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MAX CUT IN TOROIDAL GRAPHS

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Abstract

We present a polynomial time algorithm to find the maximum weight of an edge-cut in graphs embeddable on a torus, with integral weights bounded in the absolute value by a polynomial of the size of the graph. The algorithm may be easily generalized to graphs embeddable on an arbitrary orientable surface.

The algorithm has been implemented for toroidal grids using modular arithmetics and the generalized nested dissection method. The applications in statistical physics are discussed.

1. Introduction

A graph is a pair $G = (V, E)$ where V is a finite set of vertices and E is a set of *edges*, i.e., unordered pairs of vertices. We assign a variable x_e to each edge e of G and we let $x = (x_e : e \in E)$. Moreover, for any $M \subseteq E$, we denote by $x(M)$ the product of the variables of the edges of M .

A subgraph $G' = (V, E')$ of a graph $G = (V, E)$ is called *eulerian* if the degree of each vertex of G' is even.

Definition 1.1. *Let $G = (V, E)$ be a graph and $X \subseteq V$. The set of edges $C \subseteq E$ with one end in X and the other one in $V \setminus X$ is called a cut.*

Definition 1.2. *The generating function of cuts is the polynomial $\mathcal{C}(G, x)$ which equals the sum of $x(C)$ over all cuts C of G .*

Definition 1.3. *The generating function of eulerian subgraphs of G is the polynomial $\mathcal{E}(G, x)$ which equals the sum of $x(U)$ over all eulerian subgraphs U of G .*

In this paper, we consider the following combinatorial optimization problem:

Max Cut Problem: *given a graph G and a weight function w on the edges of G , find the value of the following function:*

$$w(G) = \max_{C \text{ cut}} \sum_{e \in C} w_e.$$

The Max Cut Problem is known to be *NP*-hard [9] for general graphs even when $w_e = 1$ for all edges $e \in E$. Nevertheless, due to the relevance of the problem and to its wide applicability [3], many attempts have been done to approach the problem from the algorithmic point of view. Good bounds were obtained by Delorme and Poljak [4], and Poljak and Rendl [17], using an eigenvalue minimization problem as relaxation of the max cut problem. Their relaxation is tightly connected with the semidefinite programming relaxation introduced a bit later by Goemans and Williamson in [10] who also presented a randomized algorithm with a performance guarantee of 0.878.

Barahona et al. [3] and De Simone et al. ([18], [19]) used integer programming to solve large instances of the Max Cut Problem for toroidal square lattices, for short *toroidal grids*, with general weights or with weights $+1$ and -1 only. These are important instances since the max cuts in toroidal grids correspond to the ground states of the 2-dimensional Ising problem with periodic boundary conditions. They have been extensively studied and several algorithms for solving the Max Cut Problem for toroidal grids have been proposed and implemented by physicists (see [18],[19] for the references). A successful Dagstuhl seminar on this subject was held in the end of 1997 (see [11]).

The Max Cut Problem for planar graphs may be solved efficiently as follows: let G be a planar graph and G^* its geometric dual. The cuts of G are in one-to-one correspondence with the eulerian subgraphs of G^* . A well-known transformation, described in Section 3, reduces the problem of finding a maximum eulerian subgraph of G^* to the problem of finding a maximum perfect matching in the transformed graph. The latter problem is polynomially solvable by Edmonds' algorithm [5].

In a seminal paper [12], Kasteleyn implicitly described a polynomial algorithm to determine the generating function $\mathcal{C}(G; z^{w_e})$ in the variable z , when G is planar and the edge weights w_e are bounded. Later in [13], [14], he obtained the same result for toroidal grids.

Another polynomial time algorithm to solve the Max Cut Problem in toroidal grids was proposed by Barahona in [1]. However, his arguments on page 3248 are not correct. He corrected his statements in an unpublished manuscript [2] where he also claimed to solve the case of general toroidal graphs with weights ± 1 . Let us consider the manuscript [2] in greater detail. The basic difference between Barahona's algorithm and Kasteleyn's algorithm for the Max Cut Problem in toroidal graphs lies in the different concepts of *duality* they use: Kasteleyn uses a theorem of van der Waerden [21] which relates the generating function of cuts to the generating function of eulerian subgraphs of the same graph while Barahona uses the geometric duality on torus. However, while his crucial observation on geometric duality may be easily derived for toroidal grids, the general case is not proved yet. Furthermore, the proof of another key lemma in [2] refers to pages 98-99 of the paper [14] of Kasteleyn where the needed statement is not proved.

