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CYCLES OF BINARY MATROIDS WITH NO F_7^*-MINOR

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Abstract

We study the circuit lattice of a binary matroid, i.e. the set of all integer linear combinations of circuits of a binary matroid. In this paper we prove that if a binary matroid B does not contain an F_7^*-minor then the circuit lattice of B admits a basis consisting of cycles.

1 Introduction

We will show in this paper that the lattice generated by the circuits of a binary matroid without an F_7^*-minor has a basis consisting of cycles. It may be viewed as a continuation of our paper [1] where bases of lattices of directed cycles of digraphs and cycles of graphs were described.

Let us recall some basic definitions related to binary matroids (see [7] for an exposition of the matroid theory). Let V^d denote the d-dimensional vector space over $GF[2]$ and $F^d = V^{d+1} - 0_{V^{d+1}}$ the d-dimensional projective space. F^2 is called Fano plane and denoted sometimes F_7.

Simple binary matroids are the nonempty subsets of a projective space. Each *binary matroid* is obtained from a simple binary matroid by adding new copies of some vectors or of the zero vector. Two copies of the same nonzero vector in a binary matroid are called *parallel*, each copy of the zero vector is a *loop*. The maximal sets of pairwise parallel elements are called *parallel classes*. The matrix whose columns are the vectors of a binary matroid B is called a *representation* of B. A *standard representation* of B is a representation of B of form $(I|P(B))$, where I is the identity matrix. Every binary matroid has a standard representation. $P(B)$ is called *partial representation* of B. The identity matrix corresponds to a maximal set Z of vectors of B which are linearly independent modulo 2. We also write $P(B) = P(B, Z)$. If there exists a real matrix $R(B)$ and a bijection f from the vectors of B to the columns of $R(B)$ such that $A \subset B$ is independent modulo 2 ifand only if $f(A)$ is independent over real numbers then B is called *regular matroid*. Regular matroids form an extensively studied subclass of binary matroids.

The minimal subsets of vectors of a binary matroid B which are linearly dependent modulo 2 are called *circuits* of B. Disjoint unions of circuits are *cycles*. Any modulo 2 linear combination of cycles is a cycle. Hence cycles form a space over $GF[2]$ called *cycle space* of B. A basis of the cycle space is called *cycle basis*.

Let B be a binary matroid. The binary matroid with standard representation $(P(B)^T|I)$ is called *dual* of B and is denoted by B^*. If x is an element of B and it corresponds to the i-th column of $(I|P(B))$ then let x^* be the element of B^* corresponding to the i-th column of $(P(B)^T|I)$. Observe that $B^{**} = B$. The circuits of B^* are called *cocircuits* of B and the cycles of B^* are called *cocycles* of B. Two elements x, y of B are *series* if x^*, y^* are parallel in B^*. Equivalently, x, y belong to the same cycles of B. The maximal sets of pairwise series elements are called *series classes*.

The dual of a regular matroid is regular as well. If $x \in B$ such that x^* is a loop of B^* then x is *coloop* of B. Coloops are the elements of B that do not belong to any cycle.

A circuit of a binary matroid consisting of three elements is called *triangle* and a cocircuit consisting of three elements is called *triad*.

Let B be a binary matroid and let x be a vector of B. Binary matroid B' is obtained from B by *deletion of* x if $B' = B - \{x\}$. Such B' is usually denoted by $B\backslash x$. Binary matroid B'' is obtained from B by *contraction of* x if $B'' = (B^* - \{x^*\})^*$. Such B'' is usually denoted by B/x. Binary matroid M is a *minor* of binary matroid N if M is obtained from N by deletions and contractions. M is a *restriction minor* of N if M is obtained from N by deletions only. If M is a minor of N then M^* is a minor of N^*.

Let $\mathcal{A}$ be a set of rational vectors of the same finite dimension d. *Lattice* $lat(\mathcal{A})$ generated by $\mathcal{A}$ is the set of all integer linear combinations of vectors of $\mathcal{A}$. The *dimension* of a lattice is maximum number of its linearly independent vectors. Linear dependency is taken over rational numbers in this paper. A set $\mathcal{B}$ of linearly independent vectors such that $lat(\mathcal{A}) = lat(\mathcal{B})$ is called a *basis* of $lat(\mathcal{A})$. Every lattice generated by rational vectors has a basis (see [4] also for other facts on integer lattices).

Let B be a binary matroid. The lattice generated by the circuits of B is often called the *circuit lattice* of B. The *dimension* of the circuit lattice of B equals the number of elements of its basis. This equals the number of the series classes of B. Hence a set A of cycles of B forms a basis of the circuit lattice of B if and only if $|A|$ is correct and every cycle of B is an integer linear combination of elements of A.

The lattice generated by the circuits of a binary matroid B has been studied in [3, 2]. The following conjecture was introduced in [2].

Conjecture 1.1 *Let B be a binary matroid. The lattice generated by the circuits of B has a basis consisting of cycles of B.*

This conjecture may be stated equivalently as follows (see [2]): each lattice generated by a binary space (i.e. binary code) has a basis consisting of its vectors (i.e. of codewords). The aim of this paper is to verify Conjecture 1.1 for the binary matroids without an F_7^* minor. This class contains the class of regular matroids. Our result means in particular that the conjecture is true provided the binary space has a binary basis whose vectors may be signed to form a totally unimodular matrix.

