A. Galluccio, M. Loebl

A THEORY OF PFAFFIAN ORIENTATIONS
1. PERFECT MATCHINGS AND PERMANENTS.

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Anna Galluccio — Istituto di Analisi dei Sistemi ed Informatica del CNR, Viale Manzoni 30, 00185 Roma, Italy

Martin Loebl — Department of Applied Mathematics, Charles University, 11800 Prague 1, Czech Republic.

Abstract

Kasteleyn stated that the generating function of the perfect matchings of a graph of genus g may be written as a linear combination of 4^g Pfaffians. Here we prove this statement. As a consequence we present a combinatorial way to compute permanents of square matrices.

Other consequences will be presented in a forthcoming paper.

1 Introduction

We present a theory of Pfaffian orientations of graphs, introduced by Kasteleyn ([6, 5, 4]). Our approach is an extension of the treatment of toroidal rectangular lattices (see [3, 9, 6, 5, 4]). The case of general toroidal graphs was also studied by Barahona ([1]). As a consequence, we present a new technique to compute permanents of square matrices, which completes the Polya's scheme ([8]).

$G = (V, E)$ will always be a graph and x_e will be a variable associated with each edge e of G. We let $x = (x_e : e \in E)$ and for $M \subset E$ let $x(M)$ denote the product of the variables of the edges of M. An orientation of a graph G is a digraph obtained from G by fixing an orientation of each edge of G.

Let $A\Delta B$ denote the symmetric difference of the sets A and B and let $a \stackrel{q}{=} b$ denote $a = b$ modulo 2.

Definition 1.1 *The generating function of the perfect matchings of G is the polynomial $\mathcal{P}(G, x)$ equal to the sum of $x(P)$ over all perfect matchings P of G.*

Definition 1.2 *Let G be a graph and let D be an orientation of G. Let M be a perfect matching of G. For each perfect matching P of G let $sgn(D, M\Delta P) = (-1)^n$ where n is the number of clockwise even alternating cycles of $M\Delta P$, and let $\mathcal{P}(D, M)$ equal the sum of $sgn(D, M\Delta P)x(P)$ over all perfect matchings P of G.*

Definition 1.3 *Let $G = (V, E)$ be a graph with $2n$ vertices and D an orientation of G. Denote by $A(D)$ the skew-symmetric matrix with the rows and the columns indexed by $|V|$, where $a_{vw} = x_{(v,w)}$ in case (v, w) is an arc of D, $a_{vw} = -x_{(v,w)}$ in case (w, v) is an arc of D, and $a_{vw} = 0$ otherwise.*

The Pfaffian of the skew-symmetric matrix $A(D)$ is defined as

$$Pf(A(D)) = \sum_P s^*(P) a_{i_1 j_1} \cdots a_{i_n j_n}$$

where $P = \{\{i_1 j_1\}, \cdots, \{i_n j_n\}\}$ is a partition of the set $\{1, \ldots, 2n\}$ into pairs, $i_k < j_k$ for $k = 1, \ldots, n$, and $s^*(P)$ equals to the sign of the permutation $(i_1 j_1 \ldots i_n j_n)$.

Each nonzero term of the expansion of the Pfaffian of $A(D)$ equals $x(P)$ or $-x(P)$ where P is a perfect matching of G. Let $s(D, P)$ equal the sign of the term $x(P)$ so that

$$Pf(A(D)) = \sum_P s(D,P)x(P)$$

The following theorem was proved by Kasteleyn ([4]).

Theorem 1.4 *Let G be a graph and D an orientation of G. Let P, M be two perfect matchings of G. Then*

$$s(D,P) = s(D,M)sgn(D, M\Delta P).$$

Hence,

$$Pf(A(D)) = s(D,M)\sum_P sgn(D, M\Delta P)x(P)$$

$$= s(D,M)\mathcal{P}(D,M).$$

The following theorem is well-known (see [2]).

Theorem 1.5 *Let G be a graph and let D be an orientation of G.*
Then $Pf^2(A(D)) = det(A(D))$.

Remark. This theorem has an algorithmic application: if the edge-set of G is partitioned into a bounded number of classes and the variables x_e are equal in each class, then $\mathcal{P}(D,M)$ and $Pf(A(D))$ may be determined efficiently.

Let $M = \{\{i_1 j_1\}, \ldots, \{i_n j_n\}\}$, $i_k < j_k$, be a perfect matching of G. Let x' be defined as follows: $x'_e = x_e$, if $e \notin M$ and $x'_f = x_f z$, if $f \in M$, z is a new variable. Let A' be the matrix obtained from $A(D)$ by replacing each x_e by x'_e. Using Gaussian elimination we can express efficiently $det(A')$ as a rational function, i.e. a ratio of two polynomials. Since $det(A')$ is a polynomial, its coefficients can be determined efficiently from the rational function. We can view $det(A')$ as a polynomial $det(A')(x')$ or as a polynomial $det(A')(x,z)$. By Theorem 1.5, $Pf(A')(x,z) = \pm\sqrt{det(A')(x,z)}$. Hence we can determine efficiently a polynomial $Q(x,z)$ such that $Pf(A')(x,z) = \pm Q(x,z)$. Note that $\mathcal{P}(D,M) = \pm Q(x,1)$.

There is exactly one monomial in $Q(x,z)$ containing z^n and its coefficient is $+1$ or -1. Let $Q'(x,z)$ be the unique polynomial such that $Q'(x,z) = Q(x,z)$ or $Q'(x,z) = -Q(x,z)$ and the coefficient of $Q'(x,z)$ of the term containing z^n equals to $+1$. We have $\mathcal{P}(D,M) = +Q'(x,1)$. Moreover,

$Pf(A(D)) = s(D,M)\mathcal{P}(D,M)$ and $s(D,M) = s^*(M)t^*(M)$ where $t^*(M)$ equals to the product of the signs of the elements $a_{i_k j_k}$ of the matrix $A(D)$ such that $i_k j_k \in M$. Hence $\mathcal{P}(D,M)$ and $Pf(A(D))$ may be determined efficiently.

Kasteleyn ([4]) introduced the following notion.

Definition 1.6 *A graph G is called Pfaffian if it has a Pfaffian orientation, i.e. an orientation such that each alternating cycle with respect to an arbitrary fixed perfect matching M of G is clockwise odd.*

Hence if a graph G has a Pfaffian orientation D then $s(D,P)$, P perfect matching of D, are equal and $\mathcal{P}(G,x)^2 = Pf^2(A(D)) = det(A(D))$.

