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OPTIMAL QUADRATIC SOLUTION FOR THE NONGAUSSIAN FINITE-HORIZON REGULATOR PROBLEM

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Abstract

In this paper, the problem of the optimal quadratic regulator for nongaussian discrete-time stochastic systems with a quadratic cost function is considered. The main result here obtained is that such optimal control can be derived from the classical LQG solution by substituting the linear filtering part with a quadratic optimal filter. Numerical results show high performance of this method.

Key words: discrete-time systems, LQG optimal control, Riccati equations, nonlinear filtering, separation principle, stochastic control, nongaussian systems.

1. Introduction

In this paper the optimal control problem for linear discrete-time stochastic systems with partial state information is considered. It is well known the importance of the optimal control policy in the engineering applications. From a mathematical point of view, the optimal control of discrete- time systems is a finite dimensional problem consisting in finding the minimum of a function with dynamical constraints, namely a non linear programming problem. In literature there are many papers that deal with the case in which the system to be controlled is a linear one and the performance criterion is a quadratic form in state and control. The aim is to find a minimum energy feedback control law that keeps the state of the system close to the state space origin. This problem is extensively studied both for deterministic systems [1, 2, 3, 4, 5] and for stochastic systems. For this latter case, here considered, some authors assume that only partial informations about the noises statistics are known (H_∞ [6] and minimax [7, 8, 5] approaches), other assume that the noises statistics are gaussian (L Q G control problem [9, 10, 11, 12, 13]). In many important technical areas the widely used gaussian assumption cannot be accepted as a realistic statistical description of the random quantities involved. As shown in various papers (see for istance [14, 15]), increasing attention has been paid in control engineering to nongaussian systems. In the present paper we introduce a new algoritm to solve the optimal stochastic regulator problem for nongaussian systems. Besides we consider an important class of optimal control problems which is characterized by a situation often appearing in practice. We assume that the state of the dynamic system we wish to control is not known to the controller (imperfect state information). We will prove, as a by product, also a sorte of separation principle for nongaussian systems.

The paper is organized as follows: in section 2, we recall the main definitions and properties about Kronecker algebra which are widely used in the following sections. In section 3, the control problem we wish to solve is formulated. The optimal solution among all linear output transformations is given. In section 4, in order to synthesize the optimal quadratic control, the extended state and the corresponding dynamical model of the extended system are defined. The main result of the paper is given in section 5 in which the optimal quadratic regulator for the original system is given. In section 6, a numerical simulation is presented showing high performance of the proposed quadratic optimal control with respect to the linear one. The paper ends with concluding remark in section 7.

2. Some recall on the Kronecker algebra

Throughout this paper, we will widely use Kronecker algebra [16]. In this section, for the sake of completeness, we recall just some definitions and properties used in the paper.

Definition 2.1. Let M and N be matrices of dimension $r \times s$ and $p \times q$ respectively. Then the Kronecker product $M \otimes N$ is defined as the $(r \cdot p) \times (s \cdot q)$ matrix

$$M \otimes N = \begin{bmatrix} m_{11}N & \dots & m_{1s}N \\ \dots\dots & \dots & \dots\dots \\ m_{r1}N & \dots & m_{rs}N \end{bmatrix}$$

where the m_{ij} are the entries of M.

Of course this kind of product is not commutative.

Definition 2.2. Let M be the $r \times s$ matrix

$$M = [\, m_1 \quad m_2 \quad \dots \quad m_s \,] \tag{2.1}$$

where m_i denotes i-th column of M, then the stack of M is the $r \cdot s$ vector

$$st(M) = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_s \end{bmatrix}. \tag{2.2}$$

Observe that a vector as in (2.2) can be reduced to a matrix M as in (2.1) by considering the inverse operation of the stack denoted by st^{-1}. With reference to the Kronecker product and the stack operation, the following properties hold [16]:

$$\begin{aligned}
(A+B)\otimes(C+D) &= A\otimes C + A\otimes D \\
&\quad + B\otimes C + B\otimes D && (2.3a)\\
A\otimes(B\otimes C) &= (A\otimes B)\otimes C && (2.3b)\\
(A\cdot C)\otimes(B\cdot D) &= (A\otimes B)\cdot(C\otimes D) && (2.3c)\\
(A\otimes B)^T &= A^T\otimes B^T && (2.3d)\\
st(A\cdot B\cdot C) &= (C^T\otimes A)\cdot st(B) && (2.3e)\\
u\otimes v &= st(v\cdot u^T) && (2.3f)\\
tr(A\otimes B) &= tr(A)\cdot tr(B) && (2.3g)
\end{aligned}$$

where A, B, C, D are suitably dimensioned matrices, u, v are vectors and $tr(M)$ denotes the trace of a square matrix M. The Kronecker power of the matrix M is defined as:

$$\begin{aligned}
M^{[0]} &= 1\,, \\
M^{[n]} &= M\otimes M^{[n-1]} = M^{[n-1]}\otimes M\,, \quad n>0\,.
\end{aligned}$$

