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ON THE SHAMEFUL CONJECTURE

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Abstract

Let G be a graph with n vertices and let $P(G, \lambda)$ be its chromatic polynomial. It was conjectured by Bartels and Welsh that

$$\frac{P(G, n-1)}{P(G, n)} \leq \left(\frac{n-1}{n}\right)^n .$$

This conjecture remains open and is known as the *Shameful Conjecture.* We propose two stronger conjectures and we show their validity for several classes of graphs.

1. Introduction

We assume familiarity with basic graph theory, and in particular with some well-known results on the chromatic polynomial of a graph (see for instance Chapter 15 of [1]). Let G be a graph with n vertices; if G has no edges (empty graph) then we shall denote it by O_n, if G has all possible edges then we shall denoted it by K_n, if G is a tree then we shall denote it by T_n, and if G is a cycle then we shall denote it by C_n.

A *universal* vertex in a graph is a vertex which is adjacent to all other vertices. A *clique-cutset* in a graph G is a clique whose removal from G disconnects the graph. If G has induced subgraphs G_1, G_2 and K_t such that $G = G_1 \cup G_2$ and $K_t = G_1 \cap G_2$, then we say that G arises from G_1 and G_2 by *clique identification* (see Figure 1). Clearly, if G arises by clique identification from two other graphs, then G has a clique-cutset.

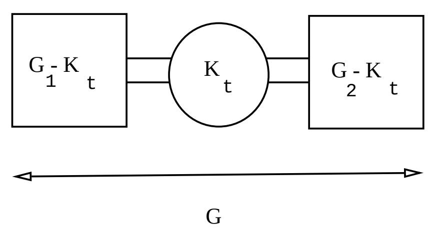

Figure 1: Clique cutset K_t

A graph is *chordal* (or *triangulated*) if it contains no induced cycles other than triangles. It is well known that a graph is chordal if it can be constructed recursively by clique identifications, starting from complete graphs.

Let uv be an edge of a graph G. By $G_{|uv}$ we denote the graph obtained from G by *contracting* the edge uv into a new vertex which becomes adjacent to all the former neighbours of u and v. We say that G is *contractable* to a graph F if G contains a subgraph that becomes F after a series of edge contractions and edge deletions. A graph is *series-parallel* if it is not contractable to K_4.

Finally, let uv be an edge of a graph G. *Subdividing* the edge uv means to delete uv and add a new vertex x which is adjacent to only u and v. It is well known that a series-parallel multigraph can be constructed recursively from a K_2 by the operations of subdiving and of doubling edges.

As usual, $\omega(G)$ denotes the clique number of G and $\chi(G)$ denotes the chromatic number of G. If we denote by $c_k(G)$ the number of ways to colour the vertices of G with exactly k colours (so that no two adjacent vertices get the same colour), then the *chromatic polynomial* of G is:

$$P(G, \lambda) = \sum_{k=1}^{n} c_k(G)(\lambda)_k,$$

where $(\lambda)_k = \binom{\lambda}{k} k!$.

Bartels and Welsh [2] made the following conjecture:

Conjecture 1. *For every graph G with n vertices,*

$$\frac{P(G, n-1)}{P(G, n)} \leq \left(\frac{n-1}{n}\right)^n \leq \frac{1}{e} = .367879...$$

This is known as the *Shameful Conjecture* and it is still open. Recently, Seymour [4] proved that

$$\frac{P(G,n-1)}{P(G,n)} \leq \frac{252}{685} = .367883...$$

approaching Conjecture 1.

The aim of this paper is to prove that the Shameful Conjecture holds for several classes of graphs. In fact, we shall prove that for these classes of graphs two stronger conjectures hold.