On page 99 of [14], Kasteleyn writes that for general toroidal graphs $\mathcal{C}(G; z^{w_e})$ may be computed analogously as for toroidal grids. In fact, he states that for graphs of genus g , $\mathcal{C}(G; z^{w_e})$ may be expressed as a linear combination of 4^g Pfaffians (see Section 5 for a definition) and hence, it may be computed efficiently for a fixed genus. We proved this statement in [7], [8].

In this paper we describe an algorithm to determine the generating function $\mathcal{C}(G; z^{w_e})$ for the graphs embedded on a torus, whose running time is bounded by a polynomial in $|E|$ and $\max |w_e|$. Hence, when the number of different weights of edges is bounded, or, more generally, when the weights are integers bounded in the absolute value by a polynomial in the size of the graph, the running time of the algorithm is polynomial. The algorithm may be easily generalized to graphs embeddable on an arbitrary orientable surface.

The reason why we restrict ourselves to consider the toroidal case only is the physical applications we have in mind. From the generating function of cuts we are able to determine the partition function of 2-dimensional Ising problem with periodic boundary conditions. This partition function generates all interesting finite temperature quantities like free energy, mean energy, specific heat, magnetization and susceptibility.

Observe that each term of the generating function of cuts is of the form $z^{\sum_{e \in C} w_e}$ where C is a cut of G . Thus, it is easy to deduce the value of a maximum cut of G by selecting the highest exponent of $\mathcal{C}(G, (z^{w_e} : e \in E))$, once we have it expressed as a polynomial in z .

So, we will consider toroidal graphs, and we will assume that the weights of edges are integers bounded in the absolute value by a polynomial in the number of edges of the graph.

We need to introduce another generating function.

Definition 1.4. *The generating function of perfect matchings of G is the polynomial $\mathcal{P}(G, x)$ which equals the sum of $x(P)$ over all perfect matchings P of G .*

In section 2 we reduce the problem of computing $\mathcal{C}(G, (z^{w_e} : e \in E))$ to the problem of computing $\mathcal{E}(G, (y_e : e \in E))$ where each y_e is a rational function of z^{w_e} . In section 3 we construct a graph G_s and a new variable y^s such that $\mathcal{E}(G, (y_e : e \in E)) = \mathcal{P}(G_s, (y_e^s : e \in E(G_s)))$. In sections 4 and 5 we present the theorem in [7] which expresses $\mathcal{P}(G, x)$, when G toroidal graph, as a linear combination of four Pfaffians. Finally in section 6 we show how to compute $\mathcal{P}(G_s, (y_e^s : e \in E(G_s)))$.

2. From $\mathcal{E}(G, x)$ to $\mathcal{C}(G, x)$

It is well-known that for a planar graph G , the cuts of G are in one-to-one correspondence with the eulerian subgraphs of its geometric dual G^* . This correspondence does not hold anymore for graphs of higher genus. Nevertheless, a duality relation exists between the generating function of cuts of a graph and the generating function of eulerian subgraphs of the same graph. This duality result was originally discovered by van der Waerden (see [21], [8]).

Theorem 2.1. *Let G be a graph and z a variable. Then*

$$\mathcal{C}(G, (z^{w_e} : e \in E)) = z^{\frac{1}{2} \sum_{e \in E} w_e} \left(\prod_{e \in E} \cosh(z, \frac{-w_e}{2}) \right) 2^{|V|-1} \mathcal{E}(G, (th(z, \frac{-w_e}{2}) : e \in E)),$$

where

$$\sinh(z, x) = \frac{z^x - z^{-x}}{2}, \quad \cosh(z, x) = \frac{z^x + z^{-x}}{2}, \quad th(z, x) = \frac{\sinh(z, x)}{\cosh(z, x)}.$$

In the next section we show how to evaluate $\mathcal{E}(G, (y_e : e \in E))$.

