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**EVEN DIRECTED CYCLES IN  $H$ -FREE DIGRAPHS**

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## Abstract

A digraph is *H-free* if its underlying graph does not contain a subgraph contractible to the graph  $H$ . We provide a polynomial-time algorithm to solve the Even Cycle Problem in the class of  $K_{3,3}$ -free digraphs and in the class of  $K_5$ -free digraphs. We also discuss the important role played by the subdivisions of  $K_{3,3}$  in solving the Even Cycle Problem in its generality.

## 1 Introduction

The problem of recognizing whether a digraph  $D$  has an odd directed cycle (triangles are allowed) can be easily solved by testing if  $D$  has an odd closed directed walk, while little is known about the complexity of testing whether a digraph has a directed cycle of even length. For this problem, called the *Even Cycle Problem*, neither a polynomial-time algorithm nor a proof of NP-completeness has been found.

Klee et al. ([5]) and Thomassen ([11]) showed that it is NP-complete to decide whether a digraph has an even directed cycle through a given arc, thus indicating that the problem might be difficult.

One class for which these modularity problems turned out to be polynomial-time solvable is the class of planar digraphs. Thomassen ([13]) provided a polynomial-time algorithm for solving the Even Cycle Problem in planar digraphs. A different algorithm is given in [2].

Seymour ([9]) proved that the problem of deciding if two distinct vertices belong to the same directed cycle is polynomially solvable for planar digraphs, while this problem is proved to be NP-complete in the general case.

Extending the results in [2], it has been proved that the problem of deciding whether all directed cycles have the same length  $p \bmod q$ ,  $p > 0$  and  $q > 1$  integers, is polynomial-time solvable in the class of planar digraphs ([1]). The polynomial solvability of the Even Cycle Problem in the class of planar digraphs easily follows as a special case of the previous result with  $p = 1$  and  $q = 2$ .

The problem considered in [2] is the following:

*Parity Dipath Problem:* given a planar digraph  $D$ , a facial cycle  $F$  of  $D$  such that  $D - F$  has no even directed cycle and two vertices  $x, y \in F$ , decide whether  $D - F$  contains a directed path of prescribed parity from  $x$  to  $y$ .

In [1], we provide a polynomial-time algorithm for solving the Parity Dipath Problem. It is not difficult to observe that a polynomial-time algorithm solving the Parity Dipath Problem can be applied recursively to solve the Even Cycle Problem. In this paper we extend that algorithm in order to describe a polynomial-time algorithm solving the Even Cycle Problem in the class of  $K_{3,3}$ -free digraphs and in the class of  $K_5$ -free digraphs. Both classes properly contain planar digraphs.

The subdivisions of  $K_{3,3}$  are linked with even directed cycles in several ways which we discuss in section 2; this section also contains basic definitions.

In section 3, we describe a polynomial-time algorithm for solving the Parity Dipath Problem and in the last section we show how to apply it in order to solve the Even Cycle Problem in the above mentioned classes of digraphs.

## 2 $K_{3,3}$ -free digraphs.

If  $G$  is a graph then  $V(G)$  will denote the set of vertices of  $G$ . If  $V'$  is a subset of  $V(G)$  then we denote by  $G(V')$  the subgraph of  $G$  whose vertex set is  $V'$  and whose edge set consists of the edges not incident with the vertices of  $V(G) - V'$ .

A *path*  $x_1x_2 \dots x_n$  consists of distinct vertices  $x_1, x_2, \dots, x_n$  and edges  $x_i x_{i+1}$ ,  $i = 1, 2, \dots, n - 1$ . A path is said to be a *cycle* if the edge  $x_n x_1$  is present.

An *edge subdivision* of a graph  $G$  is a graph  $G'$  obtained from  $G$  by replacing an edge  $uv$  by a path  $P$  from  $u$  to  $v$  whose internal vertices are different from the vertices of  $G$  and all have degree 2. If  $P$  has an even number of internal vertices, then  $G'$  is called an even edge subdivision of  $G$ . If a graph  $G'$  is obtained from  $G$  by repeated even edge subdivisions then we say that  $G'$  is an *even subdivision* of  $G$ .

A graph  $G$  is said to be *contractible* to another graph  $H$  if  $H$  can be obtained from  $G$  by sequentially contracting edges, namely deleting an edge and identifying its endvertices.

A graph  $G$  is said to be *planar* if it can be drawn in the plane with no edge crossings. Once a planar graph is drawn, the plane is subdivided into regions called *faces*. If the boundary of a face is a cycle then we call it a *facial cycle*.

A digraph  $D = (V, A)$  consists of a set of vertices  $V$  and a set  $A$  of ordered pairs of vertices, called *arcs*. Given a subset  $V'$  of  $V$ , we denote by  $D(V')$  the subdigraph of  $D$  whose vertex set is  $V'$  and whose arc set consists of the arcs not incident with vertices of  $V - V'$ . If  $A'$  is a subset of the arcs of  $D$ , we will denote by  $D - A'$  the subgraph of  $D$  obtained by removing the arcs of  $A'$  and the resulting isolated vertices.

The definition of directed paths and directed cycles are analogous to those given for the undirected case; it suffices to replace the word edge with

arc. A digraph is planar if its underlying graph is planar.

A digraph is *strongly connected* if it has a directed path from any vertex to any other vertex. Observe that if a digraph  $D$  is strongly connected and planar then there exists a facial cycle of the underlying graph of  $D$  which is a directed cycle. Unless stated differently, we will be concerned through all the paper with strongly connected digraphs.

We say that a digraph is  $H$ -free if its underlying graph does not contain a subgraph contractible to  $H$ . Note that a graph contractible to  $K_{3,3}$  induces a subdivision of  $K_{3,3}$  and hence, saying that a digraph is  $K_{3,3}$ -free is the same as saying that its underlying graph does not contain an edge subdivision of  $K_{3,3}$ . The same does not hold for  $K_5$ -free digraphs.

