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EVEN CYCLES AND H-HOMEOMORPHISMS

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Abstract

The aim of this paper is to describe a polynomial-time reduction of the Even Cycle Problem to the H-Homeomorphism Problem. As a consequence, we present a polynomial-time algorithm to solve the Even Cycle Problem in any class of digraphs which may be drawn on a fixed but arbitrary surface. The same results hold for more general modularity problems.

1 Introduction

H-HOMEOMORPHISM PROBLEM. Let $H = (V(H), A(H))$ be a fixed digraph. Given a digraph $D = (V(D), A(D))$ and a function f from $V(H)$ to $V(D)$ such that $f(x) \neq f(y)$ for $x \neq y$, decide whether there is a homeomorphism from H to D which extends f.

EVEN CYCLE PROBLEM. Decide whether a given directed graph contains a directed cycle of even length.

In this paper we reduce the Even Cycle Problem to the H-homeomorphism Problem for digraphs without directed cycles of even length, where H is a directed path of at most 5 vertices.

The homeomorphism maps the nodes of H to $f(V(H)) \subset V(D)$ and the arcs of H to the simple directed paths of D. The sets of the vertices of the paths P_1, P_2 of D corresponding to any pair e_1, e_2 of arcs of H have their intersection equal to $e_1 \cap e_2$.

The H-homeomorphism problem is known to be polynomially solvable in the class of acyclic digraphs (i.e. digraphs without directed cycles) (see [3]) and in any class of digraphs embedded on a fixed surface (see [16, 15]). On the other hand, if H is not a star where all arcs go from its center or to its center then the H-homeomorphism problem in the class of all digraphs is NP-complete (see [3]).

The Even Cycle Problem has been studied extensively partly because of its broad connections in linear algebra and combinatorics. Let us include its brief history. We say that two problems are equivalent if they are polynomially reducible to each other.

In 1913, Polya ([14]) suggested computing the permanent of a $0, 1$ matrix A by trying to change some 1 entries of A into -1 so that the determinant of the resulting matrix equals the permanent of A. Let us call a matrix A *convertible* if such a change is possible.

Szego ([19]) pointed out in the same year that not all matrices are convertible.

This may be explained nowadays using a complexity argument. There is an efficient algorithm to compute the determinant, while Valiant proved that the problem of computing the permanent of a $0, 1$ matrix is $\#P$-complete (see [23]).

The problem of computing the permanent of a $0, 1$ matrix is equivalent to the problem of counting the number of perfect matchings in a bipartite graph. This has applications in physics and chemistry.

Kasteleyn observed in 1967 that if a graph admits a Pfaffian orientation (it is defined in section 2) then the number of its perfect matchings may be determined efficiently ([9], [12]). He introduced the notion of Pfaffian orientation, proved that every planar graph admits a Pfaffian orientation and provided a polynomial-time algorithm to construct it. As a consequence, the problem of counting the number of perfect matchings in planar graphs is polynomial-time solvable.

In the beginning of seventies the Even Cycle Problem was introduced by Younger.

In 1973 Seymour ([17]) proved that the Even Cycle Problem is equivalent to the following one: given a hypergraph with the same number of vertices as hyperedges, find out whether it is 2-colourable. This problem is NP-complete for general hypergraphs.

In 1975 Little [11] observed that a matrix is convertible if and only if it is the (V_1, V_2) incidence matrix of a Pfaffian bipartite graph. He also characterized the Pfaffian bipartite graphs in terms of a forbidden structure.

In 1975 Lovasz ([13]) raised two questions:

Does there exist k such that any digraph in which there are at least k arcs leaving each vertex has a directed cycle of even length?

Does there exist k such that strongly k-connected digraph has a directed cycle of even length?

Partial results on these two questions were given by Friedland ([4]) and Alon and Linial ([1]). Thomassen disproved the first question in [20] and settled the second one affirmatively in [21]. He proved that each strongly 3-connected digraph has a directed cycle of an even length. Each such digraph must have a directed cycle of length different from p modulo q for any p, q as observed in [7].

In 1983 Klee, Ladner and Manber [10] proved that the problem of recognizing square matrices whose sign-pattern guarantees invertibility is equivalent to the Even Cycle Problem. They also pointed out that the problem to decide whether a digraph has a directed cycle of even length through a given vertex or a given arc is NP-complete. However this is true for directed cycles of odd length as well, while the problem to find out if a digraph contains a directed cycle of odd length may be solved efficiently.

In 1987 Seymour and Thomassen [18] introduced even digraphs (see section 2 for the definition). They showed that the Even Cycle Problem is equivalent to the even digraphs recognition problem and they characterised even digraphs in terms of a forbidden structure.

In 1989 Vazirani and Yannakakis [24] observed that the even digraphs recognition problem is equivalent to the convertible matrices recognition problem. Hence even digraphs and Pfaffian bipartite graphs are closely related and indeed their characterizations by Seymour, Thomassen and Little may be proved from each other (see [8]).

In 1990 a polynomial-time algorithm to solve the Even Cycle Problem in the class of planar digraphs was proposed by Thomassen ([22]) and independently in [6].

This was extended in [5, 8] by showing that the following problem may be solved in a polynomial time for $K_{3,3}$-free digraphs and for K_5-free digraphs: given integers p and q, a digraph D, does D contain a directed cycle of length different from p modulo q.

Finally let us mention that it is proved by Chung, Goddard and Kleitman ([2]) that every strongly connected digraph with n vertices and at least $(n+1)^2/4$ arcs must contain a directed cycle of even length.

Even directed cycles are closely related to Pfaffian orientations. In our considerations we make use of the proof of a characterization of Little (see [11]) of Pfaffian bipartite graphs.

The paper is organized as follows. In section 2 we state a theorem which implies the characterization of Little and may be proved using his method. This section also contains basic definitions.

In section 3 we make some technical algorithmical observations, which are crucial to reduce the Even Cycle Problem to the H-homeomorphism Problem in section 4.

Using a result of Schrijver (see [16]) this gives a polynomial-time algorithm to solve the Even Cycle Problem in any class of digraphs which may be drawn on a fixed but arbitrary surface. The same results hold for more general modularity problems.