Kasteleyn ([4]) also observed that the planar graphs have a Pfaffian orientation.

Theorem 1.7 *Each planar graph has a Pfaffian orientation.*

Proof. Let G be a planar graph, and let M be its perfect matching. Consider G drawn on the plane. Orient edges of G so that each face, except possibly the outer one, is clockwise odd. Each such face 'encircles' no vertex of G. Observe that the orientation has the property that a cycle C of G is clockwise odd if and only if C encircles an even number of vertices. Each alternating cycle with respect to M encircles an even number of vertices and hence it is clockwise odd. □

2 The Perfect Matchings

Definition 2.1 *We define surface S_g , g positive integer, as follows. It consists of a base B_0 and $2g$ bridges B^i_j, $i = 1, ..., g$ and $j = 1, 2$. B_0 is a convex $4g$-gon with vertices $a_1, ..., a_n$, $n = 4g$, numbered clockwise. Bridge B^i_1 is a 4-gon with vertices $x^i_1, x^i_2, x^i_3, x^i_4$ numbered clockwise . It is glued with B_0 so that edge (x^i_1, x^i_2) of B^i_1 is identified with edge $(a_{4(i-1)+1}, a_{4(i-1)+2})$ of B_0 and edge (x^i_3, x^i_4) of B^i_1 is identified with edge $(a_{4(i-1)+3}, a_{4(i-1)+4})$ of B_0 .*
Bridge B^i_2 is a 4-gon with vertices $y^i_1, y^i_2, y^i_3, y^i_4$ numbered clockwise. It is glued with B_0 so that edge (y^i_1, y^i_2) of B^i_2 is identified with edge $(a_{4(i-1)+2}, a_{4(i-1)+3})$ of B_0 and edge (y^i_3, y^i_4) of B^i_2 is identified with edge $(a_{4(i-1)+4}, a_{4(i-1)+5 mod 4g})$ of B_0.

The definition of surface S_g corresponds to the usual definition of an orientable surface of genus g in the following sense. Orientable surface O_g of genus g may be obtained from S_g as follows: for each bridge B, glue together the two segments in which B intersects the boundary of B_0, and delete B.

Definition 2.2 *We say that a graph G is a g-graph if it may be drawn on S_g so that all the vertices belong to the base B_0, and each edge uses at most one bridge. The set of the edges drawn on the base will be denoted by $E_0 = E_0(G)$ and the set of edges drawn on bridge B^i_j will be denoted by $E^i_j = E^i_j(G)$.*

If moreover the following conditions are satisfied then we say that G is a proper g-graph.

1. *The outer face of the subgraph embedded on B_0 is a cycle, and it is embedded on the boundary of B_0,*

2. *Each vertex is incident with at most one edge out of E_0,*

3. *The subgraph embedded on B_0 has a perfect matching.*

If G is a proper g-graph then we denote by C_0 the cycle which forms the outer face of E_0, and we denote by M_0 a perfect matching of the subgraph of G embedded on B_0.

For each g-graph we fix its drawing on S_g.

Definition 2.3 *Let $G = (V, E)$ be a proper g-graph. The graphs $G_0 = (V, E_0)$ and $G^i_j = (V, E_0 \cup E^i_j)$ are planar. We define orientations of G_0 and G^i_j as follows: orientation D_0 of G_0 is Pfaffian and such that each face of G_0 is clockwise odd (as in the proof of Theorem 1.7). Next we draw G^i_j on the plane so that the drawing of G_0 is unchanged, and edge (x^i_1, x^i_4) ((y^i_1, y^i_4) respectively)of B^i_j belongs to the outer face of the drawing of G^i_j. Now complete D_0 to an orientation of G^i_j so that each face is clockwise odd. This defines orientation $+D^i_j$ of E^i_j.*

$-D^i_j$ is defined by reversing the orientation of D^i_j.

Remark 2.4 *If G is a proper g-graph and Pfaffian orientation D_0 is fixed, then D^i_j is uniquelly determined for each ij.*

Definition 2.5 *Let G be a proper g-graph, $g \geq 1$. An orientation D of G which equals to D^i_j or $-D^i_j$ on E^i_j and to D_0 on E_0 is called relevant. We define its type $r(D) \in \{+1, -1\}^{2g}$ as follows. For $i = 0, \ldots, g-1$ and $j = 1, 2$, $r(D)_{2i+j}$ equals to $+1$ or -1 according to the sign of D^{i+1}_j in D.*

Definition 2.6 *Let $G = (V, E)$ be a proper g-graph and let A be a subset of its edges. We define its type $t(A) \in \{+1, -1\}^{2g}$ as follows. For $i = 0, .., g-1$ and $j = 1, 2$, we let $t(A)_{2i+j}$ equal to $(-1)^{s(A)_{2i+j}}$, where $s(A)_{2i+j}$ equals to the number of edges of A which belong to E^{i+1}_j.*

Let $CR(A) \stackrel{q}{=} \sum_{i=0}^{g-1} s(A)_{2i+1} \cdot s(A)_{2i+2}$.

Let $BR(A)$ be the subset of edges of A which do not belong to E_0.

For each $e \in BR(A)$, let $d(e) = 2i + j$ such that $e \in E^{i+1}_j$.

Let G be a proper g-graph. Any alternating cycle with respect to M_0 will be called *alternating cycle*. If C is a cycle of even length or a cycle embedded in the plane, then we denote by $l(C)$ the number of arcs of C directed clockwise, modulo 2.

We want to show that $sgn(D, M_0 \Delta P)$ depends only on $t(M_0 \Delta P)$ and $r(D)$.