As an easy consequence of (2.3b) and (2.3g) it follows

$$tr(A^{[h]}) = \big(tr(A)\big)^h. \tag{2.3h}$$

It is easy to verify that for $u \in \mathbb{R}^r$, $v \in \mathbb{R}^s$, the i-th entry of $u \otimes v$ is given by

$$(u\otimes v)_i = u_l \cdot v_m; \quad l = \left[\frac{i-1}{s}\right] + 1, \quad m = |i-1|_s + 1 \tag{2.4}$$

where $[\cdot]$ and $|\cdot|_s$ denote integer part and s-modulo respectively. Even if the Kronecker product is not commutative, the following property holds [17, 18].

Theorem 2.3. *For any given pair of matrices $A \in \mathbb{R}^{r\times s}$, $B \in \mathbb{R}^{n\times m}$, we have*

$$B\otimes A = C_{r,n}^T (A\otimes B) C_{s,m} \tag{2.5}$$

where the commutation matrix $C_{u,v}$ is the $(u\cdot v)\times(u\cdot v)$ matrix such that its (h,l) entry is given by:

$$\{C_{u,v}\}_{h,l} = \begin{cases} 1, & \text{if } l = (|h-1|_v)u + \left(\left[\frac{h-1}{v}\right]+1\right); \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

Observe that $C_{1,1} = 1$, hence in the vector case when $a \in \mathbb{R}^r$ and $b \in \mathbb{R}^n$, (2.5) becomes

$$b\otimes a = C_{r,n}^T (a\otimes b). \tag{2.7}$$

Other properties of the Kronecker algebra can be found in [17, 19, 20].

3. Linear quadratic control problem

The control problem we are going to solve is relative to the class of systems described by the following equations

$$x(k+1) = Ax(k) + Bu(k) + FN_k, \quad x(0) = \bar{x} \tag{3.1}$$
$$y(k) = Cx(k) + GN_k \tag{3.2}$$

with the corresponding quadratic criterion:

$$J = \frac{1}{2}E\{x^T(N)Sx(N) + \sum_{k=0}^{N-1}(x^T(k)Qx(k) + u^T(k)Ru(k))\}, \tag{3.3}$$

where , for any k,

$$x(k) \in \mathbb{R}^n, \quad y(k) \in \mathbb{R}^q, \quad u(k) \in \mathbb{R}^p.$$

In equations (3.1)-(3.2), $\{N_k\}$ is a sequence of independent zero mean nongaussian random variables, independent on the initial state, characterized by the knowledge of all moments up to the fourth one:

$$E(N_k^{[i]}) = \Psi_{N,i}, \quad i = 2, 3, 4. \tag{3.4}$$

Moreover, without less of generality, we can assume that

$$st^{-1}\Psi_{N,2} = E(N_k \cdot N_k^T) = I. \tag{3.5}$$

The initial state $\bar{x}$ is a random variable such that:

$$E(\bar{x}) = m_0, \tag{3.6}$$
$$E((\bar{x} - m_0)^{[i]}) = \Psi_{x,i}, \quad i = 2, 3, 4. \tag{3.7}$$

Moreover, we assume independence of the state noise FN_k with respect to the measurement noise GN_k with the hypothesis that $FG^T = 0$. Moreover GG^T is assumed full rank. The weight matrix R is assumed to be positive definite, whereas the matrices Q and S are at least nonnegative definite.

It is well known [21] that the output feedback optimal control $\hat{u}(k)$ of (3.1), (3.2), (3.3), among all linear transformations, is given by:

$$\hat{u}(k) = -M(k)\hat{x}(k/k-1), \tag{3.8}$$
$$M(k) = R^{-1}B^T P_c(k+1)[I + BR^{-1}B^T P_c(k+1)]^{-1}A, \tag{3.9}$$
$$P_c(k) = A^T P_c(k+1)[I + BR^{-1}B^T P_c(k+1)]^{-1}A + Q, \tag{3.10}$$
$$P_c(N) = S, \tag{3.11}$$
$$\hat{x}(k/k-1) = A\hat{x}(k-1/k-2) + AK(k-1)[y(k-1) - C\hat{x}(k-1/k-2)] + B\hat{u}(k-1), \tag{3.12}$$
$$K(k) = P(k)C^T[GG^T]^{-1}, \tag{3.13}$$
$$P(k) = [I + H(k-1)C^T[GG^T]^{-1}C]^{-1}H(k-1), \tag{3.14}$$
$$H(k-1) = AP(k-1)A^T + FF^T. \tag{3.15}$$

We recall that the control (3.8)-(3.15) is optimal with respect to all linear output transformations provided the noises are white [21, 24], independently on their distributions. It is well known that [21, 24], when the disturbances are gaussian, the control (3.8)-(3.15) is optimal among all the Borel output transformations.