We actually prove a stronger statement.

Definition 1.2 *Let B be a binary matroid and let X be a series class of B. X is called proper if there are circuits C_1, C_2, C_3 such that $C_1 \cap C_2 = X$ and C_3 is modulo 2 sum of C_1, C_2.*

Definition 1.3 *Let B be a binary matroid and let X be a proper series class of B. Let $\mathcal{C}$ be a basis of the circuit lattice of B consisting of cycles of B. $\mathcal{C}$ is called X-based if there are C_1, C_2, C_3 in $\mathcal{C}$ such that $C_1 \cap C_2 = X$ and C_3 is modulo 2 sum of C_1, C_2.*

Theorem 1.4 *Let B be a binary matroid without an F_7^* minor and let X be a proper series class of B. Then the circuit lattice of B has an X-based basis. Moreover, if B has no loop nor coloop nor a series class which is a circuit then each series class of B is proper.*

Proof. The thesis follows from Theorems 2.6, 5.5, 5.11 and 5.12 of next sections. □

The characteristic vector of a set X will also be denoted by X unless a confusion might arise. Binary matroid B is called *eulerian* if $2X$ belongs to the lattice generated by the cocircuits of B, for each parallel class X which belongs to at least one cycle. Eulerian binary matroids were studied in [3] where it is pointed out that the binary matroids without an F_7 minor are eulerian. Theorem 1.4 stated dually asserts that the lattice generated by the cocircuits of a binary matroid without an F_7 minor has a basis consisting of cocycles. It remains an interesting open problem to verify the same statement for the eulerian matroids. F^{d^*} is eulerian for any d, and it has an F_7 minor for $d \geq 3$. Winfried Hochstaettler observed that the cocircuit lattice of F^{d^*} has a basis of cocycles as well.

There are several ways to attack Conjecture 1.1. In this paper we use the seminal decomposition theorem of Seymour (see [5]) which characterizes the binary matroids without an F_7^* minor. However the results of [2] indicate that 'less local' approach may be necessary to answer the general conjecture 1.1.

The paper is organized as follows.

In section 2 we state a decomposition theorem for the binary matroids without an F_7^* minor. The theorem is a reformulation of the decomposition theorems of Seymour [5], see also [6].

In section 3 we prove Theorem 1.4 for the basic pieces of the decomposition: graphic and cographic matroids of 3-connected graphs, F_7, R_{10}.

The statement for graphic matroids has been proved in [1] and we include a sketch of proof here only.

Finally in section 4 we show that the operations of the decomposition keep Theorem 1.4 true.

2 The Decomposition Theorem

Let B be a binary matroid and let $P(B)$ be its partial representation. $P(B)$ may be viewed as the (V_1, V_2) incidence matrix of a bipartite graph, called *fundamental graph* of B and denoted by $G(B)$. Note that the fundamental graphs of B need not be isomorphic. Each fundamental graph of B is a fundamental graph of B^* as well. See next section for needed terminology of graph theory.

Definition 2.1 *Let B_1, B_2 be disjoint binary matroids. Binary matroid B is* 1*-sum of B_1 and B_2 if the set of the circuits of B is the union of the sets of the circuits of B_1 and B_2.*

Definition 2.2 *Let B_1, B_2 be disjoint binary matroids with no loop or coloop, each having at least three elements. Let r be a vertex of $G(B_1)$ corresponding to a row of $P(B_1)$ and let c be a vertex of $G(B_2)$ corresponding to a column of $P(B_2)$. Let $L(r)$ and $L(c)$ denote the set of the neighbours of r in $G(B_1)$ and of c in $G(B_2)$, respectively. Binary matroid B is* 2*-sum of (B_1, B_2) if a fundamental graph of B is obtained from $G(B_1), G(B_2)$ by deleting vertices r and c and by adding the edge from each element of $L(r)$ to each element of $L(c)$. Moreover the elements of $L(r)$ correspond to columns of $P(B)$ (as in $P(B_1)$).*

Next we define ΔY-sum of two binary matroids. The definition is more technical.

We remind that a circuit of a binary matroid consisting of three elements is called *triangle* and a cocircuit consisting of three elements is called *triad*.

Definition 2.3 *Let B_1, B_2 be disjoint binary matroids with at least* 6 *elements each. Let $G(B_1)$ have three vertices c_0, c_1, c_2 corresponding to columns of $P(B_1)$. Let $i = 1, 2$ and let $L(c_i)$ denote the set of the vertices of $G(B_1)$ connected by edges to c_i but not to c_{3-i}. Let $L(c_1, c_2)$ denote the set of the vertices of $G(B_1)$ connected by edges to both c_1, c_2. Let $G(B_2)$ have three vertices r_0, r_1, r_2 corresponding to rows of $P(B_2)$. Let $L(r_1), L(r_2), L(r_1, r_2)$ be defined analogously as $L(c_1), L(c_2), L(c_1, c_2)$.*

Binary matroid B is ΔY-sum of (B_1, B_2) if the following conditions are satisfied:

1. *$B_1 = M(G)$ or $B_1 = M^*(G)$ where G is a 3-connected graph without parallel edges. Moreover B_1 has a restriction minor $M(H)$ where H is a subdivision of K_4 and c_0, c_1, c_2 form a triangle of H.*
2. *B_2 has no loops, coloops, and series elements, and r_0, r_1, r_2 form a triad of B_2. Moreover B_2 is not a k-sum, $k < 3$.*
3. *$L(c_i) \neq \emptyset$ and $L(r_i) \neq \emptyset$, $i = 1, 2$.*
4. *A fundamental graph of B is obtained from $G(B_1), G(B_2)$ by deleting vertices $r_0, r_1, r_2, c_0, c_1, c_2$ and by adding all the edges between $L(r_1)$ and $L(c_1) \cup L(c_1, c_2)$, $L(r_2)$ and $L(c_2) \cup L(c_1, c_2)$, $L(r_1, r_2)$ and $L(c_1) \cup L(c_2)$. Moreover the elements of $L(r_1)$ correspond to columns of $P(B)$.*

Definition 2.4 *B_1 from the above definition is called triangle part and B_2 is called triad part of the ΔY-sum.*

Definition 2.5 *R_{10} is the binary matroid whose partial representation is given by the following matrix:*

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

The following theorem is a reformulation of the decomposition theorem of Seymour [5], see also [6].

Theorem 2.6 *Let B be a binary matroid with no F_7^*-minor.*

1. *Let B be not a k-sum, $k < 3$. Let x be an element of B. Then B is regular and $B = R_{10}$ or it is possible to decompose B using ΔY-sums into graphic and cographic matroids of 3-connected graphs without parallel edges. Moreover, x belongs to the triad part of the first ΔY-sum applied.*

2. *Let B be not a 1-sum. Then it is possible to decompose B using 2-sums into copies of F_7, graphic matroids of circuits, and regular matroids which are not k-sums, $k < 3$.*

Proof. Theorem 2.6 follows from Theorem 11.3.14 and Lemma 11.3.18 of [6]. □

3 Graphic and Cographic Matroids

The *degree* of a vertex in a graph equals the number of the edges incident with the vertex. A graph is called *eulerian* if all of its vertices have even degree.

A graph G is *k-connected* if there exist k vertex-disjoint paths connecting any pair of vertices of G. The graphs considered in this section are 2-connected unless specified otherwise.

Graph H is a *subdivision* of graph G if H is obtained from G by a repeated replacement of an edge by a path of lenght 2 whose terminal vertices coincide with the terminal vertices of the edge and its intermediate vertex has degree 2 in H.

Graph H is *multiplication* of a graph G if H is obtained from G by adding parallel edges. *Contracting an edge* of a graph consists in deleting that edge and identifying its endvertices. Let K_4 denote the complete graph on 4 vertices. If A is a subset of vertices then let $N(A)$ denote the set of edges with exactly one end in A.

Let $G = (V, E)$ be a graph with parallel edges, i.e. edges with the same endvertices, but without loops. The (V, E) incidence matrix of G represents binary matroid $M(G)$ whose elements are edges of G and a set of edges is independent if and only if it does not contain a circuit of G. $M(G)$ is a *graphic matroid*. The dual of $M(G)$, denoted by $M^*(G)$, is a *cographic matroid*.

The circuits of $M(G)$ are the circuits of G. The cycles of $M(G)$ are the eulerian subgraphs of G. The cocycles of $M(G)$ are the edge-cuts of G and the cocircuits of $M(G)$ are the minimal edge-cuts of G (with respect to inclusion). A 'cut' will always mean 'edge-cut' in this paper. $M(G)$ has no loops and the coloops of $M(G)$ are the 1-edge-cuts of G, i.e. the edges of G which belong to no cycles.

An *ear decomposition* of a (2-connected) graph G is a sequence $G_1, \dots, G_t = G$ of subgraphs of G such that G_1 is a cycle and each G_i, $i > 1$, arises from

G_{i-1} by adding a path P_i whose endvertices are distinct and belong to G_{i-1} while the edges and intermediate vertices of P_i do not. The paths P_i are called *ears* and the endvertices of P_i are called *initial vertices* of the ear.

A graph is 2-connected if and only if it has an ear decomposition, and from each ear decomposition it is possible to obtain a cycle basis, i.e. a basis of the vector space over $GF[2]$ generated by the incidence vectors of the circuits, by completing each new ear to a circuit using a path in the already built subgraph. Such cycle bases are called *ear-bases*.

An ear decomposition $G_1, \ldots, G_t = G$ of G will be called *correct ear decomposition* if each G_i, $i = 2, \ldots, t$, is a subdivision of a 3-edge-connected graph.

Theorem 3.1 *Let G be a subdivision of a 3-edge-connected graph. Then G has a correct ear-decomposition. Moreover the following two stronger statements hold:*

1. *If C_1, C_2 are two circuits of G such that $C_1 \cap C_2$ is a series class then G has a correct ear decomposition $G_1, \ldots, G_t$ such that $G_2 = C_1 \cup C_2$.*

2. *If K is a subgraph of G which is a subdivision of K_4 then G has a correct ear decomposition $G_1, \ldots, G_t$ such that $G_3 = K$.*

Proof. Let G_i, ≥ 2, be a subdivision of a 3-edge-connected graph and let G_i be a subgraph of G. Call an ear P_{i+1} *correct* if the initial vertices of P_{i+1} are not intermediate vertices of the same series class of G_i. Observe if P_{i+1} is correct then G_{i+1} is a subdivision of a 3-edge-connected graph.