An *arc subdivision* of a digraph  $D$  is a digraph obtained from  $D$  by replacing an arc  $uv$  by a directed path  $P$  from  $u$  to  $v$  whose internal vertices are different from the vertices of  $D$  and all have indegree and outdegree 1.

A digraph  $D$  is called *even* if every arc subdivision of  $D$  contains a directed cycle of even length. Even digraphs have been studied for their interesting connections with the Even Cycle Problem and other algebraic problems ([11, 10, 15]). For sake of completeness, we summarize here the basic definitions and the main results on even digraphs.

The *splitting* of a vertex  $v$  of a digraph  $D$  consists of replacing  $v$  by two vertices  $v_1$  and  $v_2$  so that  $(v_1v_2)$  is an arc, all arcs entering  $v$  enter  $v_1$  and all arcs leaving  $v$  leave  $v_2$ . The  $k$ -double-cycle  $C_k^*$  is the digraph arising from the cycle  $C_k$  by replacing each edge by two arcs with the same end-vertices, oppositely directed. A *weak  $k$ -double-cycle* is a digraph obtained from  $C_k^*$  by splitting of some vertices and subdividing arcs. If  $k$  is odd then a weak  $k$ -double-cycle is also called a weak odd-double-cycle.

In [10], Seymour and Thomassen characterized even digraphs.

**Theorem 2.1** *A digraph is even if and only if it contains a weak odd-double-cycle.*

Given a graph  $G$ , we say that an even cycle  $C$  of  $G$  is *nice* if  $G(V - C)$  has a perfect matching.

A digraph  $D$  is called *Pfaffian* if each nice cycle  $C$  of its underlying graph has an odd number of edges clockwise directed. A graph is called *Pfaffian* if it has a Pfaffian orientation.

Kasteleyn ([4]) observed that if a graph admits a Pfaffian orientation then the number of its perfect matchings may be determined efficiently ([4],

[8]). He introduced the notion of Pfaffian orientation, proved that every planar graph admits a Pfaffian orientation and provided a polynomial-time algorithm to construct it. As a consequence, the problem of counting the number of perfect matchings in planar graphs is polynomial-time solvable.

The Kasteleyn's method has been extended by Little ([6]) to  $K_{3,3}$ -free graphs; Little proved that any  $K_{3,3}$ -free graph admits a Pfaffian orientation. This implies that the number of perfect matchings of a  $K_{3,3}$ -free graph can be computed in polynomial time (see also [8]).

It is intriguing to see how the concepts connected with the Even Cycle Problem were considered under different perspectives and how the same results have been proved several times in a different context.

In a seminal paper of Little ([7]), Pfaffian bipartite graphs have been characterized in terms of forbidden subgraphs.

**Theorem 2.2** *A bipartite graph  $G$  is not Pfaffian if and only if it contains an even subdivision  $K^*$  of  $K_{3,3}$  such that  $G(V - K^*)$  has a perfect matching.*

This characterization has been recently rediscovered by Thomassen ([12]).

Let  $G = (V, V', E)$  be a bipartite graph with a perfect matching  $M$ . It is possible to associate to  $G$  two digraphs as follows: first permute vertices of  $V'$  so that  $ii' \in M$  and then construct digraph  $H(G, M)$  whose vertex set is  $V$  and whose arc set is  $A = \{(i, j) : ij' \in E, \text{ for } i \in V, j' \in V'\}$ . We denote by  $G_M$  the digraph with vertex set  $V \cup V'$  and arc set  $\{(i, i') : i \in V\} \cup \{(i', j) : i' \in V', j \in V, \text{ and } i'j \in E\}$ .

Vazirani and Yannakakis ([15]) pointed out that the two concepts of even digraphs and non-Pfaffian bipartite graphs are equivalent; in fact, they proved that:

**Theorem 2.3** *Let  $G$  be a bipartite graph and let  $M$  be a perfect matching of  $G$ . Then  $G$  is Pfaffian iff  $H(G, M)$  is not even.*

The theorem of Little and the characterization of even digraphs due to Seymour and Thomassen may also be deduced from each other in a direct way. The following theorem shows an explicit equivalence between the two results.

**Theorem 2.4** *Let  $G$  be a bipartite graph and  $M$  a perfect matching of  $G$ . Then  $G$  contains an even subdivision  $K^*$  of  $K_{3,3}$  such that  $G(V - K^*)$  has a perfect matching if and only if the digraph  $H(G, M)$  contains a weak odd-double-cycle.*

**Proof.** First observe two simple facts:  $H(G, M)$  has a weak  $k$ -double cycle if and only if  $G_M$  has a weak  $k$ -double cycle. Let  $R$  and  $S$  be perfect matchings of  $G$  and let  $R$  be obtained from  $S$  by alternating along cycle  $C$ . Then  $G_R$  is obtained from  $G_S$  by reversing the orientation of  $C$ .

We will use the following notation.

Let  $K^*$  be an even subdivision of  $K_{3,3}$  and let  $N_1$  be a perfect matching of  $G - K^*$ . Let  $N_2$  be a perfect matching (note that it exists) of  $K^*$ , and let  $N = N_1 \cup N_2$ .  $N$  is a perfect matching of  $G$  and we will call it an *easy matching*.

Note that  $N$  is an easy matching of  $G$  and  $K^*$  an even subdivision of  $K_{3,3}$  defining it iff  $K^*$  is a weak 3-double cycle of  $G_N$ .

Each weak  $k$ -double-cycle  $K$  consists of  $k$  *inner directed cycles*  $I_1, \dots, I_k$  and of two *outer directed cycles*  $O_1, O_2$ . Each weak  $k$ -double-cycle  $K$  is obtained from a  $k$ -double-cycle  $K'$  by splitting. The vertices of  $K'$  correspond to  $k$  disjoint directed paths  $V_1, \dots, V_k$  of  $K$  which we call *survertices* of  $K$ . The directed paths of  $K$  between its survertices will be called *surarcs* of  $K$ . Two surarcs are *neighbouring* if they belong to the same outer cycle of  $K$  and are incident to the same survertex.