Lemma 2.7 *Let G be a proper g-graph. Let $C_1, ..., C_k$ be vertex-disjoint cycles of G and let $\mathcal{C}$ denote their union. Then*

$$CR(\mathcal{C}) \stackrel{q}{=} \sum_{i=1}^{k} CR(C_i).$$

Proof. $CR(\mathcal{C})$ equals to the sum modulo 2 of $s(\mathcal{C})_{2i+1} \cdot s(\mathcal{C})_{2i+2}$, $i = 0, ..., g-1$. Now consider the drawing of cycles $C_1, ..., C_k$ in the plane, obtained by projecting each E^i_j outside of B_0. The total number of crossings modulo 2 also equals to the sum modulo 2 of $s(\mathcal{C})_{2i+1} \cdot s(\mathcal{C})_{2i+2}$, $i = 0, ..., g-1$. Each C_l, $l = 1, ..., k$ becomes a closed curve in the plane. Each pair of curves representing C_i and C_j intersect each other an even number of times. Hence we can forget these crossings without influencing the sum modulo 2 of $s(\mathcal{C})_{2i+1} \cdot s(\mathcal{C})_{2i+2}$, $i = 0, ..., g-1$. Each of the remaining crossings is a crossing of some C_l, $l = 1, ..., k$. □

Next we consider the case that there is exactly one alternating cycle of $M_0 \Delta P$.

Theorem 2.8 *Let G be a proper g-graph and let D be a relevant orientation of G. Let C be an alternating cycle of G. Then*

$$l(C) \stackrel{q}{=} [|BR(C)| - 1 - CR(C) + 1/2 \sum_{e \in BR(C)} (r(D)_{d(e)} + 1)].$$

Proof. We will assume without loss of generality that $G = C \cup C_0 \cup M_0$.

Claim 1. Let for each $i = 1, ...g$, C intersects at most one of E_1^i, E_2^i. Then

$$l(C) \stackrel{q}{=} [|BR(C)| - 1 + 1/2 \sum_{e \in BR(C)} (r(D)_{d(e)} + 1)].$$

Proof of Claim 1. C satisfying the properties of Claim 2 may be embedded in a planar way, by projecting the non-empty bridges outside of B_0. Hence $l(C) = 1$ iff $|\{e \in BR(C) : r(D)_{d(e)} = -1\}| \stackrel{q}{=} 0$. From this Claim 1 follows.

End of Claim 1.

We procede by induction on $|BR(C)|$. The case $|BR(C)| = 0$ is considered in Claim 1.

Let theorem hold for all alternating cycles C' with $|BR(C)| > |BR(C')|$. Hence for such alternating cycles, $l(C')$ depends only on $t(C')$ and $r(D)$.

We let $l(t(F), r(H))$ denote the parity of the number of edges oriented clockwise of alternating cycle F with $|BR(F)| < |BR(C)|$, in relevant orientation H of a proper g-graph.

We will make the following notational agreement: if a segment S of C_0 is traversed clockwise then we denote it by $+S$, otherwise by $-S$.

If P is a path together with a prescribed way of traversing it, we denote by $l(P)$ the parity of the number of arcs of P oriented in agreement with the way of traversing P.

Claim 2. Let there be a bridge $B = B_j^i$ containing more than one edge of C. Then 2.8 holds.

Proof of Claim 2. Let e, f be two edges of C drawn on B which 'see' each other on B, i.e. there is no other edge of C drawn between them on B.

We remind that C_0 denotes the outer face of G_0 and e, f do not belong to $M_0 \subset E_0$.

Without loss of generality let e be nearer to edge $[a_{2(i-1)+j}, a_{2(i-1)+j+3}]$ of $B = B_j^i$ than f.

Let R be the cycle of G formed by e, f and two subpaths R_1, R_2 of C_0 defined by the endvertices of e, f. By the choice of e, f, R is a face of the planar drawing of $G^i_j = (V, E_0 \cup E^i_j)$. Observe that $l(R) = 1$ in the orientation of G^i_j induced by the relevant orientation D.

Let us consider new edge g (not belonging to G), between endpoints of e, f such that one of two cycles H_1, H_2 formed by g and C and containing g is alternating. Without loss of generality, let g use vertex $u \in R_1 \cap e$. Hence we have that $g = uv_1$ or $g = uv_2$ where $u \in R_1 \cap e$ and $v_i \in R_i \cap f$, $i = 1, 2$.

We denote two opposite orientations of g by g_1, g_2. Let H_1 use g_1 and H_2 use g_2.

Observe that $l(C) = l(H_1) + l(H_2)$.

We will assume without loss of generality that H_2 is alternating. Hence H_1 contains both e, f.

We distinguish two cases now.

Case 1: $g = uv_1$**.**

Let $G' = G \cup \{g_1, g_2\}$. We consider g_1, g_2 embedded on B_0 along R_1.

Observe that $CR(C) \stackrel{q}{=} CR(H_1) + CR(H_2)$: let us project all the edges of $G' - E_0$ outside of B_0. The edge-crossings are the same as the original edge-crossings of G; hence the parity of the number of edge-crossings of G' equals to $CR(C)$.

In addition to these crossings, H_1 and H_2 share two vertices of g_1, g_2. However, in our plane drawing these two vertices are touching points between the closed curves representing H_1, H_2, not crossing points. Hence, by the same argument as in the proof of Lemma 2.7, $CR(C)$ equals the sum modulo 2 of the number of crossings of H_1 and the number of crossings of H_2. The number of crossings of H_i modulo 2 equals $CR(H_i)$, $i = 1, 2$. Hence summarising $CR(C) \stackrel{q}{=} CR(H_1) + CR(H_2)$.

We construct two digraphs D_1, D_2 as follows:

D_1 is obtained from $D - \{e, f, \}$ by adding new vertices u', v'_1 of degree 2, incident with new arcs e', f', g'_1. $e' f', g'_1$ are obtained from e, f, g_1 by replacing u by u' and v_1 by v'_1. We extend B_0 along R_2 and consider e', f', g'_1 embedded on extended B_0. Finally we add g'_1 to M_0. Let H'_1 be the cycle of D_1 obtained from H_1 by replacing e, f, g_1 by e', f', g'_1. We have $l(H_1) = l(H'_1)$.

D_2 is obtained from $D - \{e, f, \}$ by adding arc g_2. We again extend B_0 along R_1 and consider g_2 embedded on extended B_0. We let $H'_2 = H_2$.

Hence for $i = 1, 2$, D_i is orientation of a proper g-graph and H'_i is an alternating cycle of D_i. Moreover $|BR(H'_i)| < |BR(C)|$. We also have that $CR(H_2) \stackrel{q}{=} CR(H'_2)$ and $CR(H_1) \stackrel{q}{=} CR(H'_1)$.