2 A Theorem of Little.

If G is a graph then $V(G)$ will denote the set of vertices of G. If V' is a subset of $V(G)$ then we denote by $G(V')$ the subgraph of G whose vertex set is V' and whose edge set consists of the edges not incident with the vertices of $V(G) - V'$.

A *path* $x_1x_2\dots x_n$ consists of distinct vertices $x_1, x_2, \dots, x_n$ and edges x_ix_{i+1}, $i = 1, 2, \dots, n-1$. A path is said to be a *cycle* if the edge x_nx_1 is present.

An *edge subdivision* of a graph G is a graph G' obtained from G by replacing an edge uv by a path P from u to v whose internal vertices are different from the vertices of G and all have degree 2. If P has an even number of internal vertices, then G' is called an even edge subdivision of G. If a graph G' is obtained from G by repeated even edge subdivisions then we say that G' is an *even subdivision* of G.

A graph G is said to be *contractible* to another graph H if H can be obtained from G by sequentially contracting edges. Contracting an edge means deleting the edge, identifying its endvertices and the edges with the same endvertices which may arise.

If a equals b modulo 2 we write $a \stackrel{2}{=} b$.

A digraph $D = (V, A)$ consists of a set of vertices $V(D)$ and a set $A(D)$ of ordered pairs of vertices, called *arcs*. Given a subset V' of $V(D)$, we denote by $D(V')$ the subdigraph of D whose vertex set is V' and whose arc set consists of the arcs not incident with vertices of $V - V'$. This digraph may also be denoted by $D - (V(D) - V')$ if it is more convenient. If A' is a subset of the arcs of D, we will denote by $D - A'$ the subgraph of D obtained by removing the arcs of A'. If $A' = \{e\}$ then we may let $D - A' = D - e$ if no confusion arises. If G, H are digraphs then we let $G \cup H = (V(G) \cup V(H), A(G) \cup A(H))$ and $G \cap H = (V(G) \cap V(H), A(G) \cap A(H))$.

The definition of directed paths and directed cycles are analogous to those given for the undirected case; it suffices to replace the word edge with arc. A directed path P from a vertex x to a vertex y will be called a xy-dipath.

Let P be a directed path; we also denote by $s(P)$ ($t(P)$) the initial (terminal) vertex of P. The number of arcs of P is called the *length* of P and it will be denoted by $|P|$. Let $a \neq b$ be two vertices of P, a preceding b along P. We denote by $P[a, b]$ the ab-dipath of P. We also let $P[a, b) = P[a, b] - \{b\}$, $P(a, b] = P[a, b] - \{a\}$ and $P(a, b) = P[a, b] - \{a, b\}$.

A digraph is *strongly connected* if it has a directed path from any vertex to any other vertex.

An *ear decomposition* of a digraph D is a sequence $D_0, ..., D_t = D$ of subdigraphs of D such that D_0 consists of a single vertex and no arc, and each D_i arises from D_{i-1} by adding a directed path P_i whose endvertices (not necessarily distinct) belong to D_{i-1} while the arcs and intermediate vertices of P_i do not. The paths P_i are called *ears*.

A digraph is strongly connected if and only if it has an ear decomposition.

Unless stated differently, we will be concerned through all the paper with strongly connected digraphs.

We say that a digraph is *H-free* if its underlying graph does not contain a subgraph contractible to H. Note that a graph contractible to $K_{3,3}$ induces a subdivision of $K_{3,3}$ and hence, saying that a digraph is $K_{3,3}$-free is the same as saying that its underlying graph does not contain an edge subdivision of $K_{3,3}$. The same does not hold for K_5-free digraphs.

A *subdivision* of a digraph D is a digraph obtained from D by replacing some arcs uv by uv-dipaths whose internal vertices are different from the vertices of D and all have indegree and outdegree 1.

A digraph D is called *even* if every subdivision of D contains a directed cycle of even length. Even digraphs have been studied for their interesting connections with the Even Cycle Problem and other algebraic problems ([20, 18, 24]). For the sake of completeness, we summarize here the basic definitions and the main results on even digraphs.

The *splitting* of a vertex v of a digraph D consists of replacing v by two vertices v_1 and v_2 so that (v_1v_2) is an arc, all arcs entering v enter v_1 and all arcs leaving v leave v_2. The *k-double-cycle C_k^** is the digraph arising from the cycle C_k by replacing each edge by two arcs with the same endvertices, oppositely directed. A *weak k-double-cycle* is a digraph obtained from C_k^* by splitting of some vertices and subdividing arcs. If k is odd then a weak k-double-cycle is also called a weak odd-double-cycle.

Let K be a weak k-double-cycle obtained from a k-double-cycle K' by splitting and subdividing arcs. The vertices of K' correspond to k disjoint directed paths of K which we call *survertices* of K.

In [18], Seymour and Thomassen characterized even digraphs.

Theorem 2.1 *A digraph is even if and only if it contains a weak odd-double-cycle.*

A digraph D is called *Pfaffian* if each alternating cycle C with respect to a perfect matching M of its underlying graph has an odd number of edges clockwise oriented. A graph is called *Pfaffian* if it has a Pfaffian orientation. This definition is correct in the sense that it does not depend on the choice of M (see [11]).

In a seminal paper of Little ([11]), Pfaffian bipartite graphs have been characterized in terms of forbidden subgraphs.

Theorem 2.2 *A bipartite graph G is not Pfaffian if and only if it contains an even subdivision K^* of $K_{3,3}$ such that $G(V - V(K^*))$ has a perfect matching.*

In his proof of Theorem 2.2, Little deals with digraphs rather than undirected graphs. In fact, given a bipartite graph G with a perfect matching M, it is natural to construct a digraph G_M by orienting the edges of G so that the edges of M are oriented oppositely to all the remaining edges.

Before stating a modification of Theorem 2.2 let us recall two basic notions of [5] needed in its proof.