If a correct ear P_{i+1} does not exist then let S be a series class of G_i with an intermediate vertex connected by an edge to a vertex of $G - G_i$. The terminal edges of S form a 2-edge-cut of G, which is a contradiction. □

Definition 3.2 *Let G be a subdivision of a 3-edge-connected graph. Let $G_1, \ldots, G_t = G$ be a correct ear decomposition of G.*
An improved ear-basis $\mathcal{A}(G) = \mathcal{A}(G_t)$ is recursively defined as follows:

1. *$\mathcal{A}(G_2)$ consists of all three cycles of G_2.*

2. *Let $i > 2$ and G_i be obtained from G_{i-1} by adding the ear P_i.*

   We distinguish three cases:

(i) if the endvertices of P_i have degree greater than 2 in G_{i-1} then $\mathcal{A}(G_i)$ is obtained from $\mathcal{A}(G_{i-1})$ by adding an arbitrary circuit C_i^1 of G_i containing P_i;

(ii) if one endvertex of P_i have degree 2 in G_{i-1} then let e_1, e_2 be two edges of G_{i-1} incident with that vertex. Then $\mathcal{A}(G_i)$ is obtained from $\mathcal{A}(G_{i-1})$ by adding two circuits C_i^1, C_i^2 of G_i, C_i^1 containing P_i and e_1 and C_i^2 containing P_i and e_2;

(iii) if both endvertices of P_i have degree 2 in G_{i-1} then let e_1, e_2 and f_1, f_2 be two pairs of edges of G_{i-1} incident with each endvertex of P_i. Since the ear decomposition is correct, e_1, e_2, f_1, f_2 do not belong to the same series class of G_{i-1}.

Then $\mathcal{A}(G_i)$ is obtained from $\mathcal{A}(G_{i-1})$ by adding three circuits C_i^1, C_i^2, C_i^3 of G_i, C_i^1 containing P_i and e_1, f_1, C_i^2 containing P_i and e_2, f_1, and C_i^3 containing P_i and e_1, f_2.

The following Theorem is proved in [1].

Theorem 3.3 *Let G be a subdivision of a 3-edge-connected graph. Any improved ear-basis of G is a basis of the lattice generated by the circuits of G. Moreover this lattice contains all vectors of form $2E$, where E is a series class of the graphic matroid $M(G)$. Hence $M(G)$ is the dual of an eulerian matroid.*

The following Corollary follows from Theorems 3.1 and 3.3, and from Menger's theorem.

Corollary 3.4 *Let G be a 3-connected graph. Let X be a series class of $M(G)$. Then X is proper and the circuit lattice of $M(G)$ has an X-based basis. Secondly let T be a triangle of G which belongs to a subdivision H of K_4. Then there is a basis of the circuit lattice of G consisting of cycles only, and moreover it contains arbitrary 6 circuits of H (note that H has 7 cycles in total).*

Next we turn our attention to the cuts of graphs, namely the circuits of the cographic matroid $M^*(G)$. We remind that the parallel classes of a graphic matroid are exactly the maximum sets of parallel edges of the graph and they correspond to the series classes of $M^*(G)$.

Definition 3.5 *Let G be a graph and let x be a vertex of G. A graph G' is called split of x if it is obtained from G so that vertex x is replaced by two new vertices x_1, x_2 joined by a parallel class of G' and the edges of G incident with x are partitioned into two (possibly empty) sets N_1 and N_2 and the elements of N_i are made incident with x_i, $i = 1, 2$.*

Definition 3.6 *A split decomposition of a graph G is a sequence $G_1, \ldots, G_t = G$ of subgraphs of G such that G_1 consists of a set of parallel edges and each G_i, $i > 1$, is a split of a vertex of G_{i-1}.*

Theorem 3.7 *Each connected graph has a split decomposition.*

Proof. Let G be a contraexample with minimum number of vertices. Let $\mathcal{C}$ be a minimal cut of G. $G - \mathcal{C}$ has exactly two connectivity components C_1 and C_2. Let G_1 be the parallel class obtained by contraction of C_1 and C_2 into single vertices. This is the beginning of a split decomposition of G. Moreover, both G_1 and G_2 have a split decomposition by the minimality assumption. Adding these together provides a split decomposition of G, a contradiction. □

It follows from the proof of 3.7 that a statement similar to 3.1 holds for cuts as well.

Corollary 3.8 *Let G be a connected graph.*

1. *If C_1, C_2 are two minimal cuts of G such that $C_1 \cap C_2$ is a parallel class of G then G has a split decomposition $G_1, \ldots, G_t$ such that $G_2 = C_1 \cup C_2$.*

2. *If K is a multiplication of K_4 obtained from G by contraction of some edges then G has a split decomposition $G_1, \ldots, G_t$ such that $G_3 = K$.*

Definition 3.9 *Let $G_1, \ldots, G_t = G$ be a split decomposition of a graph G. A split-basis $\mathcal{C}(G) = \mathcal{C}(G_t)$ is recursively defined as follows:*

1. *$\mathcal{C}(G_1)$ consists of the parallel class of G_1.*

2. *Let $i > 1$ and let G_i be obtained from G_{i-1} by splitting of vertex x_i. Let P_i be the parallel class of G_i consisting of edges $(x_{i,1}, x_{i,2})$.*

   We distinguish two cases:

(i) If the parallel classes of G_i are the parallel classes of G_{i-1} and P_i then $\mathcal{C}(G_i)$ is obtained from $\mathcal{C}(G_{i-1})$ by adding a minimal cut C_i^1 of G_i containing P_i.