We remind that $l(R) = 1$ in the orientation of G^i_j induced by D. Hence, exactly one of g_i is oriented so that both cycles it makes with R are clockwise odd. Let it be g_2. Then D_2 is a relevant orientation and D_1 becomes relevant after reversing orientation of g_1. By the induction assumption for H_1', H_2', $l(H_2') = l(t(H_2'), r(D_2))$ and $l(H_1') \overset{q}{=} l(t(H_1'), r(D_1)) + 1$. We have that

$$l(C) \overset{q}{=} l(H_1) + l(H_2) \overset{q}{=} l(H_1') + l(H_2') \overset{q}{=}$$
$$l(t(H_1'), r(D_1)) + l(t(H_2'), r(D_2)) + 1 \overset{q}{=}$$

$$[|BR(C)| - 2 - 2 - CR(H_1') - CR(H_2') + 1/2 \sum_{h \in BR(C) - \{e,f\}} (r(D)_{d(h)} + 1)] + 1 \overset{q}{=}$$

$$[|BR(C)| - 1 - CR(C) + 1/2 \sum_{h \in BR(C)} (r(D)_{d(h)} + 1)].$$

End of Case 1.

Case 2: $g = uv_2$.

Let $G' = G \cup \{g_1, g_2\}$. We consider g_1, g_2 embedded on B.

Observe that $CR(C) + 1 \overset{q}{=} CR(H_1) + CR(H_2)$: let us project all the edges of $G' - E_0$ outside of B_0. The edge-crossings are: the original crossings of G; the crossings of g_i and an edge $h \neq g_{2-i}$: but the total number of these crossings is even since each such edge h intersects both g_1, g_2.

Hence the parity of the number of edge-crossings equals $CR(C)$.

In addition to these crossings, H_1 and H_2 share two vertices of g_1, g_2. In our plane drawing one of them is a crossing point and the other one a touching point between the closed curves representing H_1, H_2.

Hence, by the same argument as in the proof of Lemma 2.7, $CR(C) + 1$ equals the sum modulo 2 of the number of crossings of edges of H_1 and the number of crossings of the edges of H_2. The number of crossings of H_i modulo 2 equals to $CR(H_i)$, $i = 1, 2$. Hence summarising $CR(C) + 1 \overset{q}{=} CR(H_1) + CR(H_2)$.

We construct two digraphs D_1, D_2 as follows:

D_1 is obtained from $D - \{e, f\}$ by adding new arc g_1' between v_1 and the endvertex u' of e different from u. If $l(fg_1e) = 1$ then we let $g_1' = (v_1u')$. If $l(fg_1e) = 0$ then $l(eg_1f) = 1$ and we let $g_1' = (u', v_1)$.

We consider g_1' embedded on B. Let H_1' be obtained from H_1 by replacing f, g_1, e by g_1'. We have $l(H_1) = l(H_1')$.

D_2 is obtained from $D - \{e, f\}$ by adding arc g_2. We consider g_2 embedded on bridge B. We let $H_2 = H_2'$.

Hence for $i = 1, 2$, D_i is orientation of a proper g-graph and H'_i is a cycle of D_i. Moreover $|BR(H'_i)| < |BR(C)|$. We also have that $CR(H_2) = CR(H'_2)$ and $CR(H_1) = CR(H'_1)$.

We remind that $l(R) = 1$ in the orientation of G^i_j induced by D. Hence exactly one of g_i is oriented so that both cycles it makes with R are clockwise odd. Let it be g_2. Hence D_2 is relevant with $r(D_2) = r(D)$ if and only if $r(D)_{2(i-1)+j} = 1$, and we have:

$l(H_2) = l(t(H'_2), r(D))$ if and only if $r(D)_{2(i-1)+j} = 1$. Otherwise $l(H_2) \stackrel{q}{=} l(t(H'_2), r(D)) + 1$.

We will show that the same holds for H'_1: let R_3 be the segment of C_0 such that $H = (e, -R_1, -R_3, -R_2)$ is a cycle.
We show that $l(g'_1, -R_3, -R_2) = 1$: $l(g'_1, -R_3, -R_2) = l(e, g_1, f, -R_3, -R_2) \stackrel{q}{=} l(f, -R_3) + l(e, g_1, -R_2)$.
If $r(D)_{2(i-1)+j} = 1$ then $l(f, -R_3) = 1$ and $l(e, g_1, -R_2) = 0$ since $l(e, g_2, -R_2) = 1$.

If $r(D)_{2(i-1)+j} = -1$ then $l(f, -R_3) = 0$ and $l(e, g_1, -R_2) = 1$ since $l(e, g_2, -R_2) = 0$.

Summarising $l(g'_1, -R_3, -R_2) = 1$.

Hence D_1 is relevant with $r(D_1) = r(D)$ if and only if $r(D)_{2(i-1)+j} = 1$, and we again have:

$l(H_1) = l(H'_1) = l(t(H'_1), r(D))$ if and only if $r(D)_{2(i-1)+j} = 1$. Otherwise $l(H_1) = l(H'_1) \stackrel{q}{=} l(t(H'_1), r(D)) + 1$.

Summarising and using the induction assumption of 2.8 for H'_1, H'_2 we get:

$$l(C) \stackrel{q}{=} l(H_1) + l(H_2) \stackrel{q}{=}$$

$$l(t(H'_1), r(D)) + l(t(H'_2), r(D)) \stackrel{q}{=}$$

$$[|BR(C)| - 2 - CR(C) - 1 +$$

$$1/2 \sum_{h \in BR(C) - \{e,f\}} (r(D)_{d(h)} + 1) +$$

$$1/2(r(D)_{d(g_2)} + 1) +$$

$$1/2(r(D)_{d(g'_1)} + 1)] \stackrel{q}{=}$$

$$[|BR(C)| - 1 - CR(C) + 1/2 \sum_{h \in B(C)} (r(D)_{d(h)} + 1)].$$

End of Case 2.
End of Claim 2.

Claim 3. Let there be i such that C contains exactly one edge from both E_1^i and E_2^i. Then theorem 2.8 holds.

Proof of Claim 3. Let e, f be the two edges on bridges E_1^i and E_2^i, respectively, and let C_1 and C_2 be two paths such that $C = (C_1, e, C_2, f)$.

The end-vertices of e, f belong to C_0. Let us assume that along the boundary of B_0 from $a_{4(i-1)+1}$ to a_{4i+1}, the endvertices of e, f appear in the order (v_1, u_1, v_2, u_2) where $e = u_1u_2$ and $f = v_1v_2$.