3. From $\mathcal{P}(G_s, x^s)$ to $\mathcal{E}(G, x)$

We generalize a construction of Fisher [6] which yields an algorithm to compute $\mathcal{E}(G, x)$, once an algorithm for computing the generating function of perfect matchings is known. Note that the construction preserves the genus of the graph.

Definition 3.1. *Let $G = (V, E)$ be a graph and let $v \in V$. Let e_1, \dots, e_k be an order of the edges of G incident with v . Even splitting of v is a graph $G' = (V', E')$ such that $V' = V - \{v\} \cup \{v_1, \dots, v_{6k}\}$, and $E' = E - \{e_1, \dots, e_k\} \cup \{e'_1, \dots, e'_k\} \cup \{v_t v_{t+1} : 0 < t < 6k\} \cup \{v_{3j+1} v_{3j+3} : j = 0, \dots, 2k-1\}$ where e'_i is obtained from e_i by replacing v by $v_{6(i-1)+2}$, $i = 1, \dots, k$. We say that e'_i is the image of e_i in G' .*

Definition 3.2. *Let $G = (V, E)$ be a graph. We denote by $G_s = (V_s, E_s)$ the graph obtained from G by even splitting all vertices of V . If edge f' of G_s is the image of edge f of G then we let $x_{f'}^s = x_f$. We let $x_e^s = 1$ for the remaining edges e of G_s .*

Theorem 3.3. *Let G be a graph embeddable on a torus \mathcal{T} and let each even splitting of a vertex be performed in the clockwise order of the embeddings of its incident edges. Then G_s is also embeddable on \mathcal{T} . Moreover $\mathcal{P}(G_s, x^s) = \mathcal{E}(G, x)$.*

Proof. The first statement follows from the definition of even splitting. Next, observe that each eulerian subgraph W of G is in one-to-one correspondence with a perfect matching P_W of G_s which contains the set of the images of the edges of W . This together with the choice of x^s implies that $x(W) = x^s(P_W)$, for each eulerian subgraph W . ■

4. Embedding of graphs on torus

An *embedding* of a graph on a surface is defined in a natural way: the vertices are embedded as points, and each edge is embedded as a continuous non-self-intersecting curve connecting the embeddings of its endvertices. The interiors of the embeddings of the edges are pairwise disjoint and do not contain any point representing a vertex.

A graph is called *planar* if it may be embedded on the plane. The embedding of a plane graph partitions the plane into connected regions called *faces*. The (unique) unbounded face is called *outer face* and the bounded faces are called *inner faces*. If a plane graph is 2-connected then each face is a cycle.

A torus is represented on a plane as a rectangular region and its space representation can be obtained by identifying its opposite sides. An embedding of a graph on a torus may be obtained by drawing some of the edges of the graph through the sides of the rectangular region.

Mohar [16] presented a linear time algorithm for finding an embedding of a graph of genus k when k is fixed. The embedding is described by assigning a permutation of the edges incident at each vertex.

Definition 4.1. *A graph G is called a 1-graph if it may be embedded on a torus so that all vertices are embedded on rectangular region R and the embedding of each edge crosses either the horizontal or the vertical boundaries of R at most once. The set of the edges embedded entirely inside the rectangular region R will be denoted by E_0 and the set of edges crossing the horizontal (vertical) boundaries of R will be denoted by E_1 (E_2 respectively). If a 1-graph G satisfies the following further conditions:*

1. *the outer face of $G_0 = (V, E_0)$ is a cycle, and it is embedded on the boundary of R ,*
2. *each vertex is incident with at most one edge which does not belong to E_0 ,*
3. *G_0 has a perfect matching,*

then we say that G is a proper 1-graph.

Given a proper 1-graph G , we denote by C_0 the cycle which forms the outer face of G_0 , and we denote by M_0 a perfect matching of G_0 .