(ii) Otherwise let $S^1, ..., S^l$ be the parallel classes of G_{i-1} which were split by the split of x_i into two parallel classes S_1^j, S_2^j of G_i, S_1^j incident with $x_{i,1}$ and S_2^j incident with $x_{i,2}$, $j = 1, ..., l$. Let $y_j \neq x_i$ be the other vertex incident with the edges of S^j in G_{i-1}. $\mathcal{C}(G_i)$ is obtained from $\mathcal{C}(G_{i-1})$ by adding the cut $C_i^0 = N(\{x_{i,2}, y_1\}))$ and the cuts $C_i^j = N(\{x_{i,1}, y_1, ..., y_j\})$, $j = 1, ..., l$.

Theorem 3.10 *Let G be a connected graph. Any split-basis of G is a basis of the lattice generated by the cuts of G. Moreover this lattice contains all vectors of form $2E$, where E is a parallel class of G. Hence $M^*(G)$ is the dual of an eulerian matroid.*

Proof. It is not difficult to observe that the vectors of a split-basis are linearly independent over rationals. Let $G_1, \ldots, G_m = G$ be a split decomposition of G. The proof is by induction on i. For $i = 1$, the thesis follows trivially. So, let us suppose that all incidence vectors of cuts of G_{i-1} and all vectors $2E$, E parallel class of G_{i-1}, are integer linear combinations of vectors of the split-basis, $i > 1$, and let G_i be obtained from G_{i-1} by a split of vertex x_i. Let P_i denote the parallel class of G_i consisting of edges $(x_{i,1}, x_{i,2})$.

Then one of the following cases may occur: C_i^1 or $C_i^0, ..., C_i^{l_i}$, $l_i \geq 1$, are added to the split-basis.

Let C_i^1 be the only cut added to the split-basis. Then the parallel classes of G_i are the parallel classes of G_{i-1} and P_i. The nonzero entries of vector $z = 2C_i^1 - 2P_i$ form a union of parallel classes of G_{i-1} and they are all equal to 2. By the induction hypothesis, it follows that $2P_i$ is an integer linear combination of vectors of the split-basis. Let C be a cut of G_i which is not a cut of G_{i-1}, i.e. $P_i \subset C$. Let C' denote symmetric difference of C and C_i^1. We have $C = C' - C_i^1 + 2(C \cap C_i^1)$; C' is a cut contained in G_{i-1} and $C \cap C_i^1$ is union of parallel classes of G_i. It follows that C is an integer linear combination of vectors of the split-basis. Hence, Theorem 3.10 holds.

Secondly let $C_i^0, ..., C_i^{l_i}$, $l_i \geq 1$, be added to the split-basis. Then the parallel classes of G_i are the parallel classes of G_{i-1} which were not split by the split of x_i, the parallel classes S_1^j, S_2^j, $j = 1, .., l_i$ and P_i. Let $y_j \neq x_i$ be the other vertex incident with the edges of S^j in G_{i-1}. We get $2P_i =$

$C_i^0 + C_i^1 - X - 2Y$, where X is the cut of G_{i-1} consisting of all edges incident with x_i and Y is the set of the edges of G_{i-1} incident with y_1 but not x_i. Using the induction assumption $2P_i$ is an integer linear combination of vectors of the split basis. Moreover, $C_i^{l_i} - C_i^{l_i-1} + N(\{y_{l_i}\}) = 2S_2^{l_i} + 2Y_{l_i}$ where Y_{l_i} is the set of the edges of G_i incident with y_{l_i} but not with a vertex of $\{x_{i,1}, x_{i,2}, y_1, ..., y_{l_i-1}\}$. Hence $2S_2^{l_i}$ is an integer linear combination of vectors of the split basis. Similarly, $2S_2^j$, and $2S_1^j$, $j > 1$, may be expressed from the split basis. Finally $2S_2^1$ may be obtained from $2C_i^1$ by subtracting the rest. $S_1^1 = S^1 - S_2^1$. □

Next Corollary follows from Theorem 3.10 and Corollary 3.8.

Corollary 3.11 *Let G be a connected graph with no parallel edges and let X be a series class of $M^*(G)$. Then the circuit lattice of $M^*(G)$ has an X-based basis. Secondly let T be a triad of G which belongs to a graph H obtained from G by contraction of some edges and let H be a multiplication of K_4. Then there is a basis of the circuit lattice of $M^*(G)$ consisting of cycles of $M^*(G)$ only, and moreover it contains arbitrary 6 cuts of H (note that H has 7 cuts in total).*

4 R_{10} and F_7 Matroids

We show now that Conjecture 1.1 holds for the remaining basic pieces of the decomposition.