Let R_1, R_2 be two disjoint subpaths of the segment of C_0 between $a_{4(i-1)+1}$ and a_{4i+1}, which cover the endvertices of e, f. R_1, R_2 contain no other vertex of G incident with an edge out of E_0, by the choice of i. R_1, R_2, e, f form a cycle R, and without loss of generality, let $R = (R_1, e, -R_2, f)$ where R_1 is traversed clockwise, i.e. in agreement with the indices of the vertices along C_0, and R_2 is traversed anticlockwise.

Let us consider new adge g (not belonging to G), between endpoints of e, f such that one of two cycles I_1, I_2 formed by g and C and containing g is alternating. Without loss of generality let g use vertex $u_1 \in R_1 \cap e$. Hence we have that $g = u_1v_1$ or $g = u_1v_2$ where $u_1 \in R_1 \cap e$ and $v_i \in R_i \cap f$, $i = 1, 2$.

We denote two opposite orientations of g by g_1, g_2. Let I_1 use g_1 and I_2 use g_2.

Observe that $l(C) \stackrel{q}{=} l(I_1) + l(I_2)$.

We will assume without loss of generality that I_2 is alternating. Hence I_1 contains both e, f.

Let R_3 denote the segment of C_0 between u_1 and v_2.

Again we distinguish two cases.

Case 1: $g = u_1v_1$**.**

In this case g forms a cycle with R_1.

Let $G' = G \cup \{g_1, g_2\}$. We consider g_1, g_2 embedded on B_0 along R_1.

Observe that $CR(C) \stackrel{q}{=} CR(I_1) + CR(I_2)$: let us project all the edges of $G' - E_0$ outside of B_0. The edge-crossings are the same as the original edge-crossings of G. Hence the parity of the number of edge-crossings equals $CR(C)$.

In addition to these crossings, I_1 and I_2 share two vertices of g_1, g_2. However, in our plane drawing these two vertices are touching points between the closed curves representing I_1, I_2, not crossing points. Hence, by the same argument as in the proof of Lemma 2.7, $CR(C)$ equals the sum modulo 2 of the number of crossings of I_1 and the number of crossings of I_2. The number of crossings of I_i equals to $CR(I_i)$, $i = 1, 2$. Hence summarising

$CR(C) \stackrel{q}{=} CR(I_1) + CR(I_2)$.

We construct two digraphs D_1, D_2 as follows:

D_1 is obtained from $D - \{e, f\}$ by adding new vertices u_1', v_1' of degree 2, incident with new arcs e', f', g_1'. $e' f', g_1'$ are obtained from e, f, g_1 by replacing u_1 by u_1' and v_1 by v_1'. We extend B_0 along R_2 and consider e', f', g_1' embedded on extended B_0. Finally we add g_1' to M_0. Let I_1' be the cycle of D_1 obtained from I_1 by replacing e, f, g_1 by e', f', g_1'. We have $l(I_1) = l(I_1')$.

D_2 is obtained from $D - \{e, f\}$ by adding arc g_2. We again extend B_0 along R_1 and consider g_2 embedded on extended B_0. We let $I_2' = I_2$.

Hence for $i = 1, 2$, D_i is orientation of a proper g-graph and I_i' is an alternating cycle of D_i. Moreover $|BR(I_i')| < |BR(C)|$. We also have that $CR(I_2) = CR(I_2')$ and $CR(I_1) - 1 \stackrel{q}{=} CR(I_1')$.

Let us assume without loss of generality that g_2 is directed so that the cycle $l(-R_1, g_2) = 1$. Hence D_2 is a relevant orientation.

Observe that D_1 is a relevant orientation if and only if $r(D)_{d(e)} = r(D)_{d(f)}$:
first we prove if $r(D)_{d(e)} = r(D)_{d(f)} = 1$ then D_1 is a relevant orientation. In this case we need to show that $l(-R_2, f', g_1', e') = 1$.
We have $l(g_1', f', -R_3) = l(g_1, f, -R_3) \stackrel{q}{=} l(g_2, f, -R_3) + 1 = 0$ since $r(D_{d(f)}) = 1$ and thus $l(-R_3, g_2, f) = 1$. Moreover $l(g_1', f') = l(f', g_1')$ since it does not matter which way two consecutive arcs are traversed. Hence $l(f', g_1') = l(-R_3)$.

Moreover $l(-R_2, -R_3, e') = l(-R_2, -R_3, e) = 1$ since $r(D)_{d(e)} = 1$. Replacing $f' g_1'$ for $-R_3$ gives what we needed.

Analogously, if $r(D)_{d(e)} = r(D)_{d(f)} = -1$ then $l(-R_2, f', g_1', e') = 1$ and D_1 is a relevant orientation.

On the other hand, if $r(D)_{d(e)} \neq r(D)_{d(f)}$ then D_1 is obtained from a relevant orientation by reversing one arc; hence D_1 is not relevant.

Summarizing,

$$l(I_1) + l(I_2) \stackrel{q}{=} l(I_1') + l(I_2') \stackrel{q}{=} l(t(I_1'), r(D_1)) + l(t(I_2'), r(D_2)) +$$

$$1/2(r(D)_{d(e)} - 1 + r(D)_{d(f)} - 1).$$

Using the induction assumption of 2.8 for I_1', I_2' we get:

$$l(C) \stackrel{q}{=} l(I_1) + l(I_2) \stackrel{q}{=}$$

$$[l(t(I_1'), r(D_1)) + l(t(I_2'), r(D_2)) +$$

$$1/2(r(D)_{d(e)} - 1 + r(D)_{d(f)} - 1)] \stackrel{q}{=}$$

$$[|BR(C)| - 4 - CR(C) + 1 + 1/2 \sum_{h \in BR(C) - \{e,f\}} (r(D)_{d(h)} + 1) +$$

$$1/2(r(D)_{d(e)} - 1 + r(D)_{d(f)} - 1)] \stackrel{q}{=}$$

$$[|BR(C)| - 1 - CR(C) + 1/2 \sum_{h \in BR(C)} (r(D)_{d(h)} + 1)].$$

End of Case 1.

Case 2: $g = uv_2$**.**

In this case g forms a cycle with R_3.

Let $G' = G \cup \{g_1, g_2\}$. We consider g_1, g_2 embedded on B_0 along R_3.

Observe that $CR(C) \stackrel{q}{=} CR(I_1) + CR(I_2)$: Let us project all the edges of $G' - E_0$ outside of B_0. The edge-crossings are the same as the original crossings of G.