Lemma 4.2. *Let G be a graph embeddable on a torus and let $x = (x_e : e \in E)$. Then G may be transformed into a proper 1-graph G' so that $\mathcal{P}(G, x) = \mathcal{P}(G', x')$. This transformation requires a polynomial number of steps.*

Proof. If a graph G is embeddable on a torus, then we can embed it using the Mohar's algorithm. All vertices lie in the rectangular region R representing the torus on the plane. The endvertices of edges crossing the boundary of R may be drawn so that they lie on the boundary itself. Now, if the embeddings of some edges cross the horizontal and vertical boundaries of R several times, then we construct a new graph G^1 as follows: we replace each edge $e = uv$ which crosses the boundaries of R k times, by a simple path $P_e = (u, e_1, v_1, \dots, v_k, e_{2k+1}, v)$, where each v_i lies on the boundary of R , even indexed edges are embedded on R and odd indexed edges lie outside R . We let $x_{e_1}^1 = x_e$, $x_{e_i}^1 = 1$ for each $i > 1$ and $x_f^1 = x_f$ for the remaining edges. It is easy to see that there is a one-to-one correspondence between perfect matchings M of G and perfect matchings M' of G^1 such that $x(M) = x^1(M')$.

A similar construction may be implemented in the case when there is no perfect matching in the subgraph of G^1 embedded on R . In fact, construct a new graph G^2 as follows: take any perfect matching M of G^1 and replace any edge $e = uv \in M$ crossing the boundaries of R by a path $(u, e_1, y, e_2, z, e_3, v)$, and let $x_{e_1}^2 = x_e^1$, $x_{e_2}^2 = x_{e_3}^2 = 1$ and $x_f^2 = x_f^1$ for the remaining edges. Then let only edge e_2 cross the boundaries of R .

Finally, we construct G' from G^2 by adding edges so that the outer face of the planar part is a cycle, we let $x'_e = 0$ for each such edge e and $x'_f = x_f^2$ for the remaining edges. Note that G' is a proper 1-graph and $\mathcal{P}(G', x') = \mathcal{P}(G, x)$.

This completes the proof of the lemma. ■

5. Computation of $\mathcal{P}(G, x)$

In this section we suppose we are given a graph G embedded on a torus as a proper 1-graph and we show how to evaluate $\mathcal{P}(G, x)$ using “suitably defined” orientations of G . We start with some definitions.

Given two perfect matchings M and N of a graph G , the symmetric difference $M \Delta N$ consists of vertex disjoint cycles of even length, which are called *alternating cycles* of M and N .

An *orientation* of a graph $G = (V, E)$ is a *digraph* $D = (V, A)$ obtained from G by fixing an orientation of each edge of G , i.e. by ordering the elements of each edge of G . The elements of A are called *arcs*.

Definition 5.1. For digraph D we denote by $A(D)$ the matrix having rows and columns indexed by V , where $a_{vw} = x_{(v,w)}$ if (v, w) is an arc of D , $a_{vw} = -x_{(v,w)}$ if (w, v) is an arc of D , and $a_{vw} = 0$ otherwise. The Pfaffian of $A(D)$ is defined as

$$Pf(A(D)) = \sum_P s^*(P) a_{i_1 j_1} \cdots a_{i_n j_n}$$

where $P = \{\{i_1 j_1\}, \dots, \{i_n j_n\}\}$ is a partition of the set $\{1, \dots, 2n\}$ into pairs, $i_k < j_k$ for $k = 1, \dots, n$, and $s^*(P)$ equals the sign of the permutation $(i_1 j_1 \dots i_n j_n)$.

Let C be a cycle of an even length of a graph G and let D be an orientation of G . C is said to be *clockwise odd* in D if it has an odd number of edges directed in D in agreement with any of the two possible ways of traversal.

An orientation D of G is said to be *Pfaffian* if all alternating cycles with respect to a fixed perfect matching of G are clockwise odd in D .

The following two theorems are seminal results of Kasteleyn ([14]).

Theorem 5.2. Each planar graph has a Pfaffian orientation such that all inner faces are clockwise odd. Moreover, such orientation may be found in polynomial time.

Theorem 5.3. If G has a Pfaffian orientation D then $\mathcal{P}(G, x) = \pm Pf(A(D))$.