Conjecture 2. *Let G be a graph with n vertices. Then,*

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} \leq \frac{\lambda-\chi(G)}{\lambda}\left(\frac{\lambda-1}{\lambda}\right)^{n-\chi(G)} \qquad \forall\lambda \geq n; \tag{1}$$

Conjecture 3. *Let G be a graph with n vertices. Then,*

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} \leq \frac{\lambda-\omega(G)}{\lambda}\left(\frac{\lambda-1}{\lambda}\right)^{n-\omega(G)} \qquad \forall\lambda \geq n. \tag{2}$$

In the next section we shall show that Conjecture 2 is stronger than Conjecture 3 and that Conjecture 3 is stronger than the Shameful Conjecture.

Note that since $P(O_n,\lambda) = \lambda^n$, Conjecture 1 can be stated as:

$$\frac{P(G,n-1)}{P(G,n)} \leq \frac{P(O_n,n-1)}{P(O_n,n)}.$$

If for every graph G with n vertices, we denote by $\tilde{G}$ the graph given by the disjoint union of $K_{\chi(G)}$ and $O_{n-\chi(G)}$ (respectively, the disjoint union of $K_{\omega(G)}$ and $O_{n-\omega(G)}$) then (1) (respectively (2)) becomes:

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} \leq \frac{P(\tilde{G},\lambda-1)}{P(\tilde{G},\lambda)} \quad \forall\lambda \geq n.$$

Note that Conjecture 1 would immediately be true if for every edge e of G,

$$\frac{P(G,n-1)}{P(G,n)} \leq \frac{P(G'_e,n-1)}{P(G'_e,n)},$$

where G'_e denotes the graph G after the removal of edge e. Recently, Mosca [3] proved this to be false.

We close this section by noting that we could have thought of strengthening inequality (1) (and so (2)), by writing for instance,

$$\frac{P(G,\lambda-1)}{P(G,\lambda)} \leq \frac{\lambda-\chi(G)}{\lambda}\left(\frac{\lambda-1}{\lambda}\right)^{n-\omega(G)} \qquad \forall\lambda \geq n.$$

However such an inequality would be false in general; consider, for instance, the case $G = C_5$ and $\lambda = 5$.

2. The Results

In this section we shall first show that Conjecture 1 is implied by Conjecture 3 which is implied by Conjecture 2. Secondly, we shall illustrate some operations on graphs that preserve the validity of Conjecture 2 (or Conjecture 3). These operation will allow us to prove the validity of Conjecture 2 (or Conjecture 3), and so of the Shameful Conjecture for several classes of graphs.

For this purpose, let first prove the following easy inequality that will be often used in the paper:

$$\frac{m-k}{m} \leq \left(\frac{m-1}{m}\right)^k \quad \text{for every two integers } m, k \text{ with } m \geq k \geq 0. \tag{3}$$

The validity of (3) immediately comes from the following inequality, which is strict if $k > 1$:

$$\frac{m-k}{m} = \prod_{i=1}^{k} \frac{m-i}{m-i+1} \leq \prod_{i=1}^{k} \frac{m-1}{m}.$$

Now, (3) immediately implies that Conjecture 2 is stronger than Conjecture 3 (write (3) with $m = \lambda - \omega(G)$ and $k = \chi(G) - \omega(G)$). To see that Conjecture 3 implies Conjecture 1, write (2) with $\lambda = n$, that is:

$$\frac{P(G, n-1)}{P(G, n)} \leq \frac{n - \omega(G)}{n} \left(\frac{n-1}{n}\right)^{n-\omega(G)},$$

and apply inequality (3) with $m = n$ and $k = \omega(G)$. Conjecture 2 and 3 are stronger than Conjecture 1 for all graphs G except O_n and K_n (when all three conjectures are trivial), since $2 \leq \omega(G) \leq \chi(G) \leq n-1$.

The next four theorems give operations for building families of graphs which satisfy Conjecture 2 (or Conjecture 3) from the basic graph O_1.