Let  $R$  be a perfect matching of  $G$  and let  $K$  be a weak  $k$ -double cycle of  $G_R$ . A directed cycle of  $G_R$  is called an *eye* of  $K$  if it consists of two disjoint segments  $S_1, S_2$  of an inner cycle  $I_i$ ,  $i \in \{1, \dots, k\}$ , of  $K$ ,  $S_1$  subset of  $O_1 - O_2$  and  $S_2$  subset of  $O_2 - O_1$ , and two directed paths joining the endvertices of  $S_1, S_2$  whose intermediate vertices do not belong to  $K$ .

A directed cycle of  $G_R$  is called a *corner* of  $K$  if it consists of two disjoint segments  $S_1, S_2$  of neighbouring surarcs of  $K$ , and two directed paths joining the endvertices of  $S_1, S_2$  whose intermediate vertices do not belong to  $K$ .

Next Claim is not difficult to prove.

*Claim 1. Let  $R, S$  be perfect matchings of  $G$  and let  $K$  be a weak  $k$ -double-cycle of  $G_S$ ,  $k$  odd.*

1. *If  $k > 3$  and  $R$  is obtained from  $S$  by alternating along an inner cycle of  $K$  then  $G_R$  contains a weak  $(k - 2)$ -double-cycle.*

2. If  $k = 3$  and  $R$  is obtained from  $S$  by alternating along an inner cycle of  $K$  then  $G_R$  contains a weak 3-double-cycle.
3. If  $R$  is obtained from  $S$  by alternating along an eye of  $K$  then  $G_R$  contains a weak  $(k + 2)$ -double-cycle.
4. If  $K$  is a weak 3-double-cycle and  $R$  is obtained from  $S$  by alternating along a corner of  $K$  then  $G_R$  contains a weak 3-double-cycle.

First part of the theorem follows: if  $H(G, M)$  contains a weak odd-double-cycle then  $G_M$  contains a weak odd-double-cycle and alternating on its inner cycles we get by Claim 1 that  $G$  has a perfect matching  $R$  such that  $G_R$  has a weak 3-double-cycle. Hence  $G$  has an even subdivision  $K^*$  of  $K_{3,3}$  such that  $G(V - K^*)$  has a perfect matching.

If  $M$  is an easy matching then  $G_M$  and  $H(G, M)$  have a weak 3-double-cycle and the remaining part of the theorem follows. Otherwise  $M$  may be obtained from any easy matching by alternating along disjoint cycles of  $G$ . In his proof of Theorem 2.2 (see [7]), Little has implicitly shown that it is enough to alternate along special cycles.

Let  $R$  be a perfect matching of  $G$  and let  $G_R$  contains a weak odd-double-cycle. Then by  $R^0$  we denote a weak odd double cycle of  $G(R)$ .

*Claim 2. Let bipartite graph  $G$  be not Pfaffian and let  $M$  be a perfect matching of  $G$ . Then there is a sequence of perfect matchings  $(R_1, \dots, R_s)$  of  $G$  such that  $M = R_s$ ,  $R_1$  is an easy matching of  $G$  and  $R_{i+1}$ ,  $i = 1, \dots, (s - 1)$ , is obtained from  $R_i$  by one of the following operations: alternate along an inner cycle of  $R_i^0$ , alternate along an eye of  $R_i^0$  and alternate along a corner of  $R_i^0$  : in this case,  $R_i^0$  is a weak 3-double-cycle.*

Note that  $R_i^0$ ,  $i = 1, \dots, s - 1$  is defined by Claim 1. The remaining part of the theorem follows immediately from Claim 2.  $\square$

### 3 Parity Dipath Problem

An algorithm solving the Parity Dipath Problem is an algorithm such that, receiving as an input a planar digraph  $D$  with outer facial cycle  $F$  such that

$D - F$  does not contain an even directed cycle, and any two nodes  $x, y$  of  $F$ , produces one of the following outputs:

1. two  $xy$ -dipaths  $P_1$  and  $P_2$  in  $D - F$  of different length modulo 2;
2. all  $xy$ -dipaths  $P$  of  $D - F$  have the same length modulo 2;
3. indication that there is no  $xy$ -dipath in  $D - F$ .

An  $xy$ -dipath is a directed path from  $x$  to  $y$ . We denote also as  $P(x, y)$  a dipath from  $x$  to  $y$ .

In this section we describe the features of an algorithm, denoted as  $\mathcal{A}(D, F, x, y)$ , that solves the Parity Dipath Problem (the details may be found in [1] and [2]). The input of the algorithm is the quadruple  $(D, F, x, y)$ . We consider  $D$  drawn in the plane so that  $F$  is its outer face.

The algorithm consists of three basic parts.

#### *Part I. CONSTRUCTION OF A NET.*

The algorithm  $\mathcal{A}(D, F, x, y)$  constructs an acyclic subdigraph  $N$  of  $D - F$  using the following notions:

*minimum dipath:* an  $xy$ -dipath  $P$  is *minimum* (*maximum*)  $xy$ -dipath if  $P$  is the only  $xy$ -dipath of the subdigraph of  $D - F$  drawn in the area bounded by  $P$  and  $F(y, x)$  ( $F(x, y)$  respectively).

Note that there is at most one minimum  $xy$ -dipath.

*minimal dipath:* an  $xy$ -dipath  $P'$  is *minimal* with respect to an  $xy$ -dipath  $P$  if  $P' \neq P$  and  $P$  and  $P'$  are the only  $xy$ -dipaths in the subdigraph of  $D - F$  drawn in the area bounded by  $P'$  and  $P$ .

