &= \max \alpha x^1 \\ & (x^1, x^2, \ldots, x^k) \in P' \\ & x_j^i = r. \end{aligned}$$

Note that, when $r = 1$, $\eta_j^i(r)$ is equal to η_j^i of problem (3). As a consequence of the new lifting procedure, conditions (a) and (b) imply that not all tight solutions for inequality $\alpha x^h \leq \alpha_0$ have $x_j^h = u_j$, and therefore $\alpha_j^i = 0$ since $\eta_j^i(r) = \alpha_0$ for $r = 1$.
Theorem 2.3 can be proved also for polyhedra in the larger class $\mathcal{C}_B$, once the following slight modification is considered.

Modification of class $\mathcal{C}_B$:
for all variable indeces $j = 1, \ldots, n$, for all pairs of blocks $i, h = 1, \ldots, k$, and for all solutions $x^h \in P_h$ such that $x_j^h < u_j$, there exists a solution $(x^1, x^2, \ldots, x^k) \in P$ such that $x_j^i \geq 1$.

Both theorems 2.1 and 2.3 consider problems where all non trivial facet inducing inequalities for the single blocks are lifted to facets of the complete polyhedron. In the previous section we presented a simple example of a problem where some inequalities cannot be lifted to facet inducing inequalities for the block structured polyhedron, and others can. Such cases can be dealt by considering in the definition of class $\mathcal{C}_B$ only solutions x^h which are tight for a specific facet inducing inequality $\alpha x^h \leq \alpha_0$.

3. Applications

In this section we discuss some block structured problems where the single blocks are known problems whose polyhedral structure has been extensively studied in the literature. We show that the polyhedra of these problems belong to class $\mathcal{C}_B$ or $\mathcal{C}_A$ and therefore facet inducing inequalities for the single blocks also define facets for the complete problem.

3.1. The Disjoint Perfect Matchings Problem

Consider the following problem: given a graph G with an even number of nodes, find k disjoint perfect matchings minimizing a linear function on the edges of G. This problem can be formulated using a system with k blocks, where each block is described by the inequalities of the perfect matching polytope. Note that the block formulation allows to express different edge costs in different blocks.
If we restrict the problem using the same cost vector on each block and consider $k = 2$, the Disjoint Perfect Matchings Problem consists in finding a minimum cost cover of the nodes with cycles of even length. Papadimitriou proved that the problem of determining whether a graph contains a perfect binary 2-matching with no polygon of size 5 or less is NP-complete (see Theorem 5.1 in [1]). Using the same proof we can also derive that the problem of determining whether a graph contains a cover of the nodes with cycles of even lenghts is NP-complete. Thus the Disjoint Perfect Matchings Problem is NP-hard even when $k = 2$.
Another proof of NP-completeness is by reduction of the Edge Coloring Problem: let G be a graph of maximum degree k, we can reduce the Edge Coloring Problem to find k minimum disjoint perfect matchings on the complete graph K with the same number of nodes (if it is odd, add an isolated node in G). Edge costs are -1 for edges in G and 0 for edges in $K \setminus G$. Clearly

G is k edge colorable if and only if there exist k disjoint perfect matchings of value $-m$ in K, where m is the number of edges in G; otherwise G is $k+1$ edge colorable (see Vizing's theorem in [4]). The NP-completeness of the Edge Coloring Problem has been shown in [6]. As the Edge Coloring Problem is NP-complete for k-regular graphs so it is the k Disjoint Perfect Mathings Problem.
The polyhedron associated to the disjoint perfect matchings is in $\mathcal{C}_B$, assuming the graph G is complete. A problem on a generic graph G can be looked at as a problem on the complete graph K_{2n} with the same number of nodes and large weights on the edges of K_{2n} that are not in G.
To verify that the polyhedron defined by the convex hull of the solutions of the problem on graph K_{2n} belongs to $\mathcal{C}_B$, it is sufficient to show that: if there exist in K_{2n} k disjoint perfect matchings (i.e. the feasible solution set is not empty), then for each edge there exists a perfect matching containing that edge. This is obviously true since K_{2n} is symmetric.
As the Disjoint Perferct Matchings Problem is in $\mathcal{C}_B$, one can thus apply the facet inducing inequalities of the single Perfect Matching Problem (e.g. [2]).

3.2. The Peripatetic Salesman Problem

Given a set of n cities, find k edge-disjoint TSP tours minimizing a linear function. As for the previous problem, the costs may differ if edges are chosen for different tours. This problem is modeled by a system with k blocks, where each block is described by a formulation of the TSP problem. Clearly, this problem is NP-hard, since it is so when $k = 1$.
For an introduction to the Peripatetic Salesman Problem we refer to [8].
Also the polyhedron of this problem is in $\mathcal{C}_B$, as the problem is completely symmetric and for each edge there exists a TSP tour containing it; moreover, it can be completed with $k-1$ edge-disjoint TSP tours, provided that k disjoint tours exist in the complete graph K_n.
As the Peripatetic Salesman Problem is in $\mathcal{C}_B$, one can thus apply the facet inducing inequalities of the single TSP (e.g. [7]).

3.3. The Multiple Knapsack Problem

Consider a set of k knapsacks and a set of n items. Each knapsack has a capacity b^i for $i = 1, \ldots, k$. For every knapsack i, each item j has a weight $a^i_j > 0$ and a benefit c^i_j. The problem is to maximize the total benefit respecting the knapsack constraints, and assigning each item to at most one of the knapsacks. The block formulation of the problem is presented below:

$$\begin{aligned} \max \quad & \sum_{i=1}^{k}\sum_{j=1}^{n} c^i_j x^i_j \\ & \sum_{j=1}^{n} a^i_j x^i_j \le b^i && \forall\, i = 1, \ldots, k \\ & \sum_{i=1}^{k} x^i_j \le 1 && \forall\, j = 1, \ldots, n \\ & x^i_j \in \{0,1\} && \forall\, i = 1, \ldots, k \ \ \forall\, j = 1, \ldots, n. \end{aligned}$$

The polyhedral structure of the Multiple Knapsack Problem has been extensively studied in [5] and [3]. We can see that the resulting polyhedron is in $\mathcal{C}_A$, provided that the solution with no elements and the solutions with a single element selected are feasible for all knapsack subproblems. Otherwise, if an element j cannot be selected for a certain block i (i.e. $a^i_j > b^i$),

then each facet inducing inequality $\alpha x^h \leq \alpha_0$ defined on the variables of another block h can be tightened by adding x^i_j. This case can be ignored as the variable x^i_j is always zero and can be dropped from the problem. Equivalent results for the Multiple Knapsack Problem are also obtained in [5].
As a last remark, note that class $\mathcal{C}_A$ contains all block structured polyhedra where the feasible solutions of each block of the formulation can be described by a system of independent sets (e.g.[9]).

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