Theorem 1. *(Disjoint union) Let H be a graph obtained from two graphs G_1 and G_2 by disjoint union. If both G_1 and G_2 satisfy Conjecture 2 then H also satisfies Conjecture 2.*

Theorem 2. *(Add a universal vertex) Let H be a graph obtained from some graph G by adding a universal vertex. If G satisfies Conjecture 2 then H also satisfies Conjecture 2.*

Theorem 3. *(Clique identification) Let H be a graph obtained from two graphs G_1 and G_2 by clique identification. If both G_1 and G_2 satisfy Conjecture 2 then H also satisfies Conjecture 2.*

The following corollary follows immediately from Theorems 2 and 3:

Corollary 1. *Every chordal graph satisfies Conjecture 2.*

In particular, the empty graphs and the trees satisfy Conjecture 2.

Theorem 4. *(Edge subdivision) Let G be a graph with n vertices, let uv be an edge of G, let r be a positive integer, and let H be the graph obtained from G by deleting edge uv and by adding the new vertices $x_1, \cdots, x_r$ and connecting each of them to both u and v (see Figure 2). If the following two properties hold*

(a) $\min\{d_G(u), d_G(v)\} \leq \frac{n+r+1}{2}$,

(b) both G and $G_{|uv}$ satisfy Conjecture 3,

then the graph H also satisfies Conjecture 3.

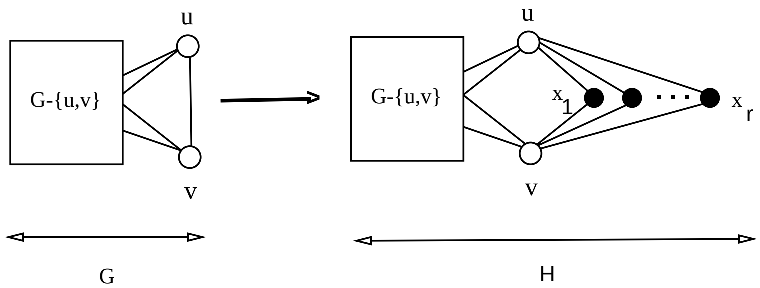

Figure 2: Subdivide edge uv

Theorems 3 and 4 can be used to prove:

Theorem 5. *Every series-parallel graph satisfies Conjecture 3.*

In [2] it was shown that the Shameful Conjecture holds also for cycles. Theorem 5 immediately implies that cycles also satisfy Conjecture 3. In fact, we can show that:

Theorem 6. *The cycles C_n satisfy Conjecture 2.*

An immediate corollary of our results is the following:

Corollary 2. *Chordal graphs and series-parallel graphs satisfy the Shameful Conjecture.*

3. The Proofs

3.1. Proof of Theorem 1

Assume that H has n vertices. Let n_i denote the number of vertices of G_i $(i = 1, 2)$ and let $\lambda \geq n (= n_1 + n_2)$. Assume that $\chi(H) = \chi(G_2) \geq \chi(G_1)$. Since

$$P(H, \lambda) = P(G_1, \lambda)P(G_2, \lambda),$$

we have

$$\frac{P(H, \lambda - 1)}{P(H, \lambda)} = \frac{P(G_1, \lambda - 1)}{P(G_1, \lambda)} \frac{P(G_2, \lambda - 1)}{P(G_2, \lambda)}.$$

Since both G_1 and G_2 satisfy Conjecture 2, we have

$$\frac{P(H, \lambda - 1)}{P(H, \lambda)} \leq \frac{\lambda - \chi(G_1)}{\lambda} \frac{\lambda - \chi(G_2)}{\lambda} \left(\frac{\lambda - 1}{\lambda}\right)^{n - \chi(G_1) - \chi(G_2)}.$$

But (3) implies that

$$\frac{\lambda - \chi(G_1)}{\lambda} \left(\frac{\lambda - 1}{\lambda}\right)^{-\chi(G_1)} \leq 1,$$

and so we are done.