The *net*  $N = \bigcup_{i=0}^m L_i$  consists of the union of the  $xy$ -dipaths  $L_i$ , called *levels*, constructed as follows:

- (i)  $L_0$  is the minimum  $xy$ -dipath of  $D - F$ ,
- (ii)  $L_i$  is a minimal  $xy$ -dipath of  $D - F$  with respect to  $L_{i-1}$ ,
- (iii)  $L_m$  is the maximum  $xy$ -dipath of  $D - F$ .

After constructing a net  $N$ , the algorithm checks whether  $N$  has two  $xy$ -dipaths of different length modulo 2. This may be done efficiently since  $N$  is acyclic.

*Part II. CHECKING FOR DIRECTED PATHS OF PRESCRIBED PARITY.*

Let  $P$  and  $Q$  be two directed paths. We say that  $P$  is *backward* with respect to  $Q$  if  $P(a, b) = Q(a, b)$  for each pair of vertices  $a, b$  such that  $a$  precedes  $b$  on both  $P$  and  $Q$ .

Let  $P$  be backward with respect to  $Q$ . Each subdipath of  $P$  whose terminal vertices belong to  $Q$  and whose intermediate vertices do not belong to  $Q$  is called a *bubble*. Hence  $P$  consists of bubbles and subdipaths of  $Q$  between their terminal vertices.

In [1], a result concerning the structure of  $xy$ -dipaths of different length modulo an integer  $q > 1$  has been proved. In the following we specialize the result to the case  $q = 2$  and we omit the proof.

**Theorem 3.1** *Let  $D$  be a planar digraph and  $F$  a facial cycle such that all directed cycles of  $D - F$  have odd length. Let us fix two vertices  $x$  and  $y$  of  $F$  and net  $N$  between them. If  $D - F$  has two  $xy$ -dipaths of different length modulo 2 and all  $xy$ -dipaths of  $N$  have the same length modulo 2 then there is an  $xy$ -dipath  $P$  in  $D - F$  which satisfies the following properties:*

1. *there exists a level  $L$  of  $N$  and two vertices  $a, b$  of  $L$  such that  $a$  precedes  $b$  along  $L$  and:*

(\*)  $P_1 = P(x, b), P_1 \cap L(b, y) = \{b\}$  and moreover  $b$  is the vertex of  $P \cap L$  nearest to  $y$  along  $L$  with the property that  $P(b, y)$  contains a vertex of  $L(x, b)$

(\*\*)  $P_3 = P(a, y), P_3 \cap L(x, a) = \{a\}$

(\*\*\*)  $P_2$  is a  $ba$ -dipath which is backward with respect to  $L$  and consists of an odd number of bubbles

2.  $|P_1| \stackrel{2}{=} |L(x, b)|$  and  $|P_3| \stackrel{2}{=} |L(a, y)|$ .

Now, assume that all  $xy$ -dipaths of  $N$  have the same length modulo 2. Then, by Theorem 3.1, if  $D - F$  has two  $xy$ -dipaths of different length modulo 2 then it has a dipath  $P$  satisfying 1-2 of Theorem 3.1. The properties of  $P$

imply that  $P$  and  $L$  have different length modulo 2. Hence, it suffices to check for each level  $L$  of  $N$  and for each pair of vertices  $a, b \in L$  whether there exist  $P_1, P_2, P_3$  satisfying 1-2 of Theorem 3.1.

Clearly, the best candidates for  $P_1$  and  $P_3$  are the minimum or the maximum  $xy$ -dipaths of  $D - F$  containing  $L(b, y)$  and the maximum or the minimum  $xy$ -dipaths of  $D - F$  containing  $L(x, a)$ , respectively. All these directed paths may be constructed in polynomial time. Hence, we may 'guess'  $P_1$  and  $P_3$  and it remains to check whether there exists a  $ba$ -dipath  $P_2$  consisting of an odd number of bubbles with respect to  $L$ .

A bubble  $B = B(s, t)$  is *minimum* if  $B$  is the minimum  $st$ -dipath of the subgraph of  $D - F$  drawn inside  $L \cup F^1$  or inside  $L \cup F^2$  where  $F^1, F^2$  are the subpaths of  $F$  between  $x$  and  $y$ .

It follows that there are at most two minimum  $st$ -bubbles. We may assume that  $P_2$  uses only minimum bubbles and so, in the rest of the section, by *bubble* we will always mean *minimum bubble*. Moreover, we consider  $L$  as fixed and we use the notation  $v \leq u$  to indicate that  $v$  precedes  $u$  along  $L$ . We will also denote by  $s(B)$  and  $t(B)$  the starting and terminal vertex of bubble  $B$ , respectively.

Given two vertices  $r$  and  $s$  such that  $s < r$ , a sequence of bubbles  $\{B_1, \dots, B_k\}$  together with the segments of  $L$  between their terminal points is a *good directed  $rs$ -walk* if  $t(B_i) \leq s(B_{i+1}) < t(B_{i-1}) \leq s(B_i)$  for  $i = 2, \dots, k-1$ ,  $t(B_k) \leq s$  and  $s(B_1) \geq r$  and moreover the sets of the intermediate vertices of two consecutive bubbles have empty intersection.

If  $k$  is odd then a good directed walk is called an *odd good directed walk*.

The next theorem states that in order to find a  $ba$ -dipath  $P_2$  of odd number of bubbles, it suffices to find an odd good directed  $ba$ -walk. Its proof may be found in [2].

**Theorem 3.2** *Let  $D$  be a planar digraph and  $F$  a facial cycle of  $D$  such that all directed cycles of  $D - F$  have odd length. Let  $x, y$  be two vertices of  $F$ , let  $L$  be an  $xy$ -dipath of  $D - F$  and  $a, b$  two vertices of  $L$ ,  $a$  preceding  $b$  along  $L$ . Then  $D - F$  has an odd good directed  $ba$ -walk if and only if  $D - F$  has a  $ba$ -dipath formed by an odd number of bubbles.*

### Part III. CHECKING FOR ODD GOOD DIRECTED WALKS.

Since there is a quadratic number of bubbles, it is possible to use a search procedure for checking the existence of an odd good directed walk.