Hence the parity of the number of edge-crossings equals $CR(C)$.

In addition to these crossings, I_1 and I_2 share two vertices of g_1, g_2. In the plane drawing both of them are touching points between the closed curves representing I_1, I_2.

Hence, by the same argument as in the proof of Lemma 2.7, $CR(C)$ equals the sum modulo 2 of the number of crossings of I_1 and the number of crossings of I_2. The number of crossings of I_i equals to $CR(I_i)$, $i = 1, 2$. Hence summarising $CR(C) \stackrel{q}{=} CR(I_1) + CR(I_2)$.

We construct two digraphs D_1, D_2 as follows:

D_1 is obtained from $D - \{e, f\}$ by adding new vertices u'_1, v'_2 of degree 2, incident with new arcs e', f', g'_1. e', f', g'_1 are obtained from e, f, g_1 by replacing u_1 by u'_1 and v_2 by v'_2. We extend B_0 along $+R_1 + R_3 + R_2$ and consider e', f', g'_1 embedded on extended B_0. Finally we add g'_1 to M_0. Let I'_1 be the cycle of D_1 obtained from I_1 by replacing e, f, g_1 by e', f', g'_1. We have $l(I_1) = l(I'_1)$.

D_2 is obtained from $D - \{e, f, \}$ by adding arc g_2. We again extend B_0 along R_3 and consider g_2 embedded on extended B_0. We let $I'_2 = I_2$.

Hence for $i = 1, 2$, D_i is orientation of a proper g-graph and I'_i is an alternating cycle of D_i. Moreover $|BR(I'_i)| < |BR(C)|$. We also have that $CR(I_2) = CR(I'_2)$ and $CR(I_1) - 1 \stackrel{q}{=} CR(I'_1)$.
Let us assume without loss of generality that g_2 is directed so that $l(-R_3, g_2) = 1$. Hence D_2 is a relevant orientation.

Observe that D_1 is a relevant orientation if and only if $r(D)_{d(e)} = r(D)_{d(f)}$:

It again suffices to consider the case that $r(D)_{d(e)} = r(D)_{d(f)} = +1$.
In this case we need to show that $l(-R_2, -R_3, -R_1, f', g'_1, e') = 1$:
We have $l(-R_1, f', g'_1) = 0$ since $r(D_{d(f)}) = 1$ and thus $l(-R_1, f, g_2) = 1$.
Moreover $l(-R_2, -R_3, e') = l(-R_2, -R_3, e) = 1$ since $r(D)_{d(e)} = +1$.
Summarising $l(-R_2, -R_3, -R_1, f', g'_1, e') = 1$.
Hence

$$l(I_1) + l(I_2) \overset{q}{=} l(I'_1) + l(I'_2) \overset{q}{=} l(t(I'_1), r(D_1)) + l(t(I'_2), r(D_2)) +$$

$$1/2(r(D)_{d(e)} - 1 + r(D)_{d(f)} - 1).$$

Using the induction assumption of 2.8 for I_1, I_2 we get:

$$l(C) \overset{q}{=} l(I_1) + l(I_2) \overset{q}{=}$$

$$l(t(I'_1), r(D_1)) + l(t(I'_2), r(D_2)) +$$
$$1/2(r(D)_{d(e)} - 1 + r(D)_{d(f)} - 1) \overset{q}{=}$$
$$|BR(C)| - 4 - CR(C) + 1 + 1/2 \sum_{h \in BR(C) - \{e,f\}} (r(D)_{d(h)} + 1) +$$
$$1/2(r(D)_{d(e)} - 1 + r(D)_{d(f)} - 1) \overset{q}{=}$$
$$|BR(C)| - 1 - CR(C) + 1/2 \sum_{h \in BR(C)} (r(D)_{d(h)} + 1).$$

End of Case 2.

End of Claim 3.

Theorem follows from Claim 1, 2, 3. □

Next we show that the statement of Theorem 2.8 holds for the set of the alternating cycles of $M_0 \Delta P$ as well.

Theorem 2.9 *Let G be a proper g-graph and let D be a relevant orientation of G. Let P be a perfect matching of G. Then*

$$sgn(D, M_0 \Delta P) = (-1)^q,$$

where

$$q \overset{q}{=} [|BR(M_0 \Delta P)| - CR(M_0 \Delta P) +$$
$$1/2 \sum_{e \in BR(M \Delta P)} (r(D)_{d(e)} + 1)].$$

Proof. Let $C_1, ..., C_k$ be the alternating cycles of $M_0 \Delta P$.
We have that $sgn(D, M_0\Delta P) = (-1)^q$, where $q \stackrel{q}{=} l(C_1) + ... + l(C_k) - k$.

Using Theorem 2.8 for $C_1, ..., C_k$ it remains to show:

$$CR(M_0\Delta P) \stackrel{q}{=} \sum_{j=1}^{k} CR(C_j).$$

This is true by Lemma 2.7.

□

Corollary 2.10 *Let G be a proper 1-graph and D a relevant orientation of G. Let C be an alternating cycle with respect to M_0. Then*

1. *if $r(D) = (1, 1)$ then $sgn(D, C) = 1$ iff $t(C) = (1, 1)$ or $t(C) = (1, -1)$ or $t(C) = (-1, 1)$.*

2. *If $r(D) = (1, -1)$ then $sgn(D, C) = 1$ iff $t(C) = (1, 1)$ or $t(C) = (-1, -1)$ or $t(C) = (-1, 1)$.*

3. *If $r(D) = (-1, 1)$ then $sgn(D, C) = 1$ iff $t(C) = (1, 1)$ or $t(C) = (-1, -1)$ or $t(C) = (1, -1)$.*

4. *If $r(D) = (-1, -1)$ then $sgn(D, C) = 1$ iff $t(C) = (1, 1)$.*

Definition 2.11 *Let G be a proper g-graph and let D be a relevant orientation of G. Let $r(D) = (r_1, ..., r_{2g})$. We let $c(r(D))$ equal to the product of c_i, $i = 0, ..., g-1$, where $c_i = c(r_{2i+1}, r_{2i+2})$ and $c(1, 1) = c(1, -1) = c(-1, 1) = 1/2$ and $c(-1, -1) = -1/2$.*

Observe that $c(r(D)) = (-1)^n 2^{-g}$, where $n = |\{i; r_{2i+1} = r_{2i+2} = -1\}|$.