Let D be a digraph and let P and Q be two directed paths of D. We say that P and Q are *backward with respect to each other* if $P(x, y) = Q(x, y)$ for all pairs of vertices x, y such that x precedes y on both P and Q. If Q is backward with respect to P and the terminal vertices of Q belong to P but the intermediate vertices of Q do not belong to P then Q is called a *bubble with respect to* P. Observe that each backward path with respect to P which starts and terminates in a vertex of P consists of bubbles with respect to P connected by subpaths of P.

Little actually proved the following theorem. We formulate it so that it contains Theorem 2.1 as well.

Theorem 2.3 *Let D be a digraph. Then D is even if and only if it is possible to reverse the orientation of the arcs of some disjoint directed cycles of D so that the resulting digraph contains a weak 3-double-cycle K^*. Reversing the orientation of the cycles back, K^* corresponds to a weak odd-double-cycle.*

Proof. We may assume that D is strongly connected. Let $P_1 \cup P_2 ... \cup P_m = D$ be its ear-decomposition. Let $P_1 \cup ... \cup P_i = D_i$.

Subdivide $P_1 = D_1$ so that it is a directed cycle of odd length. Given a subdivision D'_{i-1} of D_{i-1} with no even directed cycle, try to subdivide arcs of P_i so that the resulting digraph has no even directed cycle. Clearly, this is possible if and only if D'_{i-1} has all the directed paths from x_i to y_i of the same length modulo 2, where y_i denotes the initial vertex and x_i the terminal vertex of P_i. Since D is even, this is not the case for some $i_0 \leq m$. Let $D' = D'_{i_0-1}$ and $P = P_{i_0}$. Let y be the initial vertex and let x be the terminal vertex of P. By the choice of i_0 there are two xy-dipaths of different length modulo 2 in D'.

Claim 1. There are xy-dipaths Q', R' of different length modulo 2 and vertices x_0, y_0 in D', x_0 preceding y_0 along Q', such that $Q'[x, x_0] = R'[x, x_0]$, $Q'[y_0, y] = R'[y_0, y]$ and the set of the vertices of $Q'(x_0, y_0)$ is disjoint with the set of the vertices of $R'(x_0, y_0)$.

Let Q^* be an arbitrary xy-dipath and let R^* be an xy-dipath of different length modulo 2 than Q^* having as many vertices in common with Q^* as possible. Let x_0 be the last vertex such that $Q^*[x, x_0] = R^*[x, x_0]$. Let y_0^1 be the first vertex of $R^*(x_0, y]$ which belongs to Q^*. Let y_0 be the vertex of $R^*[y_0^1, y] \cap Q^*$ which is as near as possible to x_0 and follows it on Q^*. $Q^*[x_0, y_0]$ and $R^*[x_0, y_0]$ must have different length modulo 2 since otherwise replacing $Q^*[x_0, y_0]$ for $R^*[x_0, y_0]$ in R^* will produce another path of different length modulo 2 which shares more vertices with Q^*. Hence let $R' = R^*$ and $Q' = Q^*[x, y_0] \cup R^*(y_0, y]$. *(End of Claim 1)*

Let $Q = Q'[x_0, y_0]$ and let $R = R'[x_0, y_0]$. Moreover let us assume that Q is as small as possible with respect to inclusion. This assumption will be used later in the proof.

Let e_1 be the arc of Q terminating in y_0. Since D' is strongly connected, there is a directed cycle of D' containing e_1. Hence there is a bubble B_1 with respect to Q' in D' starting in a vertex $x_1 \in Q'[y_0, y]$ and terminating in a vertex $y_1 \in Q'[x, y_0)$. Choose B_1 so that y_1 is as near as possible to x. Let e_2 be the arc of Q' terminating in y_1 and repeat for e_2 the same argument we used for e_1. We finally get:

there is a backward path $\mathcal{B}$ with respect to Q' in D' starting in a vertex $x_1 \in Q'[y_0, y]$ and terminating in a vertex $y_j \in Q'[x, x_0]$, $j \geq 1$.

We distinguish two cases:

Case 1. *There is no bubble B of $\mathcal{B}$ such that $(B - V(Q')) \cap R \neq \emptyset$.*

Since any directed cycle of D' has odd length and $|Q| \overset{2}{\neq} |R|$, it is not difficult to see that $\mathcal{B}$ consists of an even number of bubbles.

Let B_j, $j > 1$ even, be the last bubble of $\mathcal{B}$ and let x_j be its initial and y_j its terminal vertex. Reversing the orientation of the arcs of each cycle $B_i \cup Q'(y_i, x_i)$, $0 < i < j$ and i even, produces a weak 3-double-cycle K^*, whose survertices are formed by: $Q'[y_{j-1}, x_j]$, $Q'[y_j, x_0]$, $Q'[y_0, x_1]$. When reversing the orientation of the cycles back, K^* corresponds to a weak odd-double-cycle. **(End of Case 1)**.

Case 2. *There is a bubble B of $\mathcal{B}$ such that $(B - V(Q')) \cap R \neq \emptyset$.*

Let $a \in Q'(x_0, y]$ be the initial vertex of B and let $b \in Q'[x, y_0)$ be the terminal vertex of B. Note that, by the definition of bubble, b precedes a along Q'.

Let $t_1 = a$. Let z_1 be the first vertex of $B - V(Q')$ which belongs to R and let $B'_0 = B[a, z_1]$.

Observe that $B[z_1, b]$ has a subpath B'_1 from a vertex t_2 of $R[z_1, y_0)$ to a vertex z_2 such that the intermediate vertices of B'_1 do not belong to $Q' \cup R$ and $z_2 = b$ or B'_1 is a bubble with respect to R. Hence $z_2 = b$ or z_2 precedes z_1 along R.

Choose B'_1 so that z_2 is as near as possible to Q' along R. Repeating the same construction for z_2 as we did for z_1, we finally get that:

there is $k \geq 1$, vertex t_k of R and bubble B' with respect to Q' such that $B'_0 = B[a, z_1]$, $B'_{k-1} = B[t_k, b]$ and $B'[z_1, t_k]$ is a backward path with respect to R consisting of bubbles $B'_1, ...B'_{k-2}$ which are subpaths of B.

We distinguish several cases depending on where the vertices a and b are located.