3.2. Proof of Theorem 2

Assume that H has n vertices. Write $\chi = \chi(H) = \chi(G) + 1$ and let $\lambda \geq n$. Since $P(H, \lambda) = \lambda P(G, \lambda - 1)$, we have:

$$\frac{P(H,\lambda-1)}{P(H,\lambda)} = \frac{\lambda-1}{\lambda}\,\frac{P(G,\lambda-2)}{P(G,\lambda-1)}.$$

But then, since G satisfies Conjecture 2,

$$\frac{P(H,\lambda-1)}{P(H,\lambda)} \leq \frac{\lambda-1}{\lambda}\,\frac{\lambda-\chi}{\lambda-1}\left(\frac{\lambda-2}{\lambda-1}\right)^{n-\chi},$$

and so we are done because $\dfrac{\lambda-2}{\lambda-1} < \dfrac{\lambda-1}{\lambda}$.

3.3. Proof of Theorem 3

Set $\chi = \chi(H)$. Without loss of generality, we can assume that $\chi(G_2) \geq \chi(G_1)$, and so $\chi = \chi(G_2)$. Let n_i denote the number of vertices of G_i $(i = 1, 2)$ and let $G_1 \cap G_2 = K_t$. Clearly, H has $n = n_1 + n_2 - t$ vertices. Now, let $\lambda \geq n$. Since

$$P(H,\lambda) = \frac{P(G_1,\lambda)P(G_2,\lambda)}{P(K_t,\lambda)} = \frac{P(G_1,\lambda)P(G_2,\lambda)}{(\lambda)_t},$$

we have

$$\frac{P(H,\lambda-1)}{P(H,\lambda)} = \frac{\lambda}{\lambda-t}\,\frac{P(G_1,\lambda-1)}{P(G_1,\lambda)}\,\frac{P(G_2,\lambda-1)}{P(G_2,\lambda)}.$$

Since both G_1 and G_2 satisfy (1), we have

$$\frac{P(H,\lambda-1)}{P(H,\lambda)} \leq \frac{\lambda}{\lambda-t}\,\frac{\lambda-\chi(G_1)}{\lambda}\,\frac{\lambda-\chi(G_2)}{\lambda}\left(\frac{\lambda-1}{\lambda}\right)^{n_1+n_2-\chi(G_1)-\chi(G_2)},$$

that is

$$\frac{P(H,\lambda-1)}{P(H,\lambda)} \leq \frac{\lambda-\chi(G_1)}{\lambda-t}\left(\frac{\lambda-1}{\lambda}\right)^{t-\chi(G_1)}\frac{\lambda-\chi}{\lambda}\left(\frac{\lambda-1}{\lambda}\right)^{n-\chi}.$$

Hence, to prove the theorem, we only need show that

$$\frac{\lambda-\chi(G_1)}{\lambda-t}\left(\frac{\lambda-1}{\lambda}\right)^{t-\chi(G_1)} \leq 1,$$

that is

$$\left(\frac{\lambda-1}{\lambda}\right)^{\chi(G_1)-t} \geq \frac{\lambda-\chi(G_1)}{\lambda-t}.$$

Now, since $\chi(G_1) \geq t$, (3) (with $m = \lambda$ and $k = \chi(G_1) - t$) implies that

$$\left(\frac{\lambda-1}{\lambda}\right)^{\chi(G_1)-t} \geq \frac{\lambda-\chi(G_1)+t}{\lambda}.$$

But

$$\frac{\lambda-\chi(G_1)+t}{\lambda} \geq \frac{\lambda-\chi(G_1)}{\lambda-t},$$

and so we are done.

Before proving Theorem 4, we need the following technical lemma.