Computing Pfaffians of skew-symmetric matrices has the same computational complexity as computing determinants; it is well known that determinants may be computed in polynomial time (see [20] or [15] for improvements of the Gaussian elimination algorithm).

8.

Definition 5.4. Let $G = (V, E)$ be a proper 1-graph and let $G_0 = (V, E_0)$ be as in Definition 4.1. We say that an orientation D_0 of G_0 is basic if it is Pfaffian and each inner face of G_0 is clockwise odd in D_0 .

Observe that it is always possible to construct such an orientation by Theorem 5.2.

Let R be a rectangular region of the plane representing torus, and let G be a graph of genus 1. We also assume to have G embedded on R as a proper 1-graph, as showed in Lemma 4.2 Let us denote by $G_i = (V, E_0 \cup E_i)$, $i = 1, 2$, the subgraphs of G obtained from G_0 by adding all edges of $E - E_0$ crossing, respectively, the horizontal boundaries of R and the vertical boundaries of R . It is easy to see that each of these graphs is planar and that the restriction of the embedding of G on the torus to each graph G_i provides a planar embedding of G_i , $i = 1, 2$.

Definition 5.5. Let $G = (V, E)$ be a proper 1-graph and let D_0 be a basic orientation of G_0 . We may then extend the basic orientation D_0 of G_0 to a Pfaffian orientation D_i of G_i , $i = 1, 2$, by orienting the edges of E_i so that each inner face of G_i is clockwise odd in D_i . The orientation $-D_i$ is obtained from D_i by reversing the orientation of all edges of E_i .

If G is a proper 1-graph and D_0 is a basic orientation of G_0 , then D_i is uniquely determined for $i = 1, 2$.

Definition 5.6. Let G be a proper 1-graph. An orientation D of G which equals the basic orientation D_0 on G_0 and which equals D_i or $-D_i$ on E_i , $i = 1, 2$, is called relevant. We define its type $r_D = (r_1, r_2)$ as a 2-dimensional vector whose component r_j equals $+1$ or -1 according to the sign of D_j in D , $j = 1, 2$.

The following theorem shows that the computation of $\mathcal{P}(G, x)$ may be carried out if G is a proper 1-graph. This theorem is a special case of our Corollary 2.18 in [7]. An analogous result was proved by Kasteleyn ([12]) in the case of toroidal grids.

Theorem 5.7. Let $G = (V, E)$ be a proper 1-graph and let M_0 be a perfect matching of $G_0 = (V, E_0)$. Let D^1, D^2, D^3, D^4 be the four relevant orientations of G such that $r_{D^1} = (1, 1)$, $r_{D^2} = (-1, 1)$, $r_{D^3} = (1, -1)$ and $r_{D^4} = (-1, -1)$. Then

$$\mathcal{P}(G, x) = \frac{1}{2} s^*(M_0) [Pf(A(D^1)) + Pf(A(D^2)) + Pf(A(D^3)) - Pf(A(D^4))]$$

where $s^*(M_0)$ is defined in Definition 5.1.

6. The algorithm

Let $G = (V, E)$ be a graph embeddable on torus, such that the weight w_e of each edge e of G is an integer bounded in the absolute value by a polynomial of $|E(G)|$.

As shown in section 2 we may express $\mathcal{C}(G, (z^{w_e} : e \in E))$ as the following polynomial in the variable z :

$$z^{1/2} \sum_{e \in E} w_e \left(\prod_{e \in E} \cosh(z, -w_e/2) \right) 2^{|V|-1} \mathcal{E}(G, (y_e : e \in E))$$

where each y_e equals $th(z, -w_e/2)$. In section 3 we constructed a toroidal graph G_s such that $\mathcal{P}(G_s, (y_e^s : e \in E(G_s))) = \mathcal{E}(G, (y_e : e \in E))$ and $y_e^s = y_e$ for $e \in E(G)$ and $y_e^s = 1 = z^0$ otherwise. Using the results of sections 4 and 5 we obtain a proper 1-graph H such that $\mathcal{P}(G_s, (y_e^s : e \in E(G_s))) = \mathcal{P}(H, (h_e : e \in E(H)))$ and each h_e equals some y_f^s or 1 or 0. Let $h = (h_e : e \in E(H))$. Moreover, by Theorem 5.7