First let $a \in Q'[y_0, y]$ and $b \in Q'[x, x_0]$. Observe that this is Case 1 with Q replaced by R. Hence, k is even since k odd would lead to an even directed cycle in D'. Reversing the orientation of the arcs of some disjoint directed cycles yields a weak 3-double-cycle and a weak odd-double-cycle as in Case 1.

Secondly, let both a and b belong to $Q'(x_0, y_0)$.

If k even then reversing the orientation of the arcs of $P \cup Q'$ we get similar situation as in Case 1 (in fact $|Q'(b, a)| \overset{2}{\neq} |B'(a, b)|$ since all cycles are odd in D') and same as there, a weak 3-double-cycle K^* may be found after reversing the orientation of the arcs of each cycle $B'_i \cup R(z_{i+1}, t_{i+1})$, $0 < i < k - 1$ and i odd. Note that these cycles are disjoint with $P \cup Q'$. When reversing the orientation back, K^* corresponds again to a weak odd-double-cycle.

The case k odd leads to a contradiction; in fact, by the minimality of Q, we have that $|Q[x_0, b]| \overset{2}{=} |R[x_0, t_k] \cup B'_{k-1}|$. For the same reason, $|Q[a, y_0]| \overset{2}{=} |B[a, z_1] \cup R[z_1, y_0]|$. This, together with k odd, implies that $|Q| \overset{2}{=} |R|$, which contradicts our assumption.

Thirdly let $a \in Q'(x_0, y_0)$ and $b \in Q'[x, x_0]$.

If k odd then a weak 3-double-cycle K^* may be found after reversing the orientation of the arcs of the cycles $Q'[x, a] \cup B[a, z_1] \cup R[z_1, y_0] \cup Q'[y_0, y) \cup P$, and $B'_i \cup R(z_{i+1}, t_{i+1})$, with i even and $1 < i < k$. Its survertices are formed by: $Q'[b, y_0] \cup P \cup Q'[y_0, y] \cup R[t_2, y_0]$, $R[z_{k-1}, t_k]$, $B'_0 \cup Q[x_0, a]$. When reversing the orientation back, K^* corresponds again to a weak odd-double-cycle.

The case k even is again impossible: $|Q[a, y_0]| \stackrel{2}{=} |B[a, z_1] \cup R[z_1, y_0]|$ since otherwise we get a contradiction with the minimality of Q. However this gives, together with k even, that $|Q| \stackrel{2}{=} |R|$, which contradicts our assumption.

The last case that $a \in Q'[y_0, y]$ and $b \in Q'(x_0, y_0)$ may be treated in the same way as the third case.

This finishes the proof of the theorem. □

Corollary 2.4 *Theorem 2.3 implies Theorem 2.2.*

Proof. Let G be a bipartite graph and let M be a perfect matching of G. G is not Pfaffian iff G_M is even iff it is possible to reverse the orientation of the arcs of disjoint directed cycles $C_1, ..., C_n$ of G_M so that the resulting digraph G_N contains a weak 3-double-cycle iff G has an even subdivision of $K_{3,3}$ with a perfect matching in the complement. N is the perfect matching of G obtained from M by alternating along $C_1, ..., C_n$. □

A digraph D is called (p, q)*-odd* if any subdivision of D contains a directed cycle of length different from p modulo q. Hence, even digraphs are $(1, 2)$-odd digraphs. The (p, q)-odd digraphs have been introduced and characterized in [7]. The proof of Theorem 2.3 may be modified to give another proof of that characterization.

Corollary 2.5 *A digraph is (p, q)-odd if and only if it contains a weak k-double-cycle, where $(k - 2)p \neq 0$ modulo q.*

Thomassen proved in [21] that every strongly 3-connected digraph contains a weak 3-double-cycle. It may be worthwhile to give another proof of his result using Theorem 2.3.

3 Even Cycles and Backward Paths

In this section we design a procedure *DIFFERENT PATH* which is the core of our reduction algorithm of section 4.

In this section we will use the following convention: *a digraph will mean a digraph without even directed cycles.* D' will always denote a strongly connected digraph containing vertices x and y.

A set $\mathcal{C}$ of arcs such that x and y belong to different connected components C_x and C_y of the underlying graph of $D - \mathcal{C}$ is called an *xy-cut* of D. If the arcs of $\mathcal{C}$ go from C_x to C_y only then $\mathcal{C}$ is called a *directed xy-cut* of D.

Observe that if a digraph D is strongly connected and $\mathcal{C}$ is a directed xy-cut of $D - e$, for some arc e of D, then the initial vertex of e belongs to C_y and the terminal vertex of e belongs to C_x.

Definition 3.1 *A Gadget at r is a digraph G defined as follows. Either G is obtained from a directed path of length 2 from vertex r to vertex v_r by adding arc (v_r, r) and by attaching some other vertices called leaves by vertex-disjoint directed paths going from r or from v_r. In this case v_r is called the branching vertex of G. Or, G is obtained from r by attaching some other vertices called leaves by vertex-disjoint directed paths going from r. In this case r is called the branching vertex of G and we also denote $r = v_r$.*

A Gadget to r is obtained from a Gadget at r by reversing the orientation of all arcs.

Definition 3.2 *Let Gadget_at_x be a procedure whose input consists of a digraph D', an edge e of D', a directed xy-cut $\mathcal{C}$ of $D' - e$ and an xy-dipath Q' not containing e. Let C_x denote the component of $D' - \{e\} - \mathcal{C}$ containing x.*
The output of Gadget_at_x is a quadruple (D'', e^*, Q'', W'') consisting of:

1. *digraph D'' obtained from D' by the following construction. Let t be the terminal vertex of e in D'. Let $s_1, ..., s_n$ be the initial vertices of the arcs of $\mathcal{C}$. Let us consider, for each $i \leq n$, an xs_i-dipath P'_i and a ts_i-dipath P''_i of C_x. Without loss of generality let s_i, $i \leq l \leq n$ be all the vertices such that P'_i and P''_i have the same length modulo 2. D'' is obtained from D' by replacing C_x by the Gadget at x denoted as G_x. The vertices $s_1, ..., s_n$ are the leaves of G_x. The vertices s_i, $i \leq l$, are connected by vertex-disjoint dipaths of the same length modulo 2 as P'_i to the branching vertex v_x of G_x and the vertices s_i, $l < i \leq n$, are connected by vertex-disjoint dipaths of the same length modulo 2 as P'_i to x. Moreover e is replaced by the arc e^* with the same initial vertex as e and terminal vertex equal to v_x.*

2. *arc* e^* *of* D'';

3. *xy-dipath* Q'' *of* D'' *which contains* $Q^* = Q' \cap (D' - C_x)$;

4. *backward path* W'' *with respect to* Q'' *from a vertex of* Q^* *to* x *such that its first bubble contains* e^*.