Theorem 4.1 *Let x be an element of R_{10} and $X = \{x\}$. The circuit lattice of R_{10} has an X-based basis. Hence R_{10} is the dual of an eulerian matroid.*

Proof. Let R denote the matrix of 2.5. Let $Z = \{z_1, ..., z_5\}$ denote the set of independent vectors of R_{10} such that $R = P(R_{10}, Z)$. Let $S = \{s_1, ..., s_5\}$ be the set of the columns of R. We may assume that $x \in Z$ since $P(R_{10}, Z) = P(R_{10}, S)$. Let $\mathcal{C}^{\in}$ be the set of five cycles defined by their characteristic vectors with respect to $Z \cup S$ as follows.

$$\begin{aligned} C_1 &= (0111010000) \\ C_2 &= (0011101000) \\ C_3 &= (1001100100) \\ C_4 &= (1100100010) \\ C_5 &= (1110000001) \end{aligned}$$

Note that $C_1, ..., C_5$ form the fundamental circuits of R_{10} with respect to Z and thus they generate all cycles modulo 2. Moreover, $\{z_1\} = C_3 \cap C_5$, $\{z_2\} = C_1 \cap C_4$, $\{z_3\} = C_2 \cap C_5$, $\{z_4\} = C_3 \cap C_1$ and $\{z_5\} = C_2 \cap C_4$.

Let us define cycles $C_6, ..., C_{10}$ according to these expressions: C_6 as modulo 2 sum of C_3 and C_5, C_7 as modulo 2 sum of C_1 and C_4 etc. Let $\mathcal{C} = \{C_1, ..., C_{10}\}$. We will show that $\mathcal{C}$ is an X-based basis of R_{10}. It is enough to show that it is a basis by the choice of X and $\mathcal{C}$. $\mathcal{C}$ has the right size and it contains a modulo 2 basis. Hence it remains to show that twice each element of R_{10} is an integer linear combination of vectors of $\mathcal{C}$. This immediately follows from the definition of $\mathcal{C}$ for $z_1, ..., z_5$. $2\{s_i\}$ may be obtained from $2C_i$, $i = 1, ..., 5$, by subtracting twice the corresponding subset of Z.

□

Theorem 4.2 *Let x be an element of F_7 and let $X = \{x\}$. The circuit lattice of F_7 has an X-based basis. Hence F_7 is the dual of an eulerian matroid.*

Proof. A proof similar to that for the matroid R_{10} can be done to prove that the following set of vectors:

$$
\begin{aligned}
C_1 &= (1101000) \\
C_2 &= (1010100) \\
C_3 &= (0110010) \\
C_4 &= (1110001) \\
C_5 &= (0111100) \\
C_6 &= (1011010) \\
C_7 &= (1100110)
\end{aligned}
$$

is a basis for the lattice of circuits of F_7. □

5 The Sums

In this section we assume unless stated otherwise that the binary matroids do not have loops or coloops. This means in particular that each element belongs to a cycle. Hence, any basis of the circuit lattice consisting of cycles has at least one cycle containing each element.

First we consider the ΔY-sums. The proof of the following Proposition is straightforward from the definition of ΔY-sum.

Proposition 5.1 *Let B be a ΔY-sum of (B_1, B_2). The circuits of B are:*

1. *The circuits of B_1 disjoint with $\{c_0, c_1, c_2\}$,*
2. *The circuits of B_2 disjoint with $\{r_0, r_1, r_2\}$,*
3. *Each $H = H_1 \cup H_2$ where $H_1 \cup \{c_1\}$ is a circuit of B_1 and $H_2 \cup \{r_1, r_0\}$ is a circuit of B_2.*
4. *Each $H = H_1 \cup H_2$ where $H_1 \cup \{c_2\}$ is a circuit of B_1 and $H_2 \cup \{r_2, r_0\}$ is a circuit of B_2.*
5. *Each $H = H_1 \cup H_2$ where $H_1 \cup \{c_0\}$ is a circuit of B_1 and $H_2 \cup \{r_1, r_2\}$ is a circuit of B_2.*

Proposition 5.2 *Let B be ΔY-sum of (B_1, B_2) and let $\mathcal{B}_2$ be a basis of the circuit lattice of B_2 consisting of cycles. Each cycle of $\mathcal{B}_2$ contains an even number of elements of $\{r_0, r_1, r_2\}$. Moreover there are three cycles R_0, R_1, R_2 of $\mathcal{B}_2$ such that R_0 contains r_1, r_2, R_1 contains r_0, r_2 and R_2 contains r_0, r_1.*

Proof. In each matroid, the intersection of a circuit and a cocircuit has an even number of elements. Since each element of B_2 belongs to a circuit, we get that B has a cycle containing any pair from r_0, r_1, r_2. However, any subset of such pairs does not generate the remaining pairs by integer linear combinations. This proves the second part of Proposition. □

Let B be a ΔY-sum of (B_1, B_2). Next we define ΔY-sum of a basis of the circuit lattice of B_1 and a basis of the circuit lattice of B_2, providing the bases satisfy some conditions.

Definition 5.3 *Let B be ΔY-sum of (B_1, B_2). Let $T = \{c_0, c_1, c_2\}$ and let H be a subdivision of K_4 such that $M(H)$ is a restriction minor of B_1 and T is a triangle of H (see 2.3). Let $\mathcal{B}_1$ be a basis of the circuit lattice of B_1 consisting of cycles and let $C_0, ..., C_5$ of $\mathcal{B}_1$ be 6 cycles of H, $C_i \cap T = \{c_i\}$, $i = 0, 1, 2$ and $C_3 = T$.*

Let $\mathcal{B}_2$ be a basis of the circuit lattice of B_2 consisting of cycles and let R_0, R_1, R_2 of $\mathcal{B}_2$ be such that R_0 contains r_1, r_2, R_1 contains r_0, r_2 and R_2 contains r_0, r_1. Note that R_0, R_1, R_2 must exist by 5.2.