$$\mathcal{P}(H, h) = \frac{1}{2} s^*(M_0)[Pf(A(D^1)) + Pf(A(D^2)) + Pf(A(D^3)) - Pf(A(D^4))]$$

where D^1, D^2, D^3, D^4 are the relevant orientations of H and $A(D^i)$ are the corresponding skew-symmetric matrices. The absolute value of each nonzero entry of $A(D^i)$ equals 1 or $th(z, -w_e/2) = \frac{1-z^{w_e}}{1+z^{w_e}}$, namely it is either 1 or a ratio of two polynomials in the variable z . Furthermore, each $|w_e|$ is bounded by $|E|^k$ for some positive integer k and so, $\sum_{e \in E} |w_e| \leq \sum_{e \in E} |E|^k = |E|^{k+1}$.

Hence,

$$\mathcal{C}(G, (z^{w_e} : e \in E)) = 2^{|V|-1-|E|} \prod_{e \in E} (1 + z^{w_e}) \mathcal{P}(H, h)$$

is a polynomial in z , namely $\mathcal{C}(G, (z^{w_e} : e \in E)) = \sum_{i=-m}^m a_i z^i$, where $0 \leq a_i \leq 2^{|V|}$ and $m = |E|^{k+1}$. Now, in order to evaluate the coefficients a_i of $\mathcal{C}(G, (z^{w_e} : e \in E))$, observe that a_m equals the integer part of $F_m(2^{|E||V|^k})$, where $F_m(z) = z^{-m} \mathcal{C}(G, (z^{w_e} : e \in E))$. Hence, using a polynomial algorithm to compute the Pfaffian of a matrix, we can determine a_m in polynomial time. Then we may get a_{m-1} analogously, and, in the end, find the explicit polynomial expression of $\mathcal{C}(G, (z^{w_e} : e \in E))$.

We summarize the algorithm to solve the Max Cut Problem for the toroidal graphs in the following sketch:

Input: $G = (V, E)$ and a vector $w = (w_e, e \in E)$ of integer weights bounded in the absolute value by $|E|^k$, k fixed. Let $m = |E|^{k+1}$.

1. Let $y_e = \frac{1-z^{w_e}}{1+z^{w_e}}$, for each $e \in E$;
2. Transform G into G_s and y into y_s , so that $\mathcal{P}(G_s, y^s) = \mathcal{E}(G, y)$;
3. Embed G_s on a torus as a proper 1-graph; renumber vertices of G_s so that $M_0 = \{(1, 2), \dots, (2n-1, 2n)\}$ for each $ij \in M_0$;
4. Find a *basic* orientation of G_s using the Kasteleyn algorithm;
5. Find the *relevant* orientations D^i , $i = 1, 2, 3, 4$, of G_s ;
6. Let $z = 2^{|E||V|^k}$;
7. Construct matrices $A(D^i)$, $i = 1, 2, 3, 4$, and compute $Pf(A(D^i))$;
8. Evaluate $\alpha = z^{|V|-|E|-1} (\prod_{e \in E} (1 + z^{w_e})) \frac{1}{2} [Pf(A(D^1)) + Pf(A(D^2)) + Pf(A(D^3)) - Pf(A(D^4))]$;
9. Compute $a_m = \lfloor z^{-m} \alpha \rfloor$;
10. For all $t = m-1, m-2, \dots, -m$, compute $a_t = \lfloor z^{-t} (\alpha - \sum_{i=m}^{t+1} a_i z^i) \rfloor$;
11. Express $\mathcal{C}(G, z^w) = \sum_{-m}^m a_i z^i$;

Output: evaluate the maximum exponent in the polynomial expression of $\mathcal{C}(G, z^w)$.

We remark that in the actual implementation of the above algorithm, accomplished by Jan Vondrák, student at Charles University, the part of the algorithm consisting of steps 7-11 is executed using precise modular arithmetics methods described in [22], and the generalized nested dissection method described in [15]. We will report on this elsewhere.

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