Definition 3.3 *Let Gadget_to_y be a procedure whose input consists of a digraph* D', *an edge* e *of* D', *a directed* xy*-cut* $\mathcal{C}$ *of* $D' - e$ *and an* xy*-dipath* Q' *not containing* e. *Let* C_y *denote the component of* $D' - \{e\} - \mathcal{C}$ *containing* y.

The output of Gadget_to_y is a pair (D'', Q'') *consisting of:*

1. *Digraph* D'' *obtained from* D' *by replacing* C_y *by the Gadget to* y *analogously as in 3.2.*1 C_x *is replaced by the Gadget at* x;

2. *xy-dipath* Q'' *of* D'' *which contains* $Q' \cap (D' - C_y)$.

Note that if D'' is an output of the procedure *Gadget_at_x* or *Gadget_to_y* then D'' has no even directed cycles (provided their input D' has no even directed cycles).

Definition 3.4 *An arc* e_z *of a digraph* D' *is called positive if it satisfies the following properties:*

1. $D' - e_z$ *has a directed* xy*-cut* $\mathcal{C}$ *such that the component* C_x *of* $D' - \mathcal{C} - e_z$ *containing* x *is a Gadget at* x,

2. *the terminal vertex of* e_z *is the branching vertex of* C_x,

3. *if* P *is an* xy*-dipath of* $D' - e_z$ *and* P' *is a subpath of* P *in* $D' - V(C_x)$, *then any arc of* P' *belongs to a directed cycle of* $D' - V(C_x)$.

If e_z is a positive arc of D' then we will denote by G_x the corresponding Gadget at x. For example, if D' has exactly one arc e entering x and $D' - \{x\}$ is strongly connected then e is positive.

Definition 3.5 *Let* e_z *be a positive arc of* D'. *Let* Q' *be an* xy*-dipath of* D' *which does not contain* e_z *and let* W *be a directed path from a vertex* $w \in Q' - V(G_x)$ *to* x. *A dipath* W *is called* (Q', e_z)*-active if* W *is backward with respect to* Q' *and the first bubble of* W *contains* e_z.

Note that, since e_z is positive, W consists of at most two bubbles. We usually denote by w the initial vertex of W.

Definition 3.6 *Let e_z be a positive arc of D'. Let Q' be an xy-dipath of D' which does not contain e_z and let W be a (Q', e_z)-active dipath. An xy-dipath S is called (Q', W)-critical if there are vertices s_1, s_2 of $Q' \cap S$, s_1 precedes s_2 along Q' and the following properties are satisfied:*

1. *s_1 strictly precedes w on Q',*
2. *$S[x, s_1] = Q'[x, s_1]$,*
3. *S contains no vertex of $Q'(s_1, s_2)$,*
4. *$S(s_1, s_2)$ contains a vertex of $W(w, z]$ where z is the initial vertex of e_z,*
5. *S does not contain e_z.*

The following two procedures are also fundamental to design DIFFERENT PATH algorithm. At this point we do not need to specify them otherwise then by their inputs and outputs.

Definition 3.7 *Let Shortening Procedure be an algorithm accomplishing the following task. Its input consists of a digraph D', a positive arc e_z of D', an xy-dipath Q' of D' which does not contain e_z and a (Q', e_z)-active dipath W.*

Its output is either 'NO' if there is no (Q', W)-critical dipath or 'YES' and an xy-dipath Q'' of D' which does not contain e_z and a (Q'', e_z)-active dipath W'' from a vertex w'' of Q'' to x such that $W''(w'', z]$ is a strict subset of $W(w, z]$ where z is the initial vertex of e_z.

Definition 3.8 *Let Finding R' Procedure be an algorithm accomplishing the following task. Its input is*

a digraph D',

an arc e of D',

an xy-dipath Q' of D',

a backward dipath W with respect to Q' from a vertex w of Q' to x, *a set Y consisting of at most two vertices of $W - Q'$. If $|Y \cap (V(W) - V(Q'))| = 2$ then either the vertices of $Y \cap (V(W) - V(Q'))$ belong to different bubbles of W or W consists of one bubble only and the vertices of $Y \cap (V(W) - V(Q'))$ form an arc of W.*

Its output is either 'YES' and an xy-dipath R' of $D' - e$ with different length modulo 2 than Q' (as in the proof of Theorem 2.3) such that $y_0 \in Q'(x, w]$ and $R'(x_0, y_0)$ does not contain a vertex of $W(z^*, x] - V(Q') - Y$, where z^* is the last vertex of W which belongs to $R'[y_0, y]$, or 'NO' if such R' does not exist.

Forward Procedure of 3.9 is already one-way *DIFFERENT PATH* algorithm.

Definition 3.9 *Let Forward Procedure be the following algorithm.*

Its input consists of a digraph D', a positive arc e_z of D', an xy-dipath Q' of D' which does not contain e_z, a (Q', e_z)-active dipath W and an arc $e \neq e_z$.

Forward Procedure works as follows.

1. *If $D' - e$ is strongly connected and e_z positive arc of $D' - e$ then Forward Procedure returns $(D' - e, e_z, Q', W)$,*

2. *If $D' - e$ is strongly connected and e_z is not a positive arc of $D' - e$ (hence necessarily property* 3. *of definition 3.4 is not satisfied) then Forward Procedure finds a directed xy-cut $\mathcal{C}_1$ of $D' - \{e, e_z\}$ such that the component containing y is as small as possible and returns Gadget_at_x$(D' - e, e_z, \mathcal{C}_1, Q')$.*

3. *If $D' - e$ is not strongly connected but x, y belong to the same strongly connected component D'' of $D' - e$ then Forward Procedure returns (D'', e_z, Q', W).*

4. *Finally if x, y do not belong to the same strongly connected component of $D' - e$ then Forward Procedure finds a directed xy-cut $\mathcal{C}_2$ of $D' - e$ such that the component containing y is minimal with respect to inclusion and returns Gadget_at_x$(D', e, \mathcal{C}_2, Q')$.*

Backward Procedure is the other-way *DIFFERENT PATH* algorithm.