4. The extended system

In order to synthesize the optimal quadratic control for the system (3.1)-(3.2), let us define the extended state

$$X_e(k) = \begin{bmatrix} x_u(k) \\ x_s(k) \\ x_s^{[2]}(k) \\ x^*(k) \end{bmatrix} \tag{4.1}$$

where $x_u(k)$ and $x_s(k)$ are the input and noise dependent part of the state respectively

$$x(k) = x_u(k) + x_s(k) \tag{4.2}$$

satisfying the equations:

$$x_u(k+1) = Ax_u(k) + Bu(k), \tag{4.3}$$

$$x_s(k+1) = Ax_s(k) + FN_k, \tag{4.4}$$

with the initial conditions $x_u(0) = \Psi_{x,1}$ and $x_s(0)$ a zero mean random variable with

$$E(x_s^{[i]}(0)) = \Psi_{x,i}, \quad i = 2, 3, 4 \tag{4.5}$$

and

$$x^*(k) = F^{[2]}E(N_k^{[2]}) = F^{[2]}\Psi_{N,2}. \tag{4.6}$$

Only the dynamical equation for $x_s^{[2]}(k)$ has to be computed. We have:

$$\begin{aligned} x_s^{[2]}(k+1) &= x_s(k+1) \otimes x_s(k+1) \\ &= \big[Ax_s(k) + FN_k\big] \otimes \big[Ax_s(k) + FN_k\big] \\ &= A^{[2]}x_s^{[2]}(k) + \big\{(Ax_s(k)) \otimes (FN_k) + (FN_k) \otimes (Ax_s(k)) \\ &+ F^{[2]}N_k^{[2]}\big\} \\ &= A^{[2]}x_s^{[2]}(k) + x^*(k) + f_k \end{aligned} \tag{4.7}$$

with

$$\begin{aligned} f_k &= (Ax_s(k)) \otimes (FN_k) + (FN_k) \otimes (Ax_s(k)) \\ &+ F^{[2]}(N_k^{[2]} - \Psi_{N,2}) = \\ &= (I + C_{n,n}^T)[(Ax_s(k)) \otimes (FN_k)] + F^{[2]}(N_k^{[2]} - \Psi_{N,2}) \\ &= (I + C_{n,n}^T)((Ax_s(k)) \otimes I_n)(FN_k) + F^{[2]}(N_k^{[2]} - \Psi_{N,2}) \end{aligned} \tag{4.8}$$

where we have used the property:

$$\begin{aligned} (Ax_s(k)) \otimes (FN_k) &= (Ax_s(k) \cdot 1) \otimes (I_n \cdot FN_k) \\ &= ((Ax_s(k)) \otimes I_n) \cdot (FN_k). \end{aligned}$$

In order to define the quadratic optimal controller, let us introduce the following post-processed quadratic output

$$\tilde{y}_s(k) = [y(k) - Cx_u(k)]^{[2]} - G^{[2]}\Psi_{N,2} \tag{4.9}$$

which depends on the extended state by the equation

$$\tilde{y}_s(k) = C^{[2]}x_s^{[2]}(k) + g_k \tag{4.10}$$

with

$$\begin{aligned} g_k &= (Cx_s(k)) \otimes (GN_k) + (GN_k) \otimes (Cx_s(k)) \\ &+ G^{[2]}[N_k^{[2]} - \Psi_{N,2}]. \end{aligned} \tag{4.11}$$

Now we can state the following lemma.

Lemma 4.1. *The sequences* $\{f_k\}$ *and* $\{g_k\}$ *are mutually uncorrelated and white with:*

$$E(f_k) = E(g_k) = 0 \quad \forall k = 0, 1, 2, ..., \tag{4.12}$$

$$E(f_k \cdot f_j^T) = \begin{cases} 0 & k \neq j \\ \Psi_f(k) & k = j, \end{cases} \tag{4.13}$$

$$E(g_k \cdot g_j^T) = \begin{cases} 0 & k \neq j \\ \Psi_g(k) & k = j, \end{cases} \tag{4.14}$$

where

$$\begin{aligned} \Psi_f(k) = {} & (I + C_{n,n}^T)[(A\Psi_x(k)A^T) \otimes (FF^T)](I + C_{n,n}^T)^T \\ & + F^{[2]}\{st^{-1}(\Psi_{N,4} - \Psi_{N,2}^{[2]})\}F^{[2]^T}, \end{aligned} \tag{4.15}$$

$$\begin{aligned} \Psi_g(k) = {} & (I + C_{q,q}^T)[(C\Psi_x(k)C^T) \otimes (GG^T)](I + C_{q,q}^T)^T \\ & + G^{[2]}\{st^{-1}(\Psi_{N,4} - \Psi_{N,2}^{[2]})\}G^{[2]^T}. \end{aligned} \tag{4.16}$$