With each bubble  $B$  we associate some variables which have the following meaning during the algorithm:

$pred(B)$  = the current predecessor of  $B$ , i.e. a bubble from which the search considers  $B$ ; if  $s(B) \geq b$  then  $pred(B)$  will be defined as equal to  $b$ .

$n(B)$  = the number of bubbles of the current good walk from  $b$  to  $t(B)$  that uses  $B$ ;

$act(B)$  = boolean variable whose value is *true* if there are still bubbles that have to be considered as successors of  $B$ .

We describe now a procedure that, given a bubble  $H$  and a vertex  $q \in L$ ,  $t(H) < q \leq s(H)$ , as an input, either produces a good directed walk from  $s(H)$  to  $a$  such that  $H$  is its first bubble and its second bubble starts in a vertex which precedes  $q$  along  $L$ , or states that such a walk does not exist.

**Function WALK(H,q)**

**Begin** for all bubbles  $B$ , set  $n(B) := 0$ ;  $pred(B) := \emptyset$  and  $act(B) := true$ ;  
for each bubble  $B$  such that  $t(B) < t(H) \leq s(B) < q$ , set  $n(B) := 1$   
and  $pred(B) := H$ .

**Repeat**

let  $B$  be a bubble such that  $act(B) = true$ ,  $n(B) \neq 0$  and  $n(B)$  is as small as possible;

if  $t(B) \leq a$  then  $WALK(H, q) := \{H, \dots, pred(pred(B)), pred(B), B\}$   
and STOP A:

$WALK(H, q)$  is a good directed walk from  $s(H)$  to  $a$  with the required properties

else for all bubbles  $\overline{B}$  such that  $B \cap \overline{B} = \emptyset$  and  $t(\overline{B}) < t(B) \leq s(\overline{B}) < t(pred(B))$  do

if  $n(\overline{B}) = 0$  or  $t(pred(\overline{B})) < t(B)$  then  $n(\overline{B}) := n(B) + 1$ ,  $pred(\overline{B}) := B$ ,  $act(\overline{B}) := true$ ;

$act(B) := false$ ;

**until** all bubbles  $B$  with  $n(B) \neq 0$  have  $act(B) = false$  (STOP B: no good directed walk exists).

**end.**

The algorithm for the recognition of odd good directed walks is designed with the help of the previous procedure. In fact, the Function  $WALK(H, q)$  allows us to associate to each bubble  $H$  a good directed walk  $WALK(H, t(pred(H)))$  such that  $pred(H) \cup WALK(H, t(pred(H)))$  is a good directed walk from  $s(pred(H))$  to  $a$ .

Let us describe the algorithm in a more precise way.

**Algorithm Odd Good Walk.**

**Begin** for all bubbles  $B$ , set  $n(B) = 0$ ;  $pred(B) := \emptyset$ ,  $act(B) := true$ .  
for each bubble  $B$  such that  $t(B) < b \leq s(B)$ , set  $n(B) := 1$ ,  $pred(B) := b$ .

**Repeat**

let  $B$  be a bubble such that  $act(B) = true$ ,  $n(B) \neq 0$  and  $n(B)$  is as small as possible;  
if  $t(B) \leq a$  and  $n(B)$  is odd then  $\{b, \dots, pred(B), B\}$  is an odd good directed walk and STOP A;  
if  $t(B) \leq a$  and  $n(B)$  is even then  $act(B) := false$ ;  
if  $WALK(B, t(pred(B)))$  exists and  $n(B) + (\# \text{ bubbles of } WALK(B, t(pred(B)))) - 1$  is odd then  $\{b, \dots, pred(B), WALK(B, t(pred(B)))\}$  is an odd good directed  $ba$ -walk and STOP A;  
else for all bubbles  $\bar{B}$  such that  $B \cap \bar{B} = \emptyset$ ,  $t(\bar{B}) < t(B) \leq s(\bar{B}) < t(pred(B))$   
and  $WALK(\bar{B}, t(B))$  exists, do:  
if  $n(\bar{B}) = 0$  then  $n(\bar{B}) := n(B) + 1$ ,  $pred(\bar{B}) := B$ ,  $act(\bar{B}) := true$   
else if  $n(\bar{B}) \stackrel{2}{=} n(B) + 1$  and  $t(pred(\bar{B})) < t(B)$  then  $n(\bar{B}) := n(B) + 1$ ,  
 $pred(\bar{B}) := B$ ,  $act(\bar{B}) := true$   
if  $n(\bar{B}) \stackrel{2}{\neq} n(B) + 1$  then either  $\{b, \dots, pred(\bar{B}), WALK(\bar{B}, t(pred(\bar{B})))\}$   
or  $\{b, \dots, pred(\bar{B}), WALK(\bar{B}, t(B))\}$   
or  $\{b, \dots, pred(B), B, WALK(\bar{B}, t(pred(\bar{B})))\}$  is an odd good directed  $ba$ -walk and STOP A;  
 $act(B) := false$ ;

**until** all bubbles  $B$  with  $n(B) \neq 0$  have  $act(B) = false$  (STOP B : no odd good directed walk exists)

**end.**

This completes the description of the algorithm. It is not difficult to see that the algorithm is polynomial; in fact, each time  $act(B)$  is reactivated for

a bubble  $B$ , the variable  $t(pred(B))$  is increased. While  $act(B)$  is unchanged, the number of times  $B$  is considered in the algorithm is bounded by the total number of bubbles, which is quadratic.

Applying an algorithm solving the Parity Dipath Problem to a digraph obtained from digraph  $D$  by adding the arcs  $xy$  and  $yx$  we get the following consequence which will be used in the next section.

**Consequence.** Let  $D$  be a planar digraph which does not contain an even directed cycle. Let  $F$  be a facial cycle of  $D$  and let  $x, y$  be two vertices of  $F$ . Then there is a polynomial-time algorithm to decide whether  $D$  has an  $xy$ -dipath of even (odd respectively) length.