Definition 3.10 *Let Backward Procedure be the following algorithm.*

Its Input is the same as the input of Forward Procedure. Let v_x *be the terminal vertex of* e_z*.*

Backward Procedure works as follows.

1. *If* $D' - e$ *is strongly connected and* e_z *positive arc of* $D' - e$ *then Backward Procedure returns 'NO',*

2. *If* $D' - e$ *is strongly connected and* e_z *is not a positive arc of* $D' - e$ *then Backward Procedure returns 'YES' and* (D'', Q'', W'', e'') *where* $(D'', Q'') = Gadget\_to\_y(D' - e, e_z, \mathcal{C}_1, Q')$*,* $\mathcal{C}_1$ *being the directed xy-cut found in 3.9.2.* W'' *is the only yx-dipath in* D'' : (y, v_y, v_x, x)*,* v_y *being the branching vertex of the Gadget to y of* D'' *constructed in* $Gadget\_to\_y(D' - e, e_z, \mathcal{C}_1, Q')$*. Finally* $e'' = (v_y, v_x)$*.*

3. *If* $D' - e$ *is not strongly connected but* x, y *belong to the same strongly connected component* D'' *of* $D' - e$ *then Backward Procedure returns 'NO'.*

4. *Finally if* x, y *do not belong to the same strongly connected component of* $D' - e$ *then Backward Procedure returns 'YES' and* (D'', Q'', W'', e'') *where* $(D'', Q'') = Gadget\_to\_y(D', e, \mathcal{C}_2, Q')$*,* $\mathcal{C}_2$ *being the directed xy-cut found in 3.9.4.* W'' *is a backward path with respect to* Q'' *such that* $W'' \subset W$*. We remark at this point that* W'' *exists and it is uniquelly determined: e is not an arc of* $Q'[x, w] \cup W$ *by its definition. Hence* $Q'[x, w] \cup W \subset D'' - V(G_y)$ *where* G_y *is the Gadget to y of* D'' *constructed in* $Gadget\_to\_y(D' - e, e_z, \mathcal{C}_1, Q')$*. Moreover* $W'' = P'' \cup Q'[t, w] \cup W$ *where t is the terminal vertex of e and* P'' *is the only yt-dipath of* D''*.* P'' *contains* v_y*, the branching vertex of* C_y*. We let* $e'' = e_z$*.*

Now we are ready to design the algorithm *DIFFERENT PATH*. Given D', a positive arc e_z of D', an xy-dipath Q' of D' which does not contain e_z and a (Q', e_z)-active dipath W. *DIFFERENT PATH* returns 'YES' and produces an xy-dipath R' of different length modulo 2 than Q' which does not contain e_z or it returns 'NO' if such R' does not exist.

Algorithm DIFFERENT PATH

Input: (D', e_z, Q', W).
Begin

1. While *Shortening*(D', e_z, Q', W) Do
   $(D', e_z, Q', W) := (D', e_z, Shortening(D', e_z, Q', W))$;
   If *Finding* R' $(D', e_z, Q,', W, \{v_x\})$ Then
   DIFFERENT PATH := *Finding* R' $(D', e_z, Q', W, \{v_x\})$ Else

2. If $y = w$ Then
   DIFFERENT PATH := 'NO' Else

   Let B be the first bubble of W and let $e \neq e_z$ be an arc of $D' - V(G_x)$ (see remark after 3.4 for the definition of G_x) whose initial vertex does not belong to $Q'(x, s(B)]$ and its terminal vertex belongs to $Q'(t(B), s(B)]$ (note that e exists since $w = s(B)$ belongs to $D' - V(G_x)$ by definition 3.5 and the arc of Q' leaving w satisfies 3.4.3.)

3. If *DIFFERENT PATH (Forward* (D', e_z, Q', W, e)) Then
   DIFFERENT PATH := *DIFFERENT PATH (Forward* (D', e_z, Q', W, e))
   Else

4. If *Backward* (D', e_z, Q', W, e) Then
   DIFFERENT PATH := *Finding* R' *(Backward* $(D', e_z, Q', W, e), \{v_x, v_y\}$)
   (where v_x, v_y are defined in *Backward* (D', e_z, Q', W, e)) Else
   DIFFERENT PATH:= 'NO'

end.

Theorem 3.11 *The algorithm DIFFERENT PATH is correct.*

Proof. We prove correctness of items 1.-4. of the algorithm DIFFERENT PATH.

1. Let (D', e_z, Q'', W'') be an output of *Shortening*(D', e_z, Q', W). According to Definition 3.7, W'' is a (Q'', e_z)-active dipath. Hence the *Shortening Procedure* may be applied to its output.

2. If an xy-dipath R' of different length modulo 2 than Q' exists and e_z is not an arc of R' then $(R' - V(Q')) \cap W \subset \{v_x\} \cup W(w, z]$, where z is the initial vertex of e_z. Hence, if $w = y$, there is no (Q', W)-critical path and *Finding* R' $(D', e_z, Q', W, \{v_x\})$ returns 'NO' then such R' does not exist.

3. Before moving to Step 3, the algorithm selects a bubble B as the first bubble of W and an arc $e \neq e_z$ of $D' - V(G_x)$ whose initial vertex does not belong to $Q'[x, s(B)]$ and its terminal vertex belongs to $Q'(t(B), s(B)]$.

We prove now that if there exists an xy-dipath R' of different length modulo 2 than Q' which satisfies the properties described in the proof of Theorem 2.3 and does not contain e_z, then R' does not contain e.