Proof. (4.12) is straigthforward obtained by taking into account the independence of $x_s(k)$ with N_k. To prove (4.13), by definition (4.8) for f_k, we have that, for $k \neq j$,

$$E(f_k \cdot f_j^T) = 0 \tag{4.17}$$

whereas, for $k = j$,

$$\begin{aligned} \Psi_f(k) = E(f_k \cdot f_k^T) = {} & (I + C_{n,n}^T)E\{((Ax_s(k)) \otimes I_n)(FN_k) \\ & \cdot (N_k^T F^T)[(Ax_s(k)) \otimes I_n]^T\}(I + C_{n,n}^T)^T \\ & + F^{[2]}E\{[N_k^{[2]} - \Psi_{N,2}] \cdot [N_k^{[2]} - \Psi_{N,2}]^T\}F^{[2]^T} \\ & + (I + C_{n,n}^T)E\{[(Ax_s(k)) \otimes I_n] \\ & \cdot FN_k[N_k^{[2]} - \Psi_{N,2}]^T\}F^{[2]^T}\} \\ & + F^{[2]}E\{(N_k^{[2]} - \Psi_{N,2})N_k^T \\ & \cdot F^T((Ax_s(k))^T \otimes I_n)\}(I + C_{n,n}^T)^T \\ = {} & (I + C_{n,n}^T)E\{[(Ax_s(k)) \otimes I_n]F \\ & \cdot F^T[(Ax_s(k)) \otimes I_n]^T\}(I + C_{n,n}^T)^T \\ & + F^{[2]}(st^{-1}(\Psi_{N,4} - \Psi_{N,2}^{[2]}))F^{[2]^T}. \end{aligned} \tag{4.18}$$

Moreover,

$$\begin{aligned} & E\{[(Ax_s(k)) \otimes I_n](FF^T)[(Ax_s(k)) \otimes I_n]^T\} \\ & = E\{[((Ax_s(k)) \otimes I_n)F] \cdot [((Ax_s(k)) \otimes I_n)F]^T\} \\ & = E\{[(Ax_s(k)) \otimes F] \cdot [(Ax_s(k))^T \otimes F^T]\} \\ & = \left(A \cdot E\{x_s(k)x_s^T(k)\} \cdot A^T\right) \otimes (FF^T) \\ & = (A\Psi_x(k)A^T) \otimes (FF^T), \end{aligned} \tag{4.19}$$

then

$$\begin{aligned} \Psi_f(k) = {} & (I + C_{n,n}^T)[(A\Psi_x(k)A^T) \otimes (FF^T)](I + C_{n,n}^T)^T \\ & + F^{[2]}\{st^{-1}(\Psi_{N,4} - \Psi_{N,2}^{[2]})\}F^{[2]^T}, \end{aligned} \tag{4.20}$$

where

$$\Psi_x(k) = E(x_s(k)x_s^T(k)) \tag{4.21}$$

satisfies the recursive equation

$$\Psi_x(k+1) = A\Psi_x(k)A^T + FF^T, \tag{4.22}$$
$$\Psi_x(0) = \Psi_{x(0)} = st^{-1}(\Psi_{x,2}). \tag{4.23}$$

Following the same way, we can compute the covariance of g_k.
We have $E(g_k \cdot g_j^T) = 0$ per $k \neq j$, whereas

$$\begin{aligned}\Psi_g(k) = E(g_k \cdot g_k^T) = (I + C_{q,q}^T)[(C\Psi_x(k)C^T) \otimes (GG^T)](I + C_{q,q}^T)^T \\ + G^{[2]}\{st^{-1}(\Psi_{N,4} - \Psi_{N,2}^{[2]})\}G^{[2]^T}.\end{aligned} \tag{4.24}$$

The uncorrelateness of f_k with g_j is due to the independence of the state noise FN_k with respect to the measurement noise GN_k. Therefore,

$$E(f_k \cdot g_j^T) = 0 \quad \forall k, j. \tag{4.25}$$

■

Now we can give the following proposition which summarizes the results of this section.