Lemma 1. *Let x and r be two integers with $x > r \geq 1$. Then*

$$\frac{(x-1)^r(x+1)^{r+1} - x^{2r+1}}{x[(x-1)^r x^r - (x+1)^r(x-2)^r]} < \frac{x}{2}.$$

Proof. Clearly, the lemma is true when $r = 1$. Hence, assume that $r \geq 2$, and so $x \geq 3$. Write

$$A = \frac{(x-1)^r(x+1)^{r+1} - x^{2r+1}}{x[(x-1)^r x^r - (x+1)^r(x-2)^r]}.$$

First note that

$$(x-1)^r(x+1)^{r+1} - x^{2r+1} = (x^2-1)^r(x+1) - (x^2)^r x,$$

and that

$$-(x^2)^r x < -(x^2-1)^r x.$$

Hence

$$(x-1)^r(x+1)^{r+1} - x^{2r+1} < (x^2-1)^r.$$

Moreover,

$$\begin{aligned}(x-1)^r x^r - (x+1)^r(x-2)^r &= (x^2-x)^r - (x^2-x-2)^r \\ &= \sum_{i=0}^{r} \binom{r}{i} (x^2-x-1)^i [1^{r-i} - (-1)^{r-i}].\end{aligned}$$

But then, since each term in the sum of the above expression is non negative, the sum is at least as big as the term with $i = r-1$, and so

$$(x-1)^r x^r - (x+1)^r(x-2)^r \geq \binom{r}{r-1} (x^2-x-1)^{r-1} 2 = 2r(x^2-x-1)^{r-1},$$

and so

$$A < \frac{(x^2-1)^r}{2rx(x^2-x-1)^{r-1}}.$$

But

$$\frac{(x^2-1)^r}{2rx(x^2-x-1)^{r-1}} < \frac{x^2}{2rx}\left(\frac{x^2-1}{x^2-x-1}\right)^{r-1} < \frac{x}{2r}\left(\frac{(x-1)(x+1)}{(x+1)(x-2)}\right)^{r-1},$$

and so

$$A < \frac{(x^2-1)^r}{2rx(x^2-x-1)^{r-1}} < \frac{x}{2r}\left(\frac{x-1}{x-2}\right)^{r-1}.$$

Hence, to finish the proof we only need show that

$$\left(\frac{x-1}{x-2}\right)^{r-1} \leq r.$$

This is immediate for $r = 2$. For $r \geq 3$ it follows by observing that

$$\left(\frac{x-1}{x-2}\right)^{r-1} \leq \left(\frac{r}{r-1}\right)^{r-1} \leq e < 3 \leq r.$$

∎

3.4. Proof of Theorem 4

Write $G'' = G_{|uv}$, $\omega = \omega(H)$, $\omega' = \omega(G)$, and $\omega'' = \omega(G'')$. Since

$$P(H,\lambda) = P(H+uv,\lambda) + P(H_{|uv},\lambda) = (\lambda-2)^r P(G,\lambda) + (\lambda-1)^r P(G'',\lambda),$$

we have

$$\frac{P(H,\lambda-1)}{P(H,\lambda)} = \left(\frac{\lambda-3}{\lambda-2}\right)^r \frac{P(G,\lambda-1)}{P(G,\lambda)}\alpha + \left(\frac{\lambda-2}{\lambda-1}\right)^r \frac{P(G'',\lambda-1)}{P(G'',\lambda)}(1-\alpha),$$

where

$$\alpha = \frac{(\lambda-2)^r P(G,\lambda)}{(\lambda-2)^r P(G,\lambda) + (\lambda-1)^r P(G'',\lambda)}.$$

To prove the theorem we have to show that, for every $\lambda \geq n+r$,

$$\frac{P(H,\lambda-1)}{P(H,\lambda)} \leq \frac{\lambda-\omega}{\lambda}\left(\frac{\lambda-1}{\lambda}\right)^{n+r-\omega}.$$

For this purpose, write

$$R = \left(\frac{\lambda-3}{\lambda-2}\right)^r, \quad S = \left(\frac{\lambda-2}{\lambda-1}\right)^r.$$