## 4 Even Cycle Problem in $K_{3,3}$ -free digraphs and in $K_5$ -free digraphs.

Several graph-theoretic problems, when restricted to the planar case, may be solved by polynomial-time algorithms and even efficient parallel algorithms. In some cases the algorithmic results proved for the class of planar graphs can be extended to the class of  $K_{3,3}$ -free graphs or to the class of  $K_5$ -free graphs.

Let us recall that we say that a digraph is  $H$ -free if its underlying graph does not contain a subgraph contractible to the graph  $H$ . There exists interesting decomposition theorems for  $H$ -free graphs when  $H = K_{3,3}$  or  $H = K_5$ . In the following we will explain how to use these decompositions for solving the Even Cycle Problem in these classes of digraphs.

Observe that there is no a priori algorithm to solve the Even Cycle Problem in  $K_{3,3}$ -free digraphs or  $K_5$ -free digraphs, once an algorithm to solve the Even Cycle Problem in planar digraphs is known. However such an extension exists via the Parity Dipath Problem.

Let us see first how to use the algorithm  $\mathcal{A}(D, F, x, y)$ , described in the previous section, to solve Even Cycle Problem in planar digraphs. Notice that whenever  $\mathcal{A}(D, F, x, y)$  is used to solve an instance of the Even Cycle Problem, the facial cycle  $F$  is a directed cycle.

**Function EVEN(D):boolean;**

**Begin**

$EVEN(D) := false;$

$D := D - \{\text{all arcs which do not belong to a directed cycle of } D\};$

$F :=$  a facial cycle of  $D$  which is a directed cycle;

if  $F$  has even length then  $EVEN(D) := true$  and STOP;

if  $D - F$  has a directed cycle then  $EVEN(D - F);$

**Repeat**

for each pair  $x, y \in F$  do  $\mathcal{A}(D, F, x, y);$

if  $\mathcal{A}(D, F, x, y) = \{P_1, P_2\}$  then either  $P_1 \cup F(y, x)$  or  $P_2 \cup F(y, x)$  is an even cycle. Then  $EVEN(D) := true;$

if  $\mathcal{A}(D, F, x, y) = \{P\}$  and  $P \cup F(y, x)$  is an even cycle then  $EVEN(D) := true ;$

**until**  $EVEN(D) = true;$

**if**  $EVEN(D) = true$  then **STOP**

**end.**

The recursive function  $EVEN(D)$  has true value if and only if there is an even directed cycle in  $D$ ; to see this, consider the situation that  $F$  is a directed facial cycle of  $D$  and  $EVEN(D - F) = false$ , i.e.  $D - F$  has no even directed cycle. Let  $C$  be an even directed cycle of  $D$ . Then  $C$  is formed by a nonempty subset of arcs of  $F$  and dipaths  $P_i(x_i, y_i)$ ,  $i = 1, \dots, n$  of  $D - F$  between pairs of vertices of  $F$ . Since  $D$  is planar and  $F$  has odd length, one of  $P_i(x_i, y_i)$  must have different length modulo 2 from the corresponding dipath  $F(x_i, y_i)$ . Hence the cycle  $P_i(x_i, y_i) \cup F(y_i, x_i)$  has even length. These cycles are those ones detected by  $EVEN(D)$ .

The complexity of  $EVEN(D)$  is polynomial provided that  $\mathcal{A}(D, F, x, y)$  is a polynomial-time algorithm.

Now, let us consider  $K_{3,3}$ -free digraphs.

It is a classical result of Hall ([3]) that each 3-connected component of a  $K_{3,3}$ -free graph is either planar or exactly the graph  $K_5$ .

Observe that if two planar components sharing a separating pair are glued together, the new graph is still planar. It follows that any  $K_{3,3}$ -free graph can be decomposed into *pieces* which are either planar or  $K_5$ 's. The *separation pairs* between two pieces are also separation pairs for the whole

graph. In [14], it is shown that this decomposition is unique and it does not depend on the order in which pieces are merged.

Vazirani ([14]) used this decomposition to obtain an NC algorithm for computing the number of perfect matchings of a  $K_{3,3}$ -free graph, more specifically for finding a Pfaffian orientation of a  $K_{3,3}$ -free graph.

The algorithm  $\mathcal{A}(D, F, x, y)$  may also be used as a subroutine for solving the Even Cycle Problem for  $K_{3,3}$ -free digraphs using a slightly modified decomposition.

A *double dipath* is obtained from a path by replacing each edge by two arcs with the same endvertices, oppositely directed, and by subdividing some arcs so that the resulting digraph contains no even directed cycle. Observe that given two numbers  $a, b$ , there is a double path  $P$  such that the two oppositely directed paths of  $P$  between its endvertices have length equal to  $a$  (and  $b$  respectively) modulo 2.

A digraph obtained from  $K_5$  by replacing each edge by a directed path or by a double dipath will be denoted by  $K'_5$ .

Let  $P$  be a directed path or a double dipath of a digraph  $D$ . Then  $P$  will be called *connection* of  $D$  if any arc containing an intermediate vertex of  $P$  belongs to  $P$ .

Let  $D$  be a  $K_{3,3}$ -free digraph. The description of our algorithm is simplified if we consider  $K'_5$ 's and planar digraphs as the basic pieces of the decomposition instead of  $K_5$ 's and planar digraphs.

The different pieces of the decomposition share connections of  $D$  (instead of arcs as in the Hall's decomposition) in a tree-like structure.

We will refer to leaves of the decomposition in an obvious way.

Finally observe that if  $D$  is planar, the endvertices of a connection of  $D$  belong to the same face.