Proposition 4.1. *The extended state evolves according to the system equations*

$$X_e(k+1) = \mathcal{A}X_e(k) + \mathcal{B}u(k) + f_e(k), \tag{4.26}$$
$$Y_e(k) = \mathcal{C}X_e(k) + g_e(k) \tag{4.27}$$

where

$$Y_e(k) = \begin{pmatrix} y(k) \\ \tilde{y}_s(k) \end{pmatrix}, \tag{4.28}$$

$$\mathcal{A} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & A^{[2]} & I \\ 0 & 0 & 0 & I \end{pmatrix}, \tag{4.29}$$

$$\mathcal{B} = \begin{pmatrix} B \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{4.30}$$

$$\mathcal{C} = \begin{pmatrix} C & C & 0 & 0 \\ 0 & 0 & C^{[2]} & 0 \end{pmatrix}, \tag{4.31}$$

$$f_e(k) = \begin{pmatrix} 0 \\ FN_k \\ f_k \\ 0 \end{pmatrix}, \tag{4.32}$$

$$g_e(k) = \begin{pmatrix} GN_k \\ g_k \end{pmatrix} \tag{4.33}$$

and $f_e(k)$, $g_e(k)$ are uncorrelated sequences with covariances given by:

$$cov(f_e(k)) = \Psi_{f_e}(k) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & FF^T & (st^{-1}(F^{[3]}\Psi_{N,3}))^T & 0 \\ 0 & st^{-1}(F^{[3]}\Psi_{N,3}) & \Psi_f(k) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.34}$$

$$cov(g_e(k)) = \Psi_{g_e}(k) = \begin{pmatrix} GG^T & (st^{-1}(G^{[3]}\Psi_{N,3}))^T \\ st^{-1}(G^{[3]}\Psi_{N,3}) & \Psi_g(k) \end{pmatrix}. \tag{4.35}$$

Proof. Only (4.34) and (4.35) require a proof. Let us write:

$$cov(f_e(k)) = \Psi_{f_e}(k) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Psi_r(k) & 0 \\ 0 & 0 & 0 \end{pmatrix}; \tag{4.36}$$

then, to prove (4.34) is equivalent to prove that:

$$\Psi_r(k) = \begin{pmatrix} FF^T & (st^{-1}(F^{[3]}\Psi_{N,3}))^T \\ st^{-1}(F^{[3]}\Psi_{N,3}) & \Psi_f(k) \end{pmatrix}. \tag{4.37}$$

From (4.32) we have that

$$\Psi_r(k) = \begin{pmatrix} E((FN_k)\cdot(FN_k)^T) & E((FN_k)\cdot f_k^T) \\ E(f_k\cdot(FN_k)^T) & E(f_k\cdot f_k^T) \end{pmatrix}. \tag{4.38}$$

From (3.5) it follows that

$$(\Psi_r(k))_{1,1} = FF^T, \tag{4.39}$$

whereas, from the assumptions $E(N_k) = E(x_s(0)) = 0$ and using the rule $(2.3f)$, we can calculate that

$$(\Psi_r(k))_{2,1} = [(\Psi_r(k))_{1,2}]^T = st^{-1}(F^{[3]}\Psi_{N,3}); \tag{4.40}$$

finally, taking into account (4.38) and the equation (4.13) for $k = j$, it results

$$(\Psi_r(k))_{2,2} = \Psi_f(k). \tag{4.41}$$

Equations (4.38)-(4.41) prove the thesis (4.34).

In the same way, the equation (4.35) can be derived. ∎

5. The quadratic optimal control

In this section we will define the optimal linear control for the extended system (4.26)-(4.33). Of course, because of the quadratic structure of such system with respect to the original one, the optimal linear regulator will result to be the optimal quadratic regulator for the original system. In the extended state, the index (3.3) will become:

$$J = \frac{1}{2}E\{X_e^T(N)S_eX_e(N) + \sum_{k=0}^{N-1}(X_e^T(k)Q_eX_e(k) + u^T(k)Ru(k))\}, \tag{5.1}$$

where

$$S_e = \begin{pmatrix} S & S & 0 & 0 \\ S & S & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.2}$$

$$Q_e = \begin{pmatrix} Q & Q & 0 & 0 \\ Q & Q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.3}$$

Now we are in a position to state the main result of the paper.

Theorem 5.1. *The optimal quadratic regulator for the system (3.1)-(3.2) is given by the equations*

$$\hat{u}(k) = -R^{-1}B^TP_c(k+1)[I + BR^{-1}B^TP_c(k+1)]^{-1}A\hat{x}(k/k-1), \tag{5.12}$$

$$P_c(k) = A^TP_c(k+1)[I + BR^{-1}B^TP_c(k+1)]^{-1}A + Q, \tag{5.13}$$

$$P_c(N) = S, \tag{5.14}$$

$$\hat{x}(k/k-1) = x_u(k) + \hat{x}_s(k/k-1), \tag{5.15}$$

$$\nu_o(k) = \begin{pmatrix} y(k) - C[\hat{x}_s(k/k-1) + x_u(k)] \\ [y(k) - Cx_u(k)]^{[2]} - G^{[2]}\Psi_{N,2} - C^{[2]}\widehat{x_s^{[2]}}(k/k-1) \end{pmatrix}, \tag{5.16}$$