Since by assumption both G and G'' satisfy Conjecture 3, we have

$$\frac{P(H,\lambda-1)}{P(H,\lambda)} \leq R\alpha\,\frac{\lambda-\omega'}{\lambda}\left(\frac{\lambda-1}{\lambda}\right)^{n-\omega'} + S(1-\alpha)\,\frac{\lambda-\omega''}{\lambda}\left(\frac{\lambda-1}{\lambda}\right)^{n-1-\omega''}.$$

Hence, to prove the theorem we only need show that

$$R\alpha\,\frac{\lambda-\omega'}{\lambda-\omega}\left(\frac{\lambda}{\lambda-1}\right)^{s'} + S(1-\alpha)\,\frac{\lambda-\omega''}{\lambda-\omega}\left(\frac{\lambda}{\lambda-1}\right)^{s''} \leq 1, \tag{4}$$

where $s' = r+\omega'-\omega$ and $s'' = r+\omega''-\omega+1$.

For this purpose, first note that either $\omega' = \omega$ or $\omega' = \omega+1$ and that either $\omega'' = \omega$ or $\omega'' = \omega+1$. But, since

$$\frac{\lambda-\omega^*}{\lambda-\omega}\left(\frac{\lambda}{\lambda-1}\right)^{\omega^*-\omega} \leq 1$$

where $\omega^* = \omega'$ or $\omega^* = \omega''$, it follows that we only need show the validity of (4) in the case $\omega' = \omega'' = \omega$, that is

$$\left(\frac{\lambda-3}{\lambda-2}\right)^r \left(\frac{\lambda}{\lambda-1}\right)^r \alpha + \left(\frac{\lambda-2}{\lambda-1}\right)^r \left(\frac{\lambda}{\lambda-1}\right)^{r+1} (1-\alpha) \leq 1. \tag{5}$$

Now, inequality (5) is equivalent to the following

$$\alpha\left[\left(\frac{\lambda-3}{\lambda-2}\right)^r - \left(\frac{\lambda-2}{\lambda-1}\right)^r \frac{\lambda}{\lambda-1}\right] \leq \left(\frac{\lambda-1}{\lambda}\right)^r - \left(\frac{\lambda-2}{\lambda-1}\right)^r \frac{\lambda}{\lambda-1}.$$

Since the coefficient of α in this inequality is strictly negative, we can divide both sides by this term and simplify terms to get the equivalent inequality:

$$\alpha \geq \left(\frac{\lambda-2}{\lambda}\right)^r \frac{(\lambda-2)^r\lambda^{r+1} - (\lambda-1)^{2r+1}}{(\lambda-2)^{2r}\lambda - (\lambda-3)^r(\lambda-1)^{r+1}}.$$

Replacing the expression for α, we have

$$\frac{P(G,\lambda)}{P(G'',\lambda)} \geq \frac{(\lambda-2)^r\lambda^{r+1}-(\lambda-1)^{2r+1}}{(\lambda-1)[(\lambda-2)^r(\lambda-1)^r-\lambda^r(\lambda-3)^r]}. \tag{6}$$

Hence, to prove the theorem, we only need show that (6) holds. Now, Lemma 1 implies that

$$\frac{(\lambda-2)^r\lambda^{r+1}-(\lambda-1)^{2r+1}}{(\lambda-1)[(\lambda-2)^r(\lambda-1)^r-\lambda^r(\lambda-3)^r]} \leq \frac{\lambda-1}{2}.$$

Hence it is sufficient to show that for every $\lambda \geq n+r$,

$$\frac{P(G,\lambda)}{P(G'',\lambda)} \geq \frac{\lambda-1}{2}.$$

For this purpose, consider any λ colouring of the graph G''. Since G'' has less than λ vertices, this colouring can be extended to a λ colouring of the graph G as follows: give to vertex u (respectively, v) the same colour as that given to the vertex in G'' arising from contracting uv, and give to vertex v (respectively, u) any of the $\lambda - d_G(u)$ (respectively, $\lambda - d_G(v)$) colours not used by the neighbours of vertex u (respectively, v). In other words,

$$P(G,\lambda) \geq P(G'',\lambda)(\lambda - \min\{d_G(u), d_G(v)\}).$$

Now, by assumption, $\min\{d_G(u), d_G(v)\} \leq \dfrac{n+r+1}{2}$, and so, since $\lambda \geq n+r$,

$$\lambda - \min\{d_G(u), d_G(v)\} \geq \lambda - \frac{n+r+1}{2} \geq \frac{\lambda-1}{2},$$

and we are done. The theorem follows.