The ΔY-sum $\mathcal{B}$ of $(\mathcal{B}_1, \mathcal{B}_2)$ is the set of cycles of B obtained as follows.

1. *If $D \in \mathcal{B}_2$ then let $D' \in \mathcal{B}$ where D' is obtained from D as follows:*

   $\{r_1, r_2\}$ is replaced by $C_0 - \{c_0\}$, $\{r_0, r_1\}$ is replaced by $C_1 - \{c_1\}$, $\{r_0, r_2\}$ is replaced by $C_2 - \{c_2\}$.

2. *If $D \in \mathcal{B}_1$ is disjoint with $\{c_0, c_1, c_2\}$ then let $D \in \mathcal{B}$.*

3. *If $D \in \mathcal{B}_1$, $Z = D \cap \{c_0, c_1, c_2\} \neq \emptyset$ and D is not one of $C_0, \ldots C_5$ then let $(D - Z) \cup R'_Z \in \mathcal{B}$, where R'_Z depends on Z as follows:*

   If $Z = \{c_0\}$ or $Z = \{c_1, c_2\}$ then $R'_Z = R_0 - \{r_1, r_2\}$. If $Z = \{c_1\}$ or $Z = \{c_0, c_2\}$ then $R'_Z = R_1 - \{r_0, r_2\}$. If $Z = \{c_2\}$ or $Z = \{c_0, c_1\}$ then $R'_Z = R_2 - \{r_0, r_1\}$.

Theorem 5.4 *Let B be ΔY-sum of (B_1, B_2) where B_1, B_2 are duals of eulerian matroids. Let $\mathcal{B}$ be ΔY-sum of $(\mathcal{B}_1, \mathcal{B}_2)$ where $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases of the circuit lattices of B_1 and B_2 respectively. Then $\mathcal{B}$ generates all cycles of B.*

Proof. Since $\mathcal{B}_2$ is a basis of the circuit lattice of B_2, we get that for each vector D of the circuit lattice of B_2 such that D is a cycle or $D = 2X$, X series class of B_2 disjoint with $\{r_0, r_1, r_2\}$, D' is an integer linear combination of vectors of $\mathcal{B}$ where D' is obtained from D as in 5.3.1. Using 5.3.1 and 5.3.3 we get as well that for each vector E of the circuit lattice of B_1 such that E is a cycle or $E = 2X$, X series class of B_1 disjoint with $\{c_0, c_1, c_2\}$, E' is an integer linear combination of vectors of $\mathcal{B}$, where E' is obtained from E as in 5.3.3. Hence in particular the cycles of 5.1.1 and 5.1.2 may be expressed by $\mathcal{B}$. We will show that the same is true for the cycles of 5.1.3. The remaining cases may be treated in the same way. Let H be a cycle of 5.1.3. Let $H' = H_2 \cup (C_1 - \{c_1\})$ and H'_1 be the symmetric difference of C_1 and H_1. We have $H = H' - H'_1 + 2(H - H')$. By above, H' and $2(H - H')$ may be expressed by $\mathcal{B}$ hence H does too. □

Theorem 5.5 *Let B be a binary matroid with no F_7^*-minor which is not a k-sum, $k < 3$. Let X be a series class of B. Then $|X| = 1$ and the circuit lattice of B has an X-based basis. Hence B is the dual of an eulerian matroid.*

Proof. Let B be minimal contraexample. B is neither graphic, nor cographic matroid of a 3-connected graph nor a R_{10} since the statement is proved for them in the previous section. Let X be a series class of B. Since B is not a k-sum, $k < 3$, we have that $X = \{x\}$. Using Theorem 2.6 B is a ΔY-sum of (B_1, B_2) and x belongs to B_2. Since B_2 is not a k-sum, $k < 3$, X is a series class of B_2. Let $\mathcal{B}_2$ be an X-based basis of B_2 and let $\mathcal{B}_1$ be as in 5.3. Then ΔY-sum of $(\mathcal{B}_1, \mathcal{B}_2)$ is X-based basis of B since it has the right size, and by 5.4. □

Next we consider 2-sums. The following Proposition is again straightforward.

Proposition 5.6 *Let B be 2-sum of (B_1, B_2). The circuits of B are:*

1. *The circuits of B_1 not containing r,*
2. *The circuits of B_2 not containing c,*
3. *Each $C = C_1 \cup C_2$ where $C_1 \cup \{r\}$ is a circuit of B_1 and $C_2 \cup \{c\}$ is a circuit of B_2.*

Corollary 5.7 *Let B be 2-sum of (B_1, B_2). The series classes of B are the series classes of B_1 not containing r, the series classes of B_2 not containing c and $S_1 \cup S_2$ where $S_1 \cup \{r\}$ is a series class of B_1 and $S_2 \cup \{c\}$ is a series class of B_2.*

Corollary 5.8 *Let B be a binary matroid with no F_7^*-minor and let B be not a 1-sum. Then B is a graphic matroid of a circuit or each series class of B is proper.*

Let B be 2-sum of (B_1, B_2). Let us denote $r = z_1$ and $c = z_2$. Let Z_i denote the series class containing z_i, $i = 1, 2$. Next we define 2-sum of a basis of the circuit lattice of B_1 and a basis of the circuit lattice of B_2, providing at least one of them is Z_j-based, $j \in \{1, 2\}$.