In fact, if this is not the case, then necessarily y_0 precedes the terminal vertex of e along Q'. Hence $y_0 \in Q'(x, s(B)]$. Remember $s(B) = w$. We distinguish two cases. If $R'(x_0, y_0)$ contains a vertex of $W - V(Q') - \{v_x\}$ then necessarily it contains a vertex of $W(w, z]$, where z is the initial vertex of e_z, because of the Definitions 3.1 and 3.4. Then however R' is (Q', W)-critical which contradicts the assumption that *Shortening Procedure* returned 'NO'. If $R'(x_0, y_0)$ does not contain a vertex of $W - V(Q') - \{v_x\}$ then we get a contradiction with the assumption that *Finding R'* $(D', e_z, Q', W, \{v_x\})$ returned 'NO'.

So, if the required R' exists then e is not its arc. It follows, by the construction of the Gadget at x in 3.2, that Step 3. is correct.

4. Let (D'', Q'', W'', e'') be the output of *Backward* (D', e_z, Q', W, e). Let G_y be the Gadget to y of D''. Let e^* be the arc from the branching vertex v_y of G_y to $D'' - V(G_y)$.

Since *DIFFERENT PATH (Forward Procedure* (D', e_z, Q', W, e)) returned 'NO' we know before moving to Step 4. that the required R' exists if and only if D'' has a xy-dipath R'' of different length modulo 2 than Q'' and the arcs e'' and e^* do not belong to R''.

If 3.10.2. takes place in *Backward Procedure* (D', e_z, Q', W, e) then there is exactly one yx-dipath in D'', namely (y, v_y, v_x, x), which corresponds to W''. Note that in this case $e'' = e^*$. It follows that *Finding R'* $(D'', e_z = (v_y, v_x), Q'', W'', \{v_x, v_y\})$ would find R''.

If 3.10.4. takes place in *Backward Procedure* (D', e_z, Q', W), then, by 3.10, Q'' equals Q' in $D'' - V(G_y)$ and $Q'(x, w) \cup W \subset D'' - V(G_y)$.

Let W'' be a backward path with respect to Q'' from y to x such that $W \subset W''$. Note that W'' is uniquelly determined and it has one or two more bubbles than W.

Note that v_x and v_y belong to different bubbles of W''.

We also know that, when the algorithm performs Step 4, there is no (Q', W)-critical dipath in D'. Hence, if required R'' exists, $R'' \cap (W'' - V(Q'')) \subset \{v_x\} \cup W(y, t)$ where t is the terminal vertex of e^*. However $W(y, t)$ consists of vertex v_y only. Hence the vertices of $R'' \cap (W'' - V(Q''))$ belong to $\{v_x, v_y\}$ and *Finding R'* $(D'', e'', Q'', W'', \{v_x, v_y\})$ would find R''. $\square$

Theorem 3.12 *DIFFERENT PATH is a polynomial-time algorithm providing Shortening Procedure and Finding R' Procedure are polynomial-time algorithms.*

Proof. It follows easily from the description of the algorithm *DIFFERENT PATH.* □

4 H-homeomorphism Problem

In this section we consider the H-homeomorphism Problem defined in the Introduction. We will consider only directed paths as digraphs H. In the previous section we defined Shortening Procedure and Finding R' Procedure by their inputs and outputs. Here we design them using an algorithm which solves the H-homeomorphism Problem, $H = (x_1, ..., x_n)$ being a directed path on n vertices.

Definition 4.1 *Let $(x_1, ..., x_n)$-homeomorphism(D) be a function defined as follows:*

Its input is a digraph D such that $x_1, ..., x_n$ are different vertices of D.

Its output is either 'YES' and a collection of the $x_i x_{i+1}$-dipaths of D, $i = 1, .., (n-1)$, which form a solution to the $(x_1, ..., x_n)$-homeomorphism Problem, or 'NO' if such a collection of dipaths does not exist.

Shortening Procedure (D', e_z, Q', W)
Begin
Shortening Procedure := NO;
Until STOP, do consecutively for each $s_1 \in Q'[x, w)$, $s_2 \in Q'(s_1, y]$, $s_3 \in W(w, z]$:
Begin
$D' := D' - \{e_z\} - V(Q'[x, s_1) \cup Q'(s_1, s_2))$;
If (s_1, s_3, s_2, y)-homeomorphism(D') then
Begin
Q'' := the dipath of D' found by (s_1, s_3, s_2, y)-homeomorphism(D');
$Q' := Q'[x, s_1] \cup Q''$;
Let z^* be the last vertex of $W[w, z]$ which belongs to Q';
Shortening Procedure := $(Q', W[z^*, x])$ and STOP;
End
End.

It follows immediately that the algorithm designed above accomplishes the requirements required in Definition 3.7.

Finding R' (D', e, Q', W, Y)
Begin

1. *Finding* R' := NO;

   Until STOP, do consecutively for each $y_0 \in Q'(x, w]$, $x_0 \in Q'[x, y_0)$ and $z^* \in W$ such that the first vertex of $W[z^*, x]$ which belongs to Q' precedes y_0 along Q':

   Begin

   Let W^* be the shortest subpath of W containing $W[z^*, x]$ which starts in a vertex of Q';

2. If W^* consists of an even number of bubbles then

   Begin

   $D' := D' - \{e\} - V(Q'[x, x_0) \cup Q'(x_0, y_0)) - (V(W(z^*, x]) - V(Q'));$

   If (x_0, y_0, z^*, y)-homeomorphism(D') then

   Begin

   Finding R' := 'YES' and $Q'[x, x_0] \cup (x_0, y_0, z^*, y)$-homeomorphism$(D')$;

   STOP

   End End

3. If there are vertices a, b of $W^* \cap Q'$ such that $W^*[a, b]$ is a bubble of W^*, $W^*[a, b] \cap W^*[z^*, b] \cap (Y - V(Q')) \neq \emptyset$ and

   either $a \in Q'[y_0, y]$ and $b \in Q'[x, x_0]$ or both a, b belong to $Q'(x_0, y_0)$ then

   Begin

   Until STOP, do consecutively for each vertex u of $W^*[a, b] \cap W^*[z^*, b] \cap (Y - V(Q'))$ (note that there are at most two such vertices):

   $D' := D' - \{e\} - V(Q'[x, x_0) \cup Q'(x_0, y_0)) - (V(W(z^*, x])) - V(Q') - Y);$

   If (x_0, u, y_0, z^*, y)-homeomorphism(D') then

   Begin

   Finding R' := 'YES' and $Q'[x, x_0] \cup (x_0, u, y_0, z^*, y)$-homeomorphism$(D')$;

STOP

End End End

End.