3.5. Proof of Theorem 5

Let H be a series-parallel graph with m vertices. If m is small then the theorem is obviously true. Hence, we can assume that every series-parallel graph with fewer vertices than H verifies Conjecture 3. Moreover, we can assume that H has no clique-cutset: for otherwise H would arise as clique-identification from two other series-parallel graphs and so we could apply Theorem 3.

Now, by definition, H comes from some other series-parallel graph H' by either duplicating some edge of H' or by subdividing some edge of H'. Since in the first case H will still verify Conjecture 3, we only need show the validity of the theorem when H is constructed from H' by subdividing some edge uv of H'.

Let x be the unique vertex of H that is not a vertex of H'. Set

$$T = \{y \in V(H') : d_{H'}(y) = 2, yu \in E(H'), yv \in E(H')\}.$$

Write $T = \{x_1, \cdots, x_{r-1}\}$, with $r \geq 1$. Let G denote the graph obtained from H' by removing all vertices in T. It follows that H can be built from G by subdividing edge uv with the r vertices $x_1, \cdots, x_{r-1}, x_r$ with $x_r = x$, as shown in Figure 2. Clearly, G is also series-parallel, and so it verifies Conjecture 3. Let n denote the number of vertices of G. Note that H has $n+r$ vertices. Since the graph $G_{|uv}$ is also series-parallel, we can apply Theorem 4. Hence, to prove the theorem, we only need show that

$$\min\{d_G(u), d_G(v)\} \leq \frac{n+r+1}{2}.$$

For this purpose, set

$$\begin{aligned} A &= \{y \in V(G) : yu \notin E(G), yv \notin E(G)\} \\ B &= \{y \in V(G) : yu \in E(G), yv \in E(G)\} \\ C &= V(G) - (A \cup B \cup \{u, v\}). \end{aligned}$$

Now, if B contains at most one vertex, then $d_G(u) + d_{G'}(v) \leq n + 1$ and we are done. Hence we can assume that B contains at least two vertices. Clearly, B is a stable set in G' (for otherwise, G would contain a K_4).

First, note that:

- no vertex in C is adjacent to some vertex in B.

To see this, assume the contrary: there exists some vertex $z \in C$ which is adjacent to some vertex $y \in B$. Without loss of generality, we can assume that $zu \in E(G)$. Since H has no clique-cutset, it follows that $\{u, y\}$ is not a clique-cutset in G, and so there must exists a path P in $G - \{u, y\}$ joining z to v. But then contracting all edges of $P - \{v\}$, we get a K_4, contradicting the assumption that G is series-parallel.

Next, note that

- every vertex in A is adjacent to at most one vertex in B.

This is obviously true because G is not contractable to K_4.

Since by assumption H and hence G has no clique cutset, every vertex in B is adjacent to some vertex in $A \cup C$, it follows that $|A| \geq |B|$ (recall that no vertex in B is adjacent to some vertex in C), and so $n = 2 + |A| + |B| + |C| \geq 2 + 2|B| + C$. But then $d_G(u) + d_{G'}(v) = |C| + 2|B| + 2 \leq n$, and so $\min\{d_G(u), d_G(v)\} \leq \frac{n}{2}$, and we are done.