Definition 5.9 *Let B be 2-sum of (B_1, B_2). Let $j \in \{1, 2\}$, let $\mathcal{B}_j$ be a Z_j-based basis of the circuit lattice of B_j and let C_1, C_2, C_3 of $\mathcal{B}_j$ be such that $C_1 \cap C_2 = Z_j$ and $C_3 = (C_1 \cup C_2) - Z_j$. Let $\mathcal{B}_{3-j}$ be a basis of the circuit lattice of B_{3-j} consisting of cycles of B_{3-j} and let C_4 of $\mathcal{B}_{3-j}$ contain z_{3-j}.*

The 2-sum $\mathcal{B}$ of $(\mathcal{B}_j, \mathcal{B}_{3-j})$ is the set of cycles of B obtained as follows.

1. *If* $D \in \mathcal{B}_i$ *does not contain* z_i*,* $i = 1, 2$*, then let* $D \in \mathcal{B}$*.*

2. *If* $D \in \mathcal{B}_{3-j}$ *and* $z_{3-j} \in D$ *then let* $(D - \{z_{3-j}\}) \cup (C_1 - \{z_j\}) \in \mathcal{B}$*.*

3. *If* $D \in \mathcal{B}_j$*,* $z_j \in D$ *and* $C_1 \neq D \neq C_2$ *then let* $(D - \{z_j\}) \cup (C_4 - \{z_{3-j}\}) \in \mathcal{B}$*.*

4. *If* $|Z_1 \cup Z_2| > 2$ *then let* $(C_2 - \{z_j\}) \cup (C_4 - \{z_{3-j}\}) \in \mathcal{B}$*.*

Theorem 5.10 *Let* B *be* 2*-sum of* B_1 *and* B_2 *where* B_1, B_2 *are duals of eulerian matroids. Let* $\mathcal{B}$ *be* 2*-sum of bases of the circuit lattices of* B_1 *and* B_2*. Then* $\mathcal{B}$ *is a basis of the circuit lattice of* B*.*

Proof. Observe that the number of elements of $\mathcal{B}$ equals the number of series classes of B. We are going to show that every circuit of B is an integer linear combination of elements of $\mathcal{B}$. Without loss of generality let $\mathcal{B}$ be 2-sum of $(\mathcal{B}_1, \mathcal{B}_2)$. Since the duals of B_1, B_2 are eulerian, we have that $2X$ is an integer linear combination of vectors of $\mathcal{B}_i$ for each series class X of B_i, $i = 1, 2$.

First observe that each circuit of B and twice each series class of B is an integer linear combination of $X_0 = (C_2 - \{z_1\}) \cup (C_4 - \{z_2\})$ and elements of $\mathcal{B}$. Hence if $|Z_1 \cup Z_2| > 2$ then we are done. Otherwise we have $X_1 = 2(C_1 - \{z_1\})$ is an integer linear combination of elements of $\mathcal{B}$ since $2\{z_2\}$ is an integer linear combination of elements of $\mathcal{B}_2$. Moreover, $C_3 = (C_1 - \{z_1\}) \cup (C_2 - \{z_1\}) \in \mathcal{B}$ by 5.9.1 and $X_2 = (C_4 - \{z_2\}) \cup (C_1 - \{z_1\}) \in \mathcal{B}$ by 5.9.2. Now, $X_0 = C_3 - X_1 + X_2$ which proves 5.10. □

Theorem 5.11 *Let* B *be a binary matroid with no* F_7^**-minor which is not a* 1*-sum. Let* X *be a series class of* B*. If* B *is not a graphic matroid of a circuit then the circuit lattice of* B *has an* X*-based basis. Hence,* B *is the dual of an eulerian matroid.*

Proof. Let B be a minimal contraexample. Let B be a 2-sum of (B_1, B_2). If $j \in \{1, 2\}$ and B_j is a graphic matroid of a circuit then the circuits of B are obtained from the circuits of B_{3-j} by replacing z_{3-j} by $B_j - \{z_j\}$. Hence 5.11 holds.

Otherwise each series class of B_1 and B_2 is proper by 5.8. According to 5.7 we distinguish two cases. First let X be a series class of B_1 (the case X is a series class of B_2 is handled in the same way). Let $\mathcal{B}_1$ be an X-based basis of B_1 and let $\mathcal{B}_2$ be Z_2-based basis of B_2, where Z_2 denote the series class of B_2 containing $z_2 = c$. Then 2-sum of $(\mathcal{B}_2, \mathcal{B}_1)$ is an X-based basis of B by 5.9, 5.10.

Secondly let $X = S_1 \cup S_2$ where $(S_i \cup \{z_i\}) = Z_i$ is the series class of B_i containing z_i, $i = 1, 2$. Then 2-sum of a Z_1-based basis of B_1 and Z_2-based basis of B_2 is an X-based basis of B by 5.9, 5.10. □

Finally we consider 1-sums.

Theorem 5.12 *Let B be 1-sum of B_1, B_2 and let X be a proper series class of B. If the circuit lattice of B_j, $j = 1, 2$, has an X-based basis then B has an X-based basis as well.*

Proof. Each series class of B is a series class of B_1 or B_2. Moreover, each union of a basis of the circuit lattice of B_1 and a basis of the circuit lattice of B_2 is a basis of the circuit lattice of B. This proves 5.12. □

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