$$x_u(k+1) = Ax_u(k) + B\hat{u}(k), \qquad x_u(0) = E(x(0)), \tag{5.17}$$

$$\hat{x}_s(k+1/k) = A\hat{x}_s(k/k-1) + AK_1(k)\nu_o(k), \tag{5.18}$$

$$\widehat{x_s^{[2]}}(k+1/k) = A^{[2]}\widehat{x_s^{[2]}}(k/k-1) + A^{[2]}K_2(k)\nu_o(k) + F^{[2]}\Psi_{N,2}, \tag{5.19}$$

$$K_r(k) = \begin{pmatrix} K_1(k) \\ K_2(k) \end{pmatrix} = P_r(k)C_r^T\Psi_{g_e}^{-1}(k), \tag{5.20}$$

$$P_r(k+1) = [I + H_r(k)C_r^T\Psi_{g_e}^{-1}(k+1)C_r]^{-1}H_r(k), \tag{5.21}$$

$$P_r(0) = \begin{pmatrix} \Psi_{x(0)} & st^{-1}(\Psi_{x,3}) \\ [st^{-1}(\Psi_{x,3})]^T & st^{-1}(\Psi_{x,4}) - \Psi_{x,2}\cdot\Psi_{x,2}^T \end{pmatrix}, \tag{5.22}$$

$$H_r(k) = A_rP_r(k)A_r^T + \Psi_r(k), \tag{5.23}$$

where

$$A_r = \begin{pmatrix} A & 0 \\ 0 & A^{[2]} \end{pmatrix}, \tag{5.24}$$

$$C_r = \begin{pmatrix} C & 0 \\ 0 & C^{[2]} \end{pmatrix}. \tag{5.25}$$

Proof. The proof consists in proving that equations (5.12)-(5.25) correspond to the optimal linear control for the extended system (4.26)-(4.27) with respect to the index (5.1). From the classical theory of optimal regulator with quadratic index and white disturbances, we have that the optimal linear control for (4.26)-(4.27) is given by [21]

$$\hat{u}(k) = -M(k)\hat{X}_e(k/k-1), \tag{5.26}$$
$$M(k) = R^{-1}\mathcal{B}^T\mathcal{P}_c(k+1)[I+\mathcal{B}R^{-1}\mathcal{B}^T\mathcal{P}_c(k+1)]^{-1}\mathcal{A}, \tag{5.27}$$
$$\mathcal{P}_c(k) = \mathcal{A}^T\mathcal{P}_c(k+1)[I+\mathcal{B}R^{-1}\mathcal{B}^T\mathcal{P}_c(k+1)]^{-1}\mathcal{A}+\mathcal{Q}_e, \tag{5.28}$$
$$\mathcal{P}_c(N) = \mathcal{S}_e. \tag{5.29}$$

Let us start to observe that $\mathcal{P}_c(k)$ has the structure

$$\mathcal{P}_c(k) = \begin{pmatrix} P_c(k) & P_c(k) & 0 & 0 \\ P_c(k) & P_c(k) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.30}$$

with

$$P_c(N) = S. \tag{5.31}$$

By induction, assume (5.28) satisfied in $(k+1)$. We have, by direct substitution, that

$$\begin{aligned}\mathcal{P}_c(k) &= \mathcal{A}^T\mathcal{P}_c(k+1)[I+\mathcal{B}R^{-1}\mathcal{B}^T\mathcal{P}_c(k+1)]^{-1}\mathcal{A}+\mathcal{Q}_e \\ &= \begin{pmatrix} P_c(k) & P_c(k) & 0 & 0 \\ P_c(k) & P_c(k) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\end{aligned} \tag{5.32}$$

where

$$P_c(k) = A^TP_c(k+1)[I+BR^{-1}B^TP_c(k+1)]^{-1}A+Q. \tag{5.33}$$

The control law (5.12) is readily obtained by substituting (4.29), (4.30) and (5.32) into (5.26), (5.27) and by observing that

$$\hat{X}_e(k/k-1) = \begin{pmatrix} x_u(k) \\ \hat{x}_s(k/k-1) \\ \widehat{x_s^{[2]}}(k/k-1) \\ x^*(k) \end{pmatrix} \tag{5.34}$$

and that

$$\hat{x}(k/k-1) = x_u(k) + \hat{x}_s(k/k-1). \tag{5.35}$$

Now, from the Kalman filtering theory [23] applied to the extended system (4.26), (4.27), we have:

$$\begin{aligned}\hat{X}_e(k+1/k) = \mathcal{A}\hat{X}_e(k/k-1) + \mathcal{A}\mathcal{K}(k)[Y_e(k) - \mathcal{C}\hat{X}_e(k/k-1)] \\ + \mathcal{B}\hat{u}(k),\end{aligned} \tag{5.36}$$
$$\mathcal{K}(k) = \mathcal{P}(k)\mathcal{C}^T\Psi_{g_e}^{-1}(k), \tag{5.37}$$
$$\mathcal{P}(k) = [I+\mathcal{H}(k-1)\mathcal{C}^T\Psi_{g_e}^{-1}(k)\mathcal{C}]^{-1}\mathcal{H}(k-1), \tag{5.38}$$
$$\mathcal{H}(k-1) = \mathcal{A}\mathcal{P}(k-1)\mathcal{A}^T + \Psi_{f_e}(k-1). \tag{5.39}$$