### Algorithm ECP in $K_{3,3}$ -free digraphs.

**Input.** Let  $\mathcal{W}$  be the decomposition of  $D$  into planar digraphs and  $K'_5$ 's.

**Begin**

**while**  $\mathcal{W} \neq \emptyset$  **do**

    choose a leaf  $H$  of  $\mathcal{W}$ ; let  $x, y$  be the two endvertices of the connection separating  $H$  and its father  $f(H)$ ;

    if  $EVEN(H) = true$  then **STOP**

else test whether there exist two  $xy$ -dipaths (or  $yx$ -dipaths respectively) of different length modulo 2 in  $H$  (using  $\mathcal{A}(H, F, x, y)$  in case  $H$  is planar or checking all directed paths in case  $H$  is a  $K'_5$ );

if there are two  $xy$ -dipaths (or  $yx$ -dipaths respectively)  $P$  and  $P'$  of different length modulo 2 in  $H$ , then test for the existence of a  $yx$ -dipath (or  $xy$ -dipath respectively)  $\tilde{P}$  in  $G - H$ ;

if  $\tilde{P}$  exists then  $\tilde{P} \cup P$  or  $\tilde{P} \cup P'$  is an even dicycle and **STOP**;

if all  $xy$ -dipaths (and  $yx$ -dipaths respectively) have the same length  $l_{xy}$  ( $l_{yx}$  respectively) modulo 2 then replace  $H$  by a double dipath between  $x$  and  $y$  with its  $xy$ -dipath of length  $l_{xy}$  modulo 2 and with its  $yx$ -dipath of length  $l_{yx}$  modulo 2. Such a double dipath exists for all  $l_{xy}, l_{yx}$ . If the directed paths of  $H$  between  $x$  and  $y$  go in one direction (say from  $x$  to  $y$  only) then replace  $H$  by an  $xy$ -dipath of length  $l_{xy}$  modulo 2 instead of a double dipath.

Remove  $H$  from  $\mathcal{W}$ .

**end.**

Let us consider  $K_5$ -free digraphs and their decomposition.

Let  $H_1$  and  $H_2$  be two graphs with disjoint vertex sets. Let  $T_1 = \{x_1, \dots, x_k\}$  and  $T_2 = \{y_1, \dots, y_k\}$  be complete subgraphs of  $H_1$  and  $H_2$  respectively. The graph  $H_1 +_k H_2$  is obtained by removing some pairs of edges  $x_i x_j$  and  $y_i y_j$ , for  $1 \leq i < j \leq k$ , and by identifying  $T_1$  and  $T_2$ . The graph  $H_1 +_k H_2$  is called a  $k$ -sum of  $H_1$  and  $H_2$ .

Let  $V_8$  denote the graph with vertex set  $\{v_1, v_2, \dots, v_8\}$  and edge set  $\{v_i v_{i+1} : i = 0, 2, \dots, 7 \pmod{8}\} \cup \{v_0 v_4, v_1 v_5, v_2 v_6, v_3 v_7\}$ . Note that  $V_8$  contains no triangles.

Wagner ([16]) proved that:

**Theorem 4.1** *Every connected  $K_5$ -free graph may be obtained by means of 1-, 2- and 3-sums, starting from planar graphs and copies of  $V_8$ .*

A  $k$ -sum does not create new triangles. Hence different pieces of the decomposition are again joined in a tree-like structure and we will refer to leaves of the decomposition as in the previous algorithm. Also, we proceed in the same way: we need to treat a 2-sum and a 3-sum of a leaf and the rest of the digraph, where the leaf is either planar or  $V_8$ .

It is easy to adapt the algorithm developed for  $K_{3,3}$ -free digraphs to solve the problem when the operation is a 2-sum.

Hence let  $D$  be a  $K_5$ -free digraph, let  $x_1, x_2, x_3$  be vertices of  $D$  and let  $H_1, \dots, H_m$  be the leaves of the decomposition of  $D$  connected by  $x_1, x_2, x_3$  to their common father  $F$ . Since  $V_8$  has no triangles,  $H_1, \dots, H_m$  and  $F$  are planar digraphs. We may also assume that the triangle  $T = (x_1, x_2, x_3)$  is a facial cycle of each  $H_1, \dots, H_m$  and  $F$ : if  $T$  is not a facial cycle of one of them then we may split it into two parts, one consisting of the portion of digraph drawn 'inside of  $T$ ' and the other one consisting of the portion drawn 'outside of  $T$ '.

We proceed as follows.

*Test 1.* Test if  $H_1 \cup H_2 \cup \dots \cup H_m$  contains an even cycle.

*Test 2.* If Test 1 does not succeed then for each  $r, s \in 1, 2, 3$ ,  $r \neq s$ , test if there exist two  $x_r x_s$ -dipaths of different length modulo 2 in  $H_1 \cup H_2 \cup \dots \cup H_m$ , together with an  $x_s x_r$ -dipath in the rest of  $D$  disjoint with both of them (note that only the third connecting vertex needs to be considered in order to check disjointness). The algorithm  $\mathcal{A}(H_i, T, x_r, x_s)$  may be used to accomplish this.

*Test 3.* We consider all  $x_r x_s$ -dipaths of  $H_1 \cup H_2 \cup \dots \cup H_m$  that may be completed into a directed cycle by an  $x_s x_r$ -dipath of the rest of  $D$ . If Test 1 and Test 2 do not succeed then all these  $x_r x_s$ -dipaths have the same length  $l(rs)$  modulo 2, for all  $r \neq s$ ,  $r, s \in 1, 2, 3$  for which they exist. If  $l(rs)$  is defined and there is an  $x_r x_s$ -dipath of  $H_1 \cup H_2 \cup \dots \cup H_m$  which does not contain the third vertex of  $x_1, x_2, x_3$  then we call the ordered pair  $(r, s)$  *proper*. We delete  $H_1 \cup H_2 \cup \dots \cup H_m$  from the decomposition and we replace  $F$  by planar digraph  $F'$  which is obtained from  $F - (H_1 \cup H_2 \cup \dots \cup H_m)$  by adding a digraph  $H$  such that  $V(H) \cap V(F) = \{x_1, x_2, x_3\}$  and  $H$  contains the informations about the  $l(rs)$ 's and about which ordered pairs  $(r, s)$  are proper. In the next Proposition we prove that  $H$  always exists and we show how to construct it.