Proposition 4.2 *Finding R' Procedure designed above satisfies the requirements in Definition 3.8.*

Proof. We use the proof of Theorem 2.3, where the role of $\mathcal{B}$ is played by $W[z^*, x]$.

Step 2 corresponds to Case 1 in the proof, and Step 3 corresponds to the first two subcases of Case 2.

If the vertices of $Y \cap (V(W) - V(Q'))$ belong to different bubbles of W then the remaining two subcases of Case 2 may not occur by the reasoning in the proof of Theorem 2.3. Otherwise W consists of a single bubble and the remaining two subcases of Case 2 may not occur by their definition. □

Now we are ready to describe an algorithm $Even(D)$ which solves the Even Cycle Problem. Given a digraph D as its input, it either returns a directed cycle of D of an even length or asserts that D has no such cycle. We will construct $Even(D)$ using algorithms $(x_1, ..., x_n) - homeomorphism(D')$ where $n \leq 5$, D' is a subdivision of a subdigraph of D and no directed cycle of D' has even length.

Even (D)

1. Given a digraph, we first delete the arcs which do not belong to a directed cycle. The resulting digraph D is strongly connected.

2. Let $P_1 \cup P_2 ... \cup P_m = D$ be an ear-decomposition of D. For $i = 1, ..., m$ let $P_1 \cup ... \cup P_i = D_i$, let P_i start in vertex y_i and end in vertex x_i and let C_i be a directed cycle of D_i containing P_i. The cycles $C_1, ..., C_m$ form a directed cycle basis of D. If one of them has an even length, we are done. Hence let all have an odd length. It follows from linear algebra considerations (see e.g. [18]) that a digraph D with this property has a directed cycle of an even length iff D is even.

3. Theorem 2.3 characterizes even digraphs and its proof suggests a method how to check evenness: D even iff for each $i = 2, ..., m$ all $x_i y_i$-dipaths

of D_{i-1} have the same length modulo 2. Hence this test needs to be performed for each $i = 2, ..., m$ consecutively, knowing in addition that all the directed cycles of D_{i-1} have odd lengths. In the remaining we describe how to carry out this test.

4. Let $x' = x_i$ and let D' be the strongly connected digraph obtained from D_{i-1} by splitting of x' and subdividing the new arc (x'_1, x'_2) of D' by a single vertex (see section 2 for the definitions of these operations). Note that all directed cycles of D' have an odd length. Let $x = x'_2$. Note that all $x'y$-dipaths of D_{i-1} have the same length modulo 2 iff all xy-dipaths of D' have the same length modulo 2.

5. Let e_z be the (only one) arc of D' entering x and let z be its initial vertex. Let Q' be an arbitrary xy-dipath of D'. Note that it cannot contain e_z. Let W be an arbitrary bubble with respect to Q' containing e_z. Note that W exists since D' is strongly connected and thus the first arc of Q' belongs to a directed cycle of D'. Now we will distinguish two cases.

6. If e_z is a positive arc of D' then W is (Q', e_z)-active. Hence it remains to perform *DIFFERENT PATH* (D', e_z, Q', W). It returns 'NO' iff all xy-dipath of D_{i-1} have the same length modulo 2.

7. If e_z is not a positive arc of D', then, necessarily, 3.4.3 is not satisfied. Let $\mathcal{C}$ be a directed xy-cut of $D' - e_z$ such that the component containing y is minimal with respect to inclusion. We perform the following steps:

   7a. *DIFFERENT PATH* $(Gadget\_at\_x(D', e_z, \mathcal{C}, Q'))$;

   7b. If *DIFFERENT PATH* returns 'NO' in step $7a$. then
   we procede analogously as in 3.10.2:
   Finding R' $(Gadget\_to\_y(D', e_z, \mathcal{C}, Q'), W'', (v_y, v_x), \{v_x, v_y\})$ where $W'' = (y, v_y, v_x, x)$ and v_x, v_y are the branching vertices as in 3.10.2.

8. The output of 7 is 'NO' iff all xy-dipaths of D_{i-1} have the same length modulo 2.

This completes the description of the algorithm $Even(D)$.

Theorem 4.3 *Even(D) is a polynomial-time algorithm providing each call of* $(x_1, ...x_n)$*-homeomorphism*(D') *is counted as a single step.*

Proof. This follows immediately from the description of *Even*(D). □

Corollary 4.4 *Let* $\mathcal{D}$ *be a class of digraphs such that if* $D \in \mathcal{D}$ *then each subdivision and each subdigraph of* D *belongs to* $\mathcal{D}$*. If there is a polynomial-time algorithm to solve the* $(x_1, ..., x_n)$*-homeomorphism Problem for the digraphs of* $\mathcal{D}$ *with no directed cycle of an even length,* $n \leq 5$*, then there is a polynomial-time algorithm to solve the Even Cycle Problem in* $\mathcal{D}$*. Particularly, the Even Cycle Problem is polynomially solvable in any class of digraphs which may be drawn on an arbitrary fixed surface.*

The same result holds for the problem of recognizing even digraphs.

Remark 4.5 *Let* G *be a bipartite graph and let* M *be a perfect matching of* G*.* G *is Pfaffian iff* G_M *is not even. Hence Corollary 4.4 relates to the problem of recognition of Pfaffian bipartite graphs as well. We believe the same is true for general Pfaffian graphs - this is a subject of our ongoing work.*

Using Corollary 2.5, the results of this section may be modified for general modularity. Hence Corollary 4.4 remains true if the Even Cycle Problem is replaced by the problem of finding whether a digraph contains a directed cycle of length different from p *modulo* q*, and the problem of recognizing even digraphs is replaced by the problem of recognizing* (p, q)*-odd digraphs.*

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