3.6. Proof of Theorem 6

Since $P(C_n, \lambda) = (\lambda - 1)[(\lambda - 1)^{n-1} + (-1)^n]$, we have

$$\frac{P(C_n, \lambda - 1)}{P(C_n, \lambda)} = \frac{\lambda - 2}{\lambda - 1} \frac{(\lambda - 2)^{n-1} + (-1)^n}{(\lambda - 1)^{n-1} + (-1)^n}.$$

First assume that n is even, and so $\chi(C_n) = 2$. To prove that Conjecture 2 is verified, we have to show that

$$\frac{\lambda - 2}{\lambda - 1} \frac{(\lambda - 2)^{n-1} + 1}{(\lambda - 1)^{n-1} + 1} \leq \frac{\lambda - 2}{\lambda} \left(\frac{\lambda - 1}{\lambda}\right)^{n-2}$$

that is

$$\frac{(\lambda - 2)^{n-1} + 1}{(\lambda - 1)^{n-1} + 1} \leq \left(\frac{\lambda - 1}{\lambda}\right)^{n-1}$$

For this purpose, write $x = \lambda - 1$, $h = n - 1$ and set $y = x^2 - 1$. We have by telescoping and the fact that $x \leq y$ that

$$\begin{aligned} (x+1)^h - x^h &= [(x+1) - x] \sum_{i=1}^{h} (x+1)^{h-i} x^{i-1} \\ &= \sum_{i=1}^{h} (x+1)^{h-i} x^{i-1} \\ &\leq \sum_{i=1}^{h} (y+1)^{h-i} y^{i-1} \\ &= (y+1)^h - y^h \end{aligned}$$

Therefore,

$$(x+1)^h - x^h \le (y+1)^h - y^h = (x^2)^h - (x^2-1)^h = x^h x^h - (x-1)^h (x+1)^h,$$

and so

$$\frac{(x-1)^h + 1}{x^h + 1} \le \frac{x^h}{(x+1)^h},$$

and we are done.

Next, assume that n is odd, and so $\chi(C_n) = 3$. To prove that Conjecture 2 is verified, we have to show that

$$\frac{\lambda-2}{\lambda-1} \frac{(\lambda-2)^{n-1} - 1}{(\lambda-1)^{n-1} - 1} \le \frac{\lambda-3}{\lambda} \left(\frac{\lambda-1}{\lambda}\right)^{n-3}.$$

This is obviously true if $n = 3$. Hence, assume that $n \ge 5$. Now, since

$$\frac{(\lambda-2)^{n-1} - 1}{(\lambda-1)^{n-1} - 1} \le \left(\frac{\lambda-2}{\lambda-1}\right)^{n-1},$$

we only need show that

$$\frac{\lambda-2}{\lambda-1} \left(\frac{\lambda-2}{\lambda-1}\right)^{n-1} \le \frac{\lambda-3}{\lambda} \left(\frac{\lambda-1}{\lambda}\right)^{n-3},$$

that is

$$\left(\frac{\lambda-2}{\lambda-1} \frac{\lambda}{\lambda-1}\right)^{n-3} \le \frac{\lambda-3}{\lambda} \left(\frac{\lambda-1}{\lambda-2}\right)^{3}.$$

But

$$\left(\frac{\lambda-2}{\lambda-1} \frac{\lambda}{\lambda-1}\right)^{n-3} < \frac{\lambda-2}{\lambda-1} \frac{\lambda}{\lambda-1} < \frac{\lambda-3}{\lambda} \left(\frac{\lambda-1}{\lambda-2}\right)^{3},$$

and again we are done.

4. Conclusions

In this paper we proposed two new conjectures related to the Shameful Conjecture of Bartels and Welsh. The upper bounds given in our conjectures can be considered as interpolations between the respective ratios for the empty and complete graphs, for which the conjectured bounds are tight. Their strength allowed us to define operations on graphs that maintain the validity of the conjectured bounds, allowing us to build large classes of graphs that satisfy the Shameful Conjecture. In particular, we could prove that the Shameful Conjecture is satisfied by chordal graphs and series-parallel graphs. Moreover, these classes of graphs could be used as building blocks to enlarge the class of graphs for which it is known that the Shameful Conjecture holds.

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