Corollary 2.12 *Let G be a proper 1-graph. Let D_1, D_2, D_3, D_4 be the relevant orientations of G. Then*

$$\mathcal{P}(G, x) = \sum_{i=1}^{4} c(r(D_i))\mathcal{P}(D_i, M_0).$$

Corollary 2.12 holds for all proper g-graphs. In order to deduce it we start with another corollary of Theorem 2.9.

Corollary 2.13 *Let G be a proper g-graph, D a relevant orientation of G and let P be a perfect matching of G. Then $sgn(D, M_0 \Delta P)$ is a function of $r(D)$ and $t(M_0 \Delta P)$ only. Let us denote this function by $\sigma(r(D), t(M_0 \Delta P))$.*

Lemma 2.14 *Let $r = (r_1, \ldots, r_{2g})$ and $t = (t_1, \ldots, t_{2g})$ where $r_i, t_i \in \{1, -1\}$. Let $r^j = (r_{2j+1}, r_{2j+2})$ and $t^j = (t_{2j+1}, t_{2j+2})$, $j = 0, \ldots, g-1$. Then*

$$\sigma(r, t) = \prod_{j=0}^{g-1} \sigma(r^j, t^j).$$

Proof. By Corollary 2.13, $\sigma(r, t) = sgn(D, \mathcal{C})$ where $r(D) = r$, $t(\mathcal{C}) = t$ and D is a relevant orientation of a proper g-graph G such that $G = C_0 \cup \mathcal{C}$. $\mathcal{C}$ is a set of vertex-disjoint cycles $C_1, \ldots, C_k$ with the following properties:

1. each C_i is alternating with respect to a perfect matching M_0 of G_0,
2. for each i, j $|E^i_j| \leq 1$,
3. for each i there is at most one j such that $|C_j \cap (E^i_1 \cup E^i_2)| \geq 1$,
4. for each C_j there is exactly one i such that C_j intersects $E^i_1 \cup E^i_2$.

Hence,

$$\sigma(r, t) = sgn(D, \mathcal{C}) = \prod_{i=1}^{k} sgn(D, C_i) = \prod_{i=1}^{k} sgn(D^i, C_i)$$

where D^i is the restriction of D to $C_0 \cup C_i$. Finally observe that, by Corollary 2.10, $\sigma(z_1, z_2) = 1$ if $z_2 = (1, 1)$. Hence, using Corollary 2.13, we have that $\prod_{i=1}^{k} sgn(D^i, C_i) = \prod_{j=0}^{g-1} \sigma(r^j, t^j)$, which we needed to prove. □

Theorem 2.15 *Let G be a proper g-graph. Then*

$$\mathcal{P}(G, x) = \mathcal{L}_g(G, x) = \sum_{i=1}^{4^g} c(r(D_i)) \mathcal{P}(D_i, M_0)$$

where D_i, $i = 1, \ldots, 4^g$, are the relevant orientations of G.

Proof. Let P be a perfect matching of G. In each $\mathcal{P}(D_i, M_0)$, $x(P)$ has coefficient equal to $sgn(D_i, M_0 \Delta P)$. By Corollary 2.13, $sgn(D_i, M_0 \Delta P) = \sigma(r(D_i), t(M_0 \Delta P))$.

Let

$$\mathcal{K}_g(t(M_0\Delta P) = \sum_{i=1}^{4^g} c(r(D_i))\sigma(r(D_i), t(M_0\Delta P))$$

.

Theorem follows from the following claim:

Claim. $\mathcal{K}_g(t(M_0\Delta P)) = 1$ for each $t(M_0\Delta P)$.

Proof of Claim. We proceed by induction on g.

The basis of the induction when $g = 1$ is Corollary 2.12.

To prove the induction step we introduce the following notation. If z is a vector in $\{1, -1\}^{2g}$ then we let $z = (z(0), \ldots, z(g-1))$ where $z(i) = (z_{2i+1}, z_{2i+2})$.
We call two relevant orientations D, D' of G *equivalent* if $(r(D)(1), \ldots, r(D)\,(g-1)) = (r(D')(1), \ldots, r(D')(g-1))$. Clearly, the equivalence classes consist of 4 elements; let $\mathcal{R}_1, \ldots, \mathcal{R}_{4^{g-1}}$ be the equivalence classes of the relevant orientations of G and let $\mathcal{R}_j = \{D_1^j, D_2^j, D_3^j, D_4^j\}$, $j = 1, \ldots, 4^{g-1}$.

Finally let $r(D_i^j)(k) = r_i^j(k)$, $k = 0, \ldots, g-1$ and $t = t(M_0\Delta P)$. We have that

$$\mathcal{K}_g(t) = \sum_{j=1}^{4^{g-1}} \sum_{i=1}^{4} c(r(D_i^j))\sigma(r(D_i^j), t).$$

Now, by Lemma 2.14, this equals to

$$\sum_{j=1}^{4^{g-1}} \sum_{i=1}^{4} c(r_i^j(0))c(r_i^j(1), \ldots, r_i^j(g-1)) \prod_{k=0}^{g-1} \sigma(r_i^j(k), t(k)).$$

By the definition of the equivalence classes, $r_1^j(k) = r_2^j(k) = r_3^j(k) = r_4^j(k)$ for $k \geq 1$ and $j = 1, \ldots, 4^{g-1}$. Hence, we let $r_i^j(k) = r^j(k)$ and write the above summation as:

$$\sum_{j=1}^{4^{g-1}} c(r^j(1), \ldots, r^j(g-1)) \prod_{k=1}^{g-1} \sigma(r^j(k), t(k)) \sum_{i=1}^{4} c(r_i^j(0))\sigma(r_i^j(0), t(0))$$

.

The internal sum equals to 1 for each $j = 1, \ldots, 4^{g-1}$ by the basis step of the induction, and hence, using Lemma 2.14 for the first sum, we can write the above summation as

$$\sum_{j=1}^{4^{g-1}} c(r^j(1),\ldots,r^j(g-1))\sigma((r^j(1),\ldots,r^j(g-1)),(t(1),\ldots,t(g-1))) =$$

$$\mathcal{K}_{g-1}(t(1),\ldots,t(g-1)) = 1,$$

by the induction hypothesis for $g-1$.