The proof is completed by observing that

$$\mathcal{P}(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & P_r(0) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.40)$$

with

$$P_r(0) = \begin{pmatrix} E(x_s(0)x_s^T(0)) & E(x_s(0)(x_s^{[2]}(0) - \Psi_{x,2})^T) \\ [E(x_s(0)(x_s^{[2]}(0) - \Psi_{x,2})^T)]^T & E[(x_s^{[2]}(0) - \Psi_{x,2})(x_s^{[2]}(0) - \Psi_{x,2})^T] \end{pmatrix} \quad (5.41)$$

because the first and the last components of $X_e(k)$ are deterministic at any time instant. Moreover,

$$\begin{aligned} E(x_s(0) \cdot (x_s^{[2]}(0) - \Psi_{x,2})^T) &= E(x_s(0) \cdot (x_s^{[2]}(0))^T) \\ &= st^{-1}(\Psi_{x,3}), \\ E[(x_s^{[2]}(0) - \Psi_{x,2})(x_s^{[2]}(0) - \Psi_{x,2})^T] &= st^{-1}(\Psi_{x,4}) - \Psi_{x,2} \cdot \Psi_{x,2}^T, \end{aligned}$$

then we have (5.22). Now, it is readily verified that at any time instant

$$\mathcal{P}(k) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & P_r(k) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $P_r(k)$ is given by (5.21)-(5.23). Finally, by direct computation, it results

$$\mathcal{K}(k) = \begin{pmatrix} 0 \\ K_r(k) \\ 0 \end{pmatrix},$$

with $K_r(k)$ as in (5.20). System equation (5.36) corresponds to the equations (5.17), (5.18) and (5.19) as can be easily verified. This completes the proof. ∎

Remark. As a consequence of theorem 5.1, we have that the optimal quadratic control for the nongaussian stochastic system (3.1), (3.2) is simply obtained by using the optimal quadratic filter [19] instead of the classical linear Kalman filtering and the same feedback control gain as in the linear optimal regulator. It means that the classical separation principle (see for instance [22]) continue to be true in this case.

6. Numerical results

To test the effectiveness of the proposed algorithm, let us consider the optimal control problem for a linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k) + FN_k, \quad x(0) = \bar{x} \tag{6.1}$$
$$y(k) = Cx(k) + GN_k, \tag{6.2}$$

where , for any k,

$$x(k) \in \mathbb{R}^2, \quad y(k) \in \mathbb{R}^1, \quad u(k) \in \mathbb{R}^1$$

and with

$$A = \begin{pmatrix} 0 & 1 \\ 0.765 & -0.05 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad F = \begin{pmatrix} F_1 & \vdots & 0 \end{pmatrix},$$
$$C = (1 \quad 0), \quad G = \begin{pmatrix} 0 & \vdots & G_1 \end{pmatrix}, \tag{6.3}$$

where we have assumed

$$F_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad G_1 = (5); \tag{6.4}$$

the quadratic criterion we consider has the form:

$$J = \frac{1}{2} E\{x^T(N)Sx(N) + \sum_{k=0}^{N-1} (x^T(k)Qx(k) + u^T(k)Ru(k))\}, \tag{6.5}$$

with the choice for the weight matrices:

$$S = Q = I_2, \qquad R = 0.1. \tag{6.6}$$

$\{FN_k\}$ and $\{GN_k\}$ are independent, zero mean random sequences. The noise N_k has the probability distribution shown in Table I, where the same distribution for both the noise components is assumed.

Table I

$(N_k)_1$	$-10/\bar{\sigma}$	$1/\bar{\sigma}$	0		$(N_k)_2$	$-10/\bar{\sigma}$	$1/\bar{\sigma}$	0
$p[(N_k)_1]$	1/20	1/2	9/20		$p[(N_k)_2]$	1/20	1/2	9/20

The initial value of the stochastic part of the state $x_s(0)$ is a zero mean random variable, independent on the sequence $\{N_k\}$ and with the distributions found in Table II.

Table II

$(x_s(0))_1$	2	-1		$(x_s(0))_2$	2	-1
$p[(x_s(0))_1]$	1/3	2/3		$p[(x_s(0))_2]$	1/3	2/3

The optimal linear regulator and the optimal quadratic regulator algorithms have been implemented. In order to compare the results of the two methods, numerical simulations for the problem (6.1)-(6.6) have been performed in the same conditions in both cases. The results are displayed in Fig. 1 and Fig. 2 for 50 iterations.