**Proposition 4.2** *There exists a planar digraph  $H$  such that  $x_1, x_2, x_3$  belong to the same facial cycle of  $H$ , and each  $rs$ -dipath of  $H$  has length equal to  $l(rs)$  modulo 2 and there is an  $rs$ -dipath in  $H$  which does not contain the third vertex of  $x_1, x_2, x_3$  if and only if  $(r, s)$  is proper in  $H_1 \cup H_2 \cup \dots \cup H_m$  for each ordered pair  $(r, s)$ ,  $r \neq s$ ,  $r, s \in 1, 2, 3$ .*

**Proof.** It is not difficult to observe that the thesis holds if at least one  $l(rs)$  is not defined. Hence let us assume that all  $l(rs)$ ,  $r \neq s$ ,  $r, s \in 1, 2, 3$ , are defined.

We start by constructing the  $rs$ -module as follows: let  $P$  be a path between  $x_r$  and  $x_s$  such that  $P$  consists of two edges if  $l(rs) = l(sr)(\text{mod}2)$  and  $P$  consists of three edges if  $l(rs) \neq l(sr)(\text{mod}2)$ . The  $rs$ -module is obtained from  $P$  by replacing each edge by two arcs with the same endvertices, oppositely oriented. The directed cycle of the  $rs$ -module incident with  $x_r$  or  $x_s$  is called  $x_r$ -cycle ( $x_s$ -cycle respectively) of the  $rs$ -module. We assume that the  $rs$ -module and the  $st$ -module both contain  $x_s$  and they are disjoint elsewhere.

To prove the Proposition we distinguish two cases.

First let  $l(rs) + l(st) \neq l(rt)$  for all  $r, s, t$  distinct,  $r, s, t \in 1, 2, 3$ . Then necessarily all ordered pairs  $(r, s)$  are proper. Let  $H'$  be the digraph obtained by identifying the  $x_2$ -cycle of the 12-module and the  $x_2$ -cycle of the 23-module. Then  $H$  may be obtained from  $H'$  by subdividing arcs.

Secondly let  $l(rs) + l(st) = l(rt)$  for some  $r, s, t$  distinct,  $r, s, t \in 1, 2, 3$ . Observe that  $r, s, t$  may be chosen so that the ordered pairs  $(r, s), (s, t)$  are proper. Without loss of generality assume  $r = 1, s = 2, t = 3$ .

If  $(3, 2), (2, 1)$  are proper and  $l(32) + l(21) = l(31)$  then we distinguish the following subcases.

If  $(1, 3)$  and  $(3, 1)$  are not proper then let  $H'$  be the union of the 12-module and the 23-module.

If both  $(1, 3), (3, 1)$  are proper then let  $H'$  be the union of the 12-module, the 23-module and the 13-module.

If  $(1, 3)$  is proper and  $(3, 1)$  is not proper then let  $H'$  be obtained from the union of the 12-module and the 23-module by adding the arc  $x_1x_3$ .

Similarly if  $(3, 1)$  is proper and  $(1, 3)$  is not proper then let  $H'$  be obtained from the union of the 12-module and the 23-module by adding the arc  $x_3x_1$ .

If  $(3, 2), (2, 1)$  are proper and  $l(32) + l(21) \neq l(31)$  then necessarily  $(3, 1)$  is proper. Let us subdivide the arc of the 12-module leaving  $x_2$  by vertex  $a$  and subdivide the arc of the 23-module entering  $x_2$  by vertex  $b$ . Let  $H''$  be obtained from the 12-module and the 23-module by identifying vertices  $a, b$ . If  $(1, 3)$  is not proper then let  $H' = H''$ , otherwise let  $H'$  be obtained from  $H''$  by adding the arc  $x_1x_3$ .

Finally let one of  $(3, 2), (2, 1)$  be not proper. Then necessarily  $(3, 1)$  is proper. If neither of  $(3, 2), (2, 1), (1, 3)$  is proper then let  $H'$  be the directed triangle  $(x_1, x_2, x_3)$ .

If neither  $(3, 2)$  nor  $(2, 1)$  is proper and  $(1, 3)$  is proper then let  $H'$  be obtained from the 13-module by adding the arcs  $x_1x_2$  and  $x_2x_3$ .

Otherwise assume without loss of generality that  $(3, 2)$  is not proper and  $(2, 1)$  is proper. The case that  $(1, 3)$  is proper is analogous to the cases discussed above. Hence let  $(1, 3)$  be not proper.

If  $l(23) + l(31) = l(21)$  then let  $H'$  be obtained from the 12-module by adding the arcs  $x_2x_3$  and  $x_3x_1$ . Otherwise let  $H'$  be obtained from the directed triangle  $(x_1, x_2, x_3)$  by subdividing arc  $x_2x_3$  by vertex  $a$ , subdividing arc  $x_3x_1$  by vertex  $b$  and by identifying  $a$  and  $b$ .

In all the cases we discussed,  $H$  may be obtained from  $H'$  by subdividing arcs.

This completes the proof.  $\square$

It follows that the algorithm  $\mathcal{A}(D, F, x, y)$  may also be used as a subroutine for solving the Even Cycle Problem for  $K_5$ -free digraphs using the Wagner's decomposition.

## 5 Remarks

A polynomial-time algorithm for solving the more general *Modularity Dipath Problem* where the lengths of dipaths are computed modulo an integer  $q \neq 2$  has been provided in [1]. Hence, the problem to decide whether a  $K_{3,3}$ -free digraph or a  $K_5$ -free digraph has a directed cycle of length different from  $p \pmod q$  is polynomially solvable.

The polynomial-time solvability of the Parity Dipath Problem suggests that the problem of deciding whether a planar digraph has a directed path of even/odd length between two specified vertices of the same facial cycle might be polynomially solvable.

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