□

As a consequence of Theorem 1.4 and Theorem 2.15, we get

Corollary 2.16 *Let G be a proper g-graph. Then $s(D_i, M_0) = s(D_j, M_0)$ for each $i, j \in \{1, \ldots, 4^g\}$ and*

$$\mathcal{P}(G,x) = \mathcal{L}'_g(G,x) = s(D_1, M_0)\sum_{i=1}^{4^g} c(r(D_i))Pf(A(D_i))$$

where D_i, $i = 1, \ldots, 4^g$, are the relevant orientations of G.

Corollary 2.17 *Let G be a graph embeddable on an orientable surface of genus g. Then $\mathcal{P}(G,x)$ may be expressed as a linear combination of 4^g Pfaffians of matrices $A(D)$, where each D is an orientation of G.*

Proof. Orientable surface O_g of genus g may be obtained from S_g as follows: for each bridge B, glue together the two segments in which B intersects the boundary of B_0, and delete B.

If a graph G is embeddable on an orientable surface O_g of genus g, then without loss of generality no vertex belongs to the boundary of B_0. Then, splitting the boundaries back to the bridges, this gives a drawing of the graph on S_g such that each vertex lies on B_0 but some edges may use several bridges.

We construct a graph G' so that we replace each edge e which uses k bridges, $k \geq 1$, by a path P_e of edges $(e_1, \ldots, e_{2k+1})$ and vertices $v_1, \ldots, v_{2k}$. We let $x'_{e_1} = x_e$ and $x'_{e_i} = 1$ for each $i > 1$. Next we add edges so that the outer face of the planar part is a cycle. We let $x'_e = 0$ for each such edge e.

Note that G' is a proper g-graph and $\mathcal{P}(G', x') = \mathcal{P}(G, x)$.

By Theorem 2.15, $\mathcal{P}(G', x')$ may be written as a linear combination of 4^g Pfaffians of $A(D')$, where each D' is a relevant orientation of G'.

Claim. For each relevant orientation D' of G' there is an orientation D of G so that $Pf(A(D')) = Pf(A(D))$ or $Pf(A(D')) = -Pf(A(D))$.

Proof of Claim. We construct D from D' in two steps:

1. delete the edges e of $G' - G$ with $x'_e = 0$,

2. for each edge e of G which was changed into a path P_e of odd length in the construction of G', orient e in the direction in which an odd number of edges of P_e is directed in D': this is uniquelly determined since P_e has odd length.

If P is a perfect matching of G then there is a unique perfect matching P' of G' such that $x(P) = x'(P')$.

Observe that $sgn(D, P\Delta Q) = sgn(D', P'\Delta Q')$ for each pair of perfect matchings P, Q of G. The claim now follows from Theorem 1.4.

End of Claim.

This finishes the proof of the Corollary.

□

3 Exact Matching, Pfaffian Orientation, and Permanents.

The results of the previous sections have interesting algorithmic consequences.

Theorem 3.1 *Let S be an orientable surface and k a fixed positive integer. Let $\mathcal{G}$ be the class of graphs which may be embedded on S and such that the edges are partitioned into at most k classes and variables x_e are equal in each class. Then $\mathcal{P}(G, x)$ may be determined efficiently for $G \in \mathcal{G}$. As a consequence, it is possible to verify efficiently whether $G \in \mathcal{G}$ is Pfaffian.*

Proof. It follows from Theorems 2.15 and 2.17 that $\mathcal{P}(G, x)$ may be determined efficiently.

Concerning the problem of recognizing whether G is Pfaffian, we procede as follows: it was proved by Vazirani and Yannakakis (see the proof of theorem 3.1 of [12]) that there is an orientation D of graph G so that G is Pfaffian if and only if D is its Pfaffian orientation, and moreover D may be constructed efficiently. Hence $Pf(A(D))$ equals to the number of the perfect matchings of G if and only if G is Pfaffian, and it means that we can decide efficiently whether a graph is Pfaffian once we can compute efficiently its number of perfect matchings.

□

In particular the following well-known problem may be solved efficiently for the graphs embeddable on an arbitrary fixed orientable surface:

Exact Matching Problem. Given a graph G with some edges coloured red, and a number h. Find out whether G has a perfect matching with exactly h red edges.

Next we consider permanents of square matrices.

In 1913, Polya ([8]) suggested computing the permanent of a matrix A by changing the signs of some entries of A so that the determinant of the resulting matrix equals the permanent of A. Let us call a $0,1$-matrix A *convertible* if such a change is possible.

Szego ([10]) pointed out in the same year that not all matrices are convertible.

This may be explained nowadays using a complexity argument. There is an efficient algorithm to compute the determinant, while Valiant proved that the problem of computing the permanent of a $0,1$ matrix is $\#P$-complete (see [11]).

The computational problem of recognition of convertible matrices has been proved recently to admit a polynomial algorithm by McCuaig, Robertson, Seymour and Thomas (see [7]).

The recognition of convertible matrices is equivalent to the problem of recognition of bipartite Pfaffian graphs, and to the 'Even Cycle Problem': given a directed graph, decide whether it contains a directed cycle of even length.

Let A be a square matrix. Denote by $G(A)$ the bipartite graph whose two bipartition classes are indexed by the rows and the columns of A, and for each edge ij, $a_{ij} = x_{ij}$. Then $per(A) = \mathcal{P}(G(A), x)$.

This means that Corollary 2.17 provides a new combinatorial way to compute permanents of square matrices: $per(A)$ may be written as a linear combination of 4^g terms of form $Pf(A(D))$, where D is an orientation of $G(A)$ and g is the genus of $G(A)$.

Since $G(A)$ is a bipartite graph, the non-zero entries of $A(D)$ belong to two blocks A_1, A_2, where A_1 is obtained from A by changing the sign of some entries and $A_2 = -A_1$. Moreover $|Pf(A(D)| = |det(A_1)| = |det(A_2)|$ by Theorem 1.5.

This means that the method of Polya may be completed as follows:

Corollary 3.2 *Let A be a square matrix. Then $per(A)$ may be expressed as a linear combination of terms of form $det(A^i)$, $i = 1, ..., 4^g$, where each A^i is obtained from A by changing the sign of some entries.*

Concluding Remarks. In a continuation of this paper which is in preparation we consider the following related problems:

1. The generating function of T-joins and edge-cuts,
2. The evaluation of Tutte polynomial and the weight enumerator of a binary code,
3. A theory of crystal structures and application of the proposed method to solve the Ising problem for three-dimensional crystal structures.

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