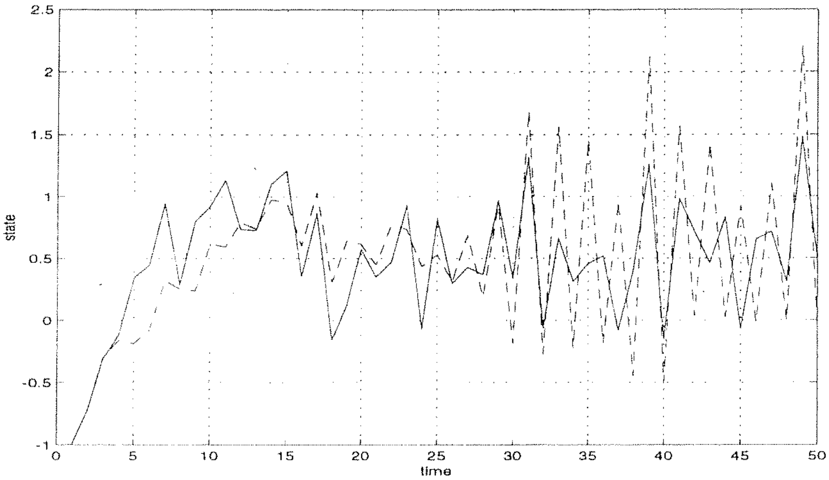

FIG. 1. *Behaviour of the state first component on $N = 50$ steps: optimal linear regulator (dashed line) and optimal quadratic regulator (solid line).*

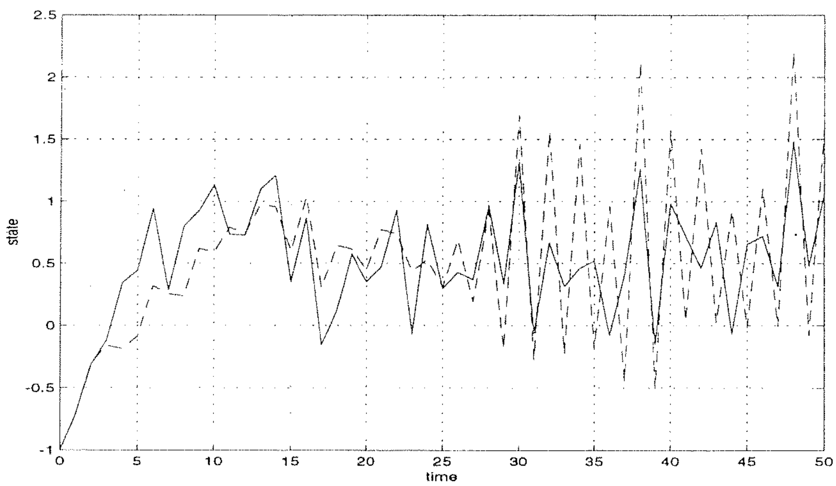

FIG. 2. *Behaviour of the state second component on $N = 50$ steps: optimal linear regulator (dashed line) and optimal quadratic regulator (solid line).*

It can be clearly seen that the state evolution of the process is closer to zero if we apply the quadratic optimal control law instead of the linear one.

In these simulations, the index J, that gives a measure of the state and control energy, has been calculated. It results:

$$J_{linear} = 38.2705;$$

$$J_{quadratic} = 25.9492.$$

The comparison of the linear case value with the quadratic case one confirms the effectiveness of the proposed method. The cost percentage reduction, in fact, results:

$$\left((J_{linear} - J_{quadratic})/J_{linear}\right) \cdot 100 = 32.19\%.$$

7. Concluding remarks

The optimal quadratic finite-horizon regulator for linear stochastic nongaussian systems is obtained in closed form by a recursive algorithm. It has been evidentiated that, in practice, it is obtained by using the optimal quadratic filter [19] instead of the linear Kalman filter and the same feedback control law as in the linear optimal regulator. As a consequence, this confirms the validity of the separation principle even in this case. Moreover, we have that the proposed algorithm does not need a considerable increase of the computational burden with respect to the linear one.

Numerical simulations, presented in section 6, show the high performance of the optimal quadratic control with respect to standard optimal linear control. It is reasonable to think that the present algorithm may be further on improved simply increasing the order of the filter used in the control system.

We think that it is important to do some remarks on the infinite-horizon controller. As in the classical L Q G problem, the existence of the steady-state solution for the optimal regulator is tied to the stabilizability and detectability of the extended system (4.26)-(4.33) united with the existence of a statistical steady-state solution for the stochastic part of the state. Hence, we have that, in the hypotesis of stability of the dinamical matrix A, the steady-state solution for the optimal regulator does certainly exist. The more general case of instability of the process will be matter